



THE
ELEMENTS
OF THAT
Mathematical Art
COMMONLY CALLED
ALGEBRA,
Expounded in Four BOOKS.

By JOHN KERSEY.

*Nil tam difficile est, quod non solertia vincat.
Dimidium facti, qui bene cœpit, habet.*



L O N D O N :

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TO
ALEXANDER DENTON
Of Hillesdon in the County of Bucks, Esquire,

AND
MR EDMUND DENTON
His Brother;

The hopeful Blossoms, and only Offspring of the
Truly Just and Vertuous

EDMUND DENTON Esq;

Son and Heir of

S^r ALEXANDER DENTON Kn^t.

A faithful Patriot, and eminent Sufferer in our late
Intestine Wars, for his Loyalty to His late MAJESTY

King **CHARLES** the First,

Of Ever-Blessed Memory:

J O H N K E R S E Y,

In testimony of his Gratitude, for signal Favours
conferr'd on him by that truly Noble Family;

Which also gave both Birth and Nourishment
to his *Mathematical* Studies,

HUMBLY DEDICATES
His Labours in this Treatise of the ELEMENTS
OF THE
ALGEBRAICAL ART.

THE P R E F A C E



It is an undoubted truth, that among all Humane Arts and Sciences, *ARITHMETICK* and *GEOMETRY* have obtain'd the greatest evidence of Certainty. This Prerogative results from the Verity and Perspicuity of their Principles; which consist of *Definitions*, *Postulates*, (or *Petitions*), and *Axioms*; for these being intelligible, reasonable and certain, are universally assented to as pure Fountains of Knowledge, and sure Foundations of right Reasoning, by all judicious and impartial Students in Sciences. Hence it is, that all Propositions which are proved by those certain Principles are likewise certain, and called Demonstrative Truths; by which are meant strictly and properly, infallible Consequences, or Conclusions, deduced from clear and undeniable Premises. For which cause, divers Philosophers have endeavour'd, as far as the quality of their Discourses would admit, to make the force of their Arguments amount to Mathematical Demonstration, which, by universal consent of the Learned, is the clearest and most convincing Proof, of the Truth of a Proposition, that can possibly be given by Humane reasoning. Nor was it without Reason, that the Ancients, (as many of the Learned affirm,) taught their Scholars *Arithmetick* and *Geometry*, next after the Rudiments of Letters, as expedients to take off their minds from Levity, and to render them capable of sound Judgement, before they sett'd upon the Study of Philosophy: Which Method of Schooling was in great esteem with *Plato*, (as his Book of *Common-weal* testifies;) who was of Opinion, That ingenious and pregnant Proficients in *Arithmetick* were apt to learn any Arts whatsoever, and he permitted no Student that was ignorant of *Geometry* to enter into his School.

This also may be added concerning the Excellency of those Twin-like Arts or Sciences, That they depend not upon any other Sciences, either for Help or Demonstration, nor do they owe their Dignity to the Suffrage or Vote of our Senses, which oft-times deceive us; but since *Quantity*, about which *Arithmetick* and *Geometry* are conversant, may be consider'd abstractively, and separate from all kind of Matter, the Verity of their Propositions is examined and proved in the Mind only; where, among all the Exercises that conduce to the search of Truth, none are found so pure, clear and comprehensible, (right Reason being Judge,) as *Arithmetick* and *Geometry*; thence they are called Pure Mathematicks, and are properly to be learnt before any of the rest of the Mathematical Arts.

Nor are *Arithmetick* and *Geometry* excellent in themselves only, but, highly esteem'd also for their manifold Utility, as well in the Employments of Men about Accompts, Trade, Building, Measuring of Land, and divers other common Affairs, as in facilitating and enlivening divers other Noble Arts; for how can Harmonical Composition in *Musick*, or exact Measure, and Proportion in *Painting* be perform'd, without the assistance of *Arithmetick* and *Geometry*. Besides, these Sciences, (as the Mathematician very

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well knows,) like the two Pillars, *Jachim* and *Booz*, in the Porch of *Solomon's Temple*, are the stability and support of all the rest of the Mathematical Arts; for if *Astronomy*, *Navigation*, *Dyalling*, *Opticks*, *Fortification*, and the rest of the Arts called Mixed Mathematicks; be stript of the Demonstrations and Operations imparted to them by *Geometry* and *Arithmetick*, that which remains will be as barren as the Earth without the Influence of the Sun, and as unactive as a humane Body without a reasonable Soul.

The premises may suffice to give a hint of the Excellency and Utility of *Arithmetick* and *Geometry*, whence we may reasonably infer; First, that so great and so profitable a Subject is worthy of the Study of all ingenious Minds, in a degree proportional to their respective Stations or Employments, as well for promoting their own, as the Publick Good. Secondly, that that Art which by a more easie, and not less sure Method than that called *Synthetic*, finds out the Solutions and Demonstrations of the more knotty Propositions, as well *Geometrical* as *Arithmetical*; (and oftentimes by the way too, discovers unexpected and admirable Speculations,) may very well deserve the Enquiry of such Lovers of Art as have hours to spare, and are desirous to be acquainted with the choicest pieces in the Common-wealth of Learning: But such an Art is that commonly called *ALGEBRA*, which first assumes the Quantity sought, whether it be a Number or a Line in a *Question*, as if it were known, and then, with the help of one or more Quantities given, proceeds by undeniable Consequences, until that Quantity which at first was but assumed or supposed to be known, is found equal to some Quantity certainly known, and is therefore known also.

Which Analytical way of Reasoning produceth in Conclusion, either a *Theorem* declaring some Property, Proportion or Equality, justly infer'd from things given or granted in a *Proposition*, or else a *Canon* directing infallibly how that may be found out or done which is desired; and discovers Demonstrations of the certainty of the resulting *Theorem* or *Canon*, in the *Synthetic* Method, or way of Composition, by the Steps of the *Analysis*, or *Resolution*. These are but glances of the many Rare Effects produced by the *Analytick* or *Algebraick* Art, which is an inexhaustible Fountain of *Theorems*, a Key truly golden for the unlocking of *Problems* as well *Geometrical* as *Arithmetical*; and not only a sure, but delightful Guide to such Students, who not being satisfied with a bare knowledge of the Truth or practical Use of those sublime Inventions that have rendred the antient Mathematicians so venerable, are desirous to know how they were found out, and how to prosecute their search of Truth, so, as to advance Knowledge upon Solid Foundations.

But the Excellency of the *Algebraical* Art is best known to those that are acquainted with the most eminent Writers upon that Subject; among which, these are deservedly Famous, namely, *Diophantus* of *Alexandria*, (the first Inventor of this rare Art, as some by his Preface to *Dionysius* do conjecture, but others give the Honour of that Noble Invention to *Geber* an Arabian Astronomer, whence, as is conceived, the word *Algebra* took rise,) *Cardanus*, *Tartaglia*, *Clavius*, *Stevinus*, *Vieta*, (the first Inventor, or at least the happy Restorer of *Specious*, or *Literal Algebra*, so called, because it operates chiefly by Alphabetical Letters,) Mr. *William Oughtred*, (our learned Country-man,) whose *Clavis Mathematicæ*, for Solid matter,

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neat Contractions, and succinct Demonstrations, is hardly to be parallel'd,) Mr. *Thomas Harriot*, (another learned Mathematician of our Nation,) *Ghetaldus*, *Andersonus*, *Bachetus*, *Herigonius*, *Cartesius*, *Fran. van Schooten*, *Florimond de Beane*, *Hugenius*, *Huddenius*, *Sylus*, *Fermatius*, *Billius*, *Rhenaudin*, and many others too numerous to be here recited, but to bring up the Rear of these renowned Analysts, I shall mention four more of our own Nation, and now living, (whose pardon I humbly begg for this my boldness,) namely, the Right Reverend Father in God, *Seth*, Lord Bishop of *Sarum*, Dr. *John Wallis*, Professor of *Geometry* in the University of *Oxford*, Dr. *Isaac Barrow* Master of *Trinity-Colledge* in *Cambridge*, and one of His Majesties Chaplains, and Dr. *John Pell*; the learned Works of which four Worthies proclaim their rare Talents in Universal Mathematicks.

Now because this excellent Art is but very sparingly treated of in our native Language, and since according to the old Maxim, *Bonum quod communius est melius*, Good the more common the better it is, I have, in imitation of the industrious Bee that gathers Honey from various Flowers, yet without any diminution either of their Beauties or Virtues, extracted out of the before-mentioned Authors, this Treatise consisting of Four Books, (the Two first of which are Printed, a good progress made in the Third, and the Fourth ready for the Press,) and have design'd it chiefly to give such of my Mathematical Country-men as are altogether strangers to, and desirous to be acquainted with the so much celebrated Art called *Algebra*, a plain and intelligible Introduction to its Doctrine, as also a considerable taste of its Use, in finding out *Theorems* and solving *Problems*, as well *Arithmetical* as *Geometrical*.

And here, to avoid the stain of Ingratitude, I cannot but declare to the World, that my old and much respected Friend, Mr. *John Collins*, a person well known to be both singularly skilfull in, and an industrious Promoter of the Mathematicks in general, hath been a principal Instrument of bringing this Work to light, as well by animating me to Compile it, as by endeavouring to procure it to be well Printed.

To conclude, I have earnestly endeavoured to render the Fundamentals, and most important Rules of the *Algebraical* Art in both kinds, to wit, Numeral and Literal, very clear and easie to capacities competently exercis'd in the Elements of *Arithmetick* and *Geometry*. And the favourable acceptance, which my Additions to Mr. *Wingate's* Treatise of Common *Arithmetick* have found, with divers eminent Mathematicians and other Lovers of Art, doth encourage me to hope, that the younger Students of Symbolical *Arithmetick* and *Analytical* Doctrine, will be well pleas'd with the following Discourse, and that my Labours therein will be as candidly accepted, as they have been cordially intended to serve my Native Country.

*From my House at the Sign of
the Globe in Shandois-Street,
in Covent-garden, the 15th
day of April, 1673.*

John Kersey.



A T A B L E O F

The Contents of the First and Second BOOKS of this TREATISE.

The Contents of the First Book.

Chap.

1. **D**efinitions, concerning the Nature, Scope, and kinds of Algebra: The Constitution of Collick Quantities or Powers; with the manner of expressing them by Alphabetical Letters: The signification of Characters used in the First Book.
2. Addition,
3. Subtraction,
4. Multiplication,
5. Division,
6. The like Operations in Algebraick Fractions.
7. The Rule of Three in Quantities represented by Letters;
8. An Introduction to the Extraction of Roots out of Algebraick Quantities; the compleat Doctrine thereof being delivered in the first, second, third and fourth Chapters of the Second Book,
9. How by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.
10. A Collection of easie Questions to exercise the preceding Rules.
11. Concerning an Equation, and the Reduction of Equations.
12. The use of the Reductions in the foregoing Chap. 11.
13. The manner of converting Analogies into Equations, and Equations into Analogies.
14. The Resolution of Simple Equations exercis'd in 28. Questions;
15. Concerning the Resolution of such Compound Equations wherein there are two different Powers of the Quantity sought, and those Powers such, that the higher of them is a Square, whose Side or Square Root is the lower Power.
16. The Equations of the foregoing Chap. 15. are exercis'd in 28. Questions, resolv'd as well by Numerals as Literal Algebra.
17. Of Arithmetical Progression, where Mr. Oughtred's Twenty Questions upon this Subject are explained.

The

The CONTENTS.

The Contents of the Second Book.

Chap.

1. **C**oncerning the Genesis or Procreation of Powers from Roots Binomial, Trinomial, &c.
2. Concerning the Composition of Powers in numbers, from a Binomial Root.
3. The extraction of all kinds of Roots out of Powers given in numbers.
4. The extraction of Roots out of Powers express'd by Letters.
5. Concerning Geometrical Proportion.
6. Various Theorems about Quantities in Continual Proportion.
7. Twenty Questions about Quantities in Continual Proportion, resolv'd by Literal Algebra.
8. The manner of finding out all the Aliquot Parts both of Numbers and Algebraical Quantities; as also the smallest Numbers that shall have given multitudes of Aliquot Parts.
9. The Arithmetick both of Surd Numbers and Surd Quantities express'd by Letters. The Constitution and Invention of six Binomials in numbers, agreeable to those expounded in the 10th. Book of Euclid's Elements; with Rules to extract the Square Root out of every one of them; as also, what Root you please out of any Binomial in numbers, having such a Binomial Root as is desired.
10. An Explication of Simon Stevin's general Rule to extract one Root out of any possible Equation in numbers, either exactly, or very nearly true.
11. Extractions out of the Algebraical Treatises of Vieta and Renates des Cartes, concerning the Constitution and Resolution of Compound Equations in numbers, especially those which have many Roots: where also, the rise of two Rules, the Invention whereof Cardanus attributes to Scipio Ferreus, concerning the Resolution of certain Cubick Equations in numbers; is clearly exhibited.
12. Of the Method of resolving Questions wherein many Quantities are sought, by assuming different Letters to represent the said Quantities severally.
13. Concerning the Resolution of such Arithmetical Questions as are capable of innumerable Answers.

ERRATA.

ERRATA.

Such hath been the exact care of the Printer, that the Faults of importance escaped in this Impression of the First and Second Books are only these fourteen.

Page, Line.	Faults,	thus to be corrected.
25 13	— 3ddd	— 3ddde
31 16	(By $a-b$)	(By $a-b$)
52 7	whole	wholes, or totals.
64 22	19	9
64 32	not exceed	be less than
77 37	Squares the Terms	Squares of the Terms
114 23	<i>Sect. 12. Chap. 2, 3, 5.</i>	<i>Sect. 2, 3, 5. Chap. 12.</i>
147 34	} a single Character	the greatest single Character
152 6		
227 4		
253 37	second	first
281 20	Proportionals	Proportionals
	<i>Chap. 15.</i>	<i>Chap. 16.</i>
300 17	444 } $\rightarrow 40$	444 } $\rightarrow 40$
	$\rightarrow 10$	$\rightarrow 10$
317 34	seventh	seventeenth
346 26	6	66

THE

Chap. 1.

A Treatise of the ELEMENTS OF THE Algebraical Art.

Book I.

CHAP. I.

*Concerning the Nature, Scope, and Kinds of ALGEBRA:
The Construction of Collick Quantities, or Powers, with
the manner of expressing them by Alphabetical Letters: The
signification of Characters used in the First Book.*



THE Mathematical Arts or Sciences are exercis'd about *Quantity*, which is compris'd under Numbers, Lines, Superficies, and Solids: These, if they be considered abstractively, and separate from all kind of Matter, are the proper Objects of *Arithmetick* and *Geometry*, which are call'd *Pure Mathematicks*.

II. The *Method* which Mathematicians are wont to use in searching out truth about Quantity, is twofold; *viz.* 1. *Synthetical*, or by way of Composition: 2. *Analytical*, or by way of Resolution.

III. Mathematical Composition, or the *Synthetical method*, argues, together with known Quantities to search out unknown, and then demonstrates that the Quantity found out will satisfy the Proposition.

IV. Mathematical Resolution, or the *Analytical Art*, commonly call'd *Algebra*, is that way of reasoning which assumes or takes the Quantity sought as if it were known, or granted; and then with the help of one or more Quantities given or known, proceeds by Consequences, until at length the Quantity first only assumed or feigned to be known, is found equal to some Quantity or Quantities certainly known, and is therefore likewise known.

V. The Scope, Drift or Office of the Analytick or Algebraick Art, is to search out three kinds of Truths, *viz.*

1. *Theorems*, which are nothing else but Declarations, or Affirmations of certain Properties, Proportions, or Equalities, justly infer'd from some Suppositions or Concessions about Quantity: Which Theorems are to be reserved in store, as ready helps to find out new, and to confirm old Truths: This kind of Resolution when it rests in a bare Invention of Truth, is call'd *Contemplative*, or *Notional*.

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21. Canon.

2. *Canons*, or infallible Rules, to direct how to solve knotty Questions, by the help of Quantities given or known, this kind of Resolution is called *Problematical*.

3. *Demonstrations*, or evident and indisputable Proofs, to manifest the truth of such Theorems and Canons as are Analytically found out.

VI. *Algebra* is by late Writers divided into two kinds, *to wit*, *Numeral*, and *Literal*, (or *Specious*.)

VII. *Numeral Algebra* is so called, because in this Method of resolving a Question, the Quantity sought or unknown is solely design'd or represented by some Alphabetical Letter, or other Character taken at pleasure, but all the Quantities given are express'd by Numbers.

VIII. *Literal*, or *Specious Algebra* is so called, because in this method of resolving a Question, as well the given or known Quantities as the unknown are all severally expressed or represented by Alphabetical Letters. Whence it comes to pass, that at the end of the Resolution of a Question, every Quantity appearing distinct under the same Letter or Form by which it was at first expressed, a *Canon* is discovered to direct how the Question propos'd may be solved, not only by the quantities first given, but by any other whatsoever that are capable of solving the Question. In this respect therefore *Literal Algebra* far exceeds the *Numeral*, for this latter serves only to solve *Arithmetical Questions*, and produceth not a *Canon* without much difficulty, in regard the numbers first given, by reiterated Multiplications, Divisions and other Arithmetical operations, will for the most part be so confounded and interwoven, that their footsteps can hardly be traced out: But *Literal* or *Specious Algebra* is applicable to the solving of *Geometrical Problems*, as well as *Arithmetical*.

IX. The *Doctrines of Algebra* is principally grounded upon the knowledge of certain Quantities called by some Authors *Cosick Quantities*, by others, *Powers*, the Construction whereof is explain'd in six Sections next following.

X. Numbers are said to be in *Geometrical Proportion continued*, when as the first is to the second, so is the second to the third, and so is the third to the fourth, &c. As, for Example, these Numbers, 1, 2, 4, 8, 16, 32, &c. are Continual Proportionals; for, as the first Term 1, is the half of the second Term 2, so is the second Term 2, the half of the third Term 4, and so is 4 the half of 8, &c. Likewise these Numbers, 3, 9, 27, 81, 243, &c. are in Geometrical Proportion continued; For as the first Term 3 is a third part of the second Term 9, so is the second Term 9 a third part of the third Term 27, and so is 27 one third of 81, &c. Also, these Numbers are Continual Proportionals, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. for as the first Term 1, is the double of the second Term $\frac{1}{2}$, so is $\frac{1}{2}$ the double of $\frac{1}{4}$, and $\frac{1}{4}$ the double of $\frac{1}{8}$, &c.

XI. In any series or rank of Numbers proceeding from Unity in a continued Geometrical proportion, whether ascending or descending, all the Numbers or Terms except the first, which is supposed to be 1, (to wit, Unity,) are called *Cosick Numbers*, or *Powers*; viz. the second Term or Proportional is called the *Root*, or first Power; the third Proportional is called the *Square*, or second Power; the fourth Proportional is called the *Cube*, or third Power; the fifth Proportional is called the *Biquadrate*, or fourth Power, the sixth Proportional, the fifth Power, &c. As for Example, in this rank of Continual Proportionals, 1, 2, 4, 8, 16, 32, &c. the second Term 2 is the Root; the third Term 4 is the second Power, or the Square of the Root 2; the fourth Term 8 is the third Power, or the Cube of the Root 2; the fifth Term 16 is the Biquadrate or fourth Power of the same Root 2, &c.

In like manner in this rank of Continual Proportionals descending from 1, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. the second Term $\frac{1}{2}$ is the Root; the third Term $\frac{1}{4}$ is the second Power; the fourth Term $\frac{1}{8}$ is the third Power, &c. The like is to be understood of any other Rank of numbers in a continued Geometrical proportion, whose first Term or Proportional is Unity.

XII. From the two last preceding Sections, (which are grounded upon 10. Prop. 8. Elem. Euclid.) it is evident that any Number whatsoever being propos'd for a Root, the second Power, or the *Square*, is produced by the multiplication of the Root by it self; the third Power, or the *Cube*, is produced by the multiplication of the second Power by the Root; the fourth Power is produced by the multiplication of the third Power by the Root, &c.

As, for Example, if 2 be given for the Root, this 2 multiplied by it self, produceth 4 for the second Power, to wit, the Square of the Root 2: Again, 4 the second Power

being

being multiplied by the Root 2 gives 8 the third Power, or the Cube; which third Power multiplied by the Root 2, produceth the fourth Power 16, &c.

In like manner, if this Fraction $\frac{1}{2}$ be prescribed for a Root, by multiplying $\frac{1}{2}$ by it self there comes forth $\frac{1}{4}$ for the second Power, or the Square of the Root $\frac{1}{2}$; Again, the second Power $\frac{1}{4}$ multiplied by the Root $\frac{1}{2}$ produceth the third Power $\frac{1}{8}$, or the Cube of the Root $\frac{1}{2}$; and the third Power $\frac{1}{8}$ multiplied by the Root $\frac{1}{2}$ gives the fourth Power $\frac{1}{16}$, &c.

But when the Root is 1, to wit, Unity, every one of its Powers will also be 1; for multiplication by 1 makes no alteration. All which will be further illustrated by the Scales of Cosick numbers or Powers in the following Table, which shews that if the Root be 5, the Square is 25, the Cube 125, the Biquadrate or fourth Power 625, the fifth Power 3125, &c.

A Table of Powers in Numbers.

The Root or first Power.	1	2	3	4	5
The Square or second Power.	1	4	9	16	25
The Cube or third Power.	1	8	27	64	125
The Biquadrate or fourth Power.	1	16	81	256	625
The fifth Power.	1	32	243	1024	3125
The sixth Power.	1	64	729	4096	15625
The seventh Power.	1	128	2187	16384	78125
The eighth Power, &c.	1	256	6561	65536	390625

XIII. The Root or first Power being given, the third, fifth, eighth, or any other Power may be found out without respect to the intermediate Power or Powers, in this manner, viz. Suppose the number 3 be prescribed for the Root, and that the fifth Power be desired: first write down the Root 3 five times thus, 3, 3, 3, 3, 3; then multiply these five equal numbers one into another according to the Rule of continual Multiplication, so the last Product 243 shall be the desired fifth Power raised from the Root 3.

In like manner, if the eighth Power of the Root 2 be desired, you may write the Root 2 eight times thus, 2, 2, 2, 2, 2, 2, 2, 2, these multiplied continually produce 256, which is the eighth Power of the Root 2. After the same manner you may find out any other Power from a number given for the Root.

XIV. If over or under any Series or Rank of Cosick numbers or Algebraick powers, constituted according to the three last foregoing Sections, there be placed a rank of Numbers beginning with Unity, and proceeding according to the natural order of numbers, as 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. these numbers so placed are usually called the *Indices*, or *Exponents* of those Powers, as well because they shew the order, seat, or place of each Power, as also its number of Degrees or Dimensions; that is, how many times the Root is involved or multiplied in producing each Power respectively: As for Example, let there be a rank or Scale of Algebraick powers raised from the root 3, as 3, 9, 27, 81, 243, 729, 2187, &c. and over them let there be so many numbers placed in an Arithmetical progression, beginning with 1, and proceeding according to the natural order of Numbers, as here you see:

INDICES.

1	2	3	4	5	6	7	8	&c.
3	9	27	81	243	729	2187	6561	&c.

POWERS.

I say the Index 4 in the Arithmetical progression, shews that the fourth Power 81, which stands under 4, is produced by the multiplication of the Root 3 four times into it self, viz. these four numbers 3, 3, 3, 3, multiplied continually will produce 81; likewise the Index 7 in the Arithmetical progression shews, that the seventh Power 2187, which stands under 7, is produced by the multiplication of the Root 3 seven times into it self, viz. these seven equal numbers, 3, 3, 3, 3, 3, 3, 3, multiplied continually produce 2187. And so of others.

To that use of *Indices*, this may be added; viz. If any two or more Indices be added together, the sum will be an Index shewing what Power will be produced by the multiplication of those Powers one into another which answer to the Indices that were added together: As for Example, if the Indices 3 and 5 be added together, the sum is the Index 8; which shews, that if the third and fifth Powers be multiplied one by the other the eighth Power will be produced: As in the rank of Powers in the preceding Table, if the third Power 27 be multiplied by the fifth Power 243, the Product will give the eighth Power 6561. In like manner, so far as the Indices 2 and 6 added together make the Index 8; therefore the second Power 9 multiplied by the sixth Power 729 will also produce the eighth Power 6561: Again, because the Indices 1, 2, and 5 added together make the Index 8; therefore the first, second and fifth Powers, to wit, 3, 9, and 243 multiplied continually will likewise produce the eighth Power 6561. And as the Index 3 added to it self makes the Index 6, so the third Power 27 multiplied by it self, or squared, will produce the sixth Power 729.

And as the Addition of Indices answers to the Multiplication of their correspondent Powers, so the subtraction of Indices answers to the division of their correspondent Powers: As, for example, because the Index 8 lessened by the Index 5, leaves for a Remainder the Index 3; therefore the eighth Power 6561 divided by the fifth Power 243 gives in the Quotient the third Power 27. Likewise, as the Index 7 lessened by the Index 3 leaves the Index 4; so the seventh Power 2187 divided by the third Power 27, gives the fourth Power 81.

XV. From the premises it is evident, that upon an Arithmetical foundation, a Scale or Rank of Algebraick Powers may be raised and continued as far as you please; the three first of which have an affinity with, and may be expounded by Geometrical dimensions: For first, we may conceive any terminated Right-line to be divided into a number of equal parts at pleasure, suppose 12; then this number 12, or that Right-line, may be esteemed as a Root: Secondly, the said 12 multiplied by it self produceth 144 the second Power, which is equal to the Area of a square Superficies whose side is 12: Thirdly, the said second Power 144 multiplied by the Root 12 produceth the third Power 1728, which is equal to the Solid content of a Cube, (to wit, a Solid in the form of a Dye) whose side is 12.

But none of the rest of the Algebraick powers can properly be explain'd by any Geometrical quantity, in regard there are but three dimensions in Geometry, to wit, Length, Breadth, and Depth (or Thickness).

XVI. In searching out the solution of a Question by the Algebraick Art, the number or line sought is usually called a *Root*, which so long as it remains unknown cannot be really express'd, and therefore it must be design'd or represented by some Symbol or Character, at the will of the Artist; also the Powers which may be imagined to proceed from the said Root in such manner as hath before been declared are likewise to be represented by Symbols or Characters; concerning which there is much diversity among *Algebraical* Writers, every one pleading his fancy in the choice of Characters: But in this matter I shall imitate Mr. Thomas Harriot in his *Art Analytica*, and Renates des Cartes in his *Geometria*, but chiefly the former; whose method of expressing Quantities by alphabetical Letters, I conceive to be the plainest for Learners, viz.

To design or represent the Root sought, whether it be a Number or a Line in a Question propounded, we may assume any Letter of the Alphabet, as *a, b, c, &c.* but for the better distinguishing of known quantities from unknown, some *Analysts* are wont to assume one of the five Vowels, as *a, e, &c.* to represent the quantity sought; and Consonants, as *b, c, d, &c.* to represent quantities known or given: Now if the letter *a* be assumed to represent the Root sought, then (according to Mr. Harriot) the second Power, or the Square raised from that Root, may be represented by *aa*; the third Power, or the Cube, by *aaa*; the fourth Power by *aaaa*, the fifth Power by *aaaaa*; and after the same manner

manner

manner any higher Power of the Root or number *a* may be represented: For so many Dimensions or Degrees as are in the Power, so many times the Letter which at first was assumed for the Root is to be repeated.

Or after the manner of *Renates des Cartes*, if the letter *a* be assumed to represent the Root, the Square may be design'd thus, *a²*, the Cube thus, *a³*, the fourth Power thus, *a⁴*, the fifth Power thus, *a⁵*. And so any other Power may be express'd by writing the Index or Exponent of the Power in a small figure next after, and near the head of the letter assumed to represent the Root. Both which ways will be further illustrated by the following Table.

A Table shewing two ways (now most in use) to express simple Powers by Alphabetical Letters.

The Root or first Power,	<i>a.</i>	<i>a</i>
The Square or second Power,	<i>aa.</i>	<i>a²</i>
The Cube or third Power,	<i>aaa.</i>	<i>a³</i>
The fourth Power,	<i>aaaa.</i>	<i>a⁴</i>
The fifth Power,	<i>aaaaa.</i>	<i>a⁵</i>
The sixth Power,	<i>aaaaaa.</i>	<i>a⁶</i>
The seventh Power,	<i>aaaaaaa.</i>	<i>a⁷</i>
The eighth Power,	<i>aaaaaaaa.</i>	<i>a⁸</i>

After the same manner, known Quantities and their Powers may be represented by Consonants; as, *b* may be put for any known number in a Question, and then its Square may be signified by *bb*, the Cube by *bbb*, the fourth Power by *bbbb*, the fifth Power by *bbbbb*, the sixth by *bbbbbb*, and so forwards: Or the Square of the Root *b* may be express'd thus, *b²*, the Cube thus, *b³*, the fourth Power thus, *b⁴*, the fifth Power thus, *b⁵*, the sixth Power thus, *b⁶*, and so forwards.

XVII. Numbers set before, that is, on the left hand of quantities express'd by letters are called Numbers prefix; but if no number be prefix to the letter, then 1 or unity must be imagined to be prefix: As, in these quantities *a*, (or *1 a*), *2 a*, *3 a*, $\frac{1}{2} a$, $\frac{2}{3} a$, *5bbb* (or *5 b³*) the numbers prefix are (as you see) 1, 2, 3, $\frac{1}{2}$, $\frac{2}{3}$, and 5, every one of which numbers (and the like so prefix) shews how often the quantity represented by the letter or letters immediately following the number is taken; so *a* or *1 a* signifies some number or line once taken, also *2 a* represents the double, $\frac{1}{2} a$ the half, and $\frac{2}{3} a$ two third parts of the number or line represented by *a*. In like manner *5bbb*, or *5 b³*, signifies that the Cube of the number or line represented by *b* is taken five times.

XVIII. All numbers express'd by figures and cyphers (as in vulgar Arithmetick) not having any letter or letters annexed to them, are for distinction sake called Absolute numbers; as these numbers, 5, 20, 105, $\frac{1}{2}$, $\frac{2}{3}$, and all others when they be not prefix or annex to any letter or letters are called Absolute numbers.

XIX. All *Algebraical operations* are perform'd in an Arithmetical manner, partly in the vulgar way by numbers, and partly by Alphabetical letters, in all the parts of Arithmetick, to wit, Addition, Subtraction, Multiplication, Division, and the Extraction of Roots: But since letters cannot be disposed like numbers to perform those operations, some Characters must of necessity be used to signify such operations. The Characters used in this first Book are explained in the following Sections.

XX. This Character $+$ is a sign of *Affirmation*, as also of *Addition*, and always belongs to the quantity that follows the sign; as, $+$ affirms the quantity denoted by *a* to be real, or greater than nothing; the like may be said of $-b$, and $+c$, &c.

When no sign is prefix before a quantity, the sign $+$ is always to be understood, and must be imagined to be prefix; so *a* implies $+$ *a*, likewise *2 b* signifies the same thing with $+$ *2 b*, the like of others.

But when the sign $-$ is placed between two quantities, it imports as much as the word *plus*,

plus, or more, and signifies that those quantities are added or to be added together: As $3+4$. (or 3 more 4.) signifies the sum of 3 and 4; or it hints that 4 is to be added to 3. In like manner $a+b$ signifies the sum of numbers or quantities represented by a and b ; and $a+b+c$ signifies the sum of quantities denoted by a , b , and c .

XXI. This Character $-$ is a sign of *Negation*, as also of *Subtraction*, and always belongs to the following quantity; as for Example, -5 is a fictitious number less than nothing by 5; viz. as -5 l. may represent five pounds in money, or the Estate of some person who is clearly worth five pounds; so -5 l. may represent a Debt of five pounds owing by some person who is worse than nothing by five pounds.

But when the sign $-$ is placed between two quantities, it imports as much as the word *minus*, or *less*; and intimates that the number or quantity following that sign is subtracted or to be subtracted from the number or quantity that stands next before the same sign: As $8-3$ (or 8 less 3) signifies that 3 is subtracted or to be subtracted from 8; or $8-3$ denotes the excess of 8 above 3, to wit, 5.

In like manner $a-b$ (or a less b) signifies that the quantity denoted by b is subtracted or to be subtracted from the quantity a ; or $a-b$ may signify the excess of the quantity a above the quantity b .

XXII. This Character ω signifies the *Difference* of two quantities, to wit, the excess of the greater above the less, when 'tis not determin'd or known in which of those quantities the excess lyeth; so $a\omega b$ signifies the difference of two quantities represented by a and b , when 'tis not known whether a be greater or less than b .

XXIII. This Character \times is a sign of *Multiplication*, and is put for the word *into*, or *by*, viz. when 'tis set between two quantities it signifies that they are multiplied, or to be multiplied mutually one by the other: As, 6×3 (or 6 into or by 3) imports the Product of the multiplication of 6 by 3, to wit, 18.

In like manner $a \times b$ signifies that the quantity represented by a is multiplied or to be multiplied by the quantity b ; also $a \times b \times c$ signifies the Product made by the continual multiplication of the quantities a , b , and c , one into another.

But for the most part the Multiplication of quantities denoted by letters is signified by the joining of letters together, like letters in a word; as ab signifies the Product of the multiplication of the quantity a by the quantity b . Also abc signifies the Product of the continual multiplication of the quantities a , b and c one into another: All which will be further illustrated in Chap. 4.

XXIV. Quantities design'd or represented by letters are either Simple or Compound.

XXV. A Simple quantity is design'd or express'd either by a single letter, or by two or more letters join'd together like letters in a word: As a (or $-a$) is a simple quantity; likewise $2aa$, $3abc$, and $dddd$ are simple quantities.

XXVI. A Compound quantity consisteth of two or more simple quantities connected or join'd one to another by $+$ or $-$; so $a+b$ is a compound quantity, likewise $a-c$, also $a+b+c$, and $a+b-c$ are compound quantities.

XXVII. Every one of these four Characters, to wit, $+$, $-$, ω , and \times (before defined in Sect. 20, 21, 22, and 23.) may sometimes have reference to such a Compound quantity as followeth the sign, and hath a Line drawn over every member of it. As, for Example, by $a+b\omega c$, you are to understand that the difference of the quantities b and c (whether the Excess be in b or in c) is added or to be added to the quantity a .

In like manner, $a-b+c$ shews that the Compound quantity $b-c$ is subtracted or to be subtracted from the quantity a ; where in regard of the line drawn over $b+c$, the sign $-$ hath reference to the subtraction of c as well as b from the quantity a . But if that line were omitted, then the sign $-$ would only refer to the next following simple quantity: As, $a-b+c$, (or $a+c-b$) signifies the subtraction of b only from $a+c$.

Moreover, $a\omega b+c$ signifies the difference between the simple quantity a , and the compound quantity $b+c$.

And $a \times b-c$ signifies that the quantity a is multiplied or to be multiplied by the excess of the quantity b above the quantity c .

XXVIII. This Character $\sqrt{}$ is called a Radical sign, and signifies that the Square root of the number or quantity that stands next after the said sign $\sqrt{}$, is extracted, or to be extracted; as $\sqrt{25}$ signifies the Square root of 25, to wit, 5; and $\sqrt{36}$ signifies the Square root of 36, to wit, 6.

Like-

Likewise \sqrt{ab} signifies the Square root of the quantity ab . So that when a number or quantity immediately follows the said radical sign $\sqrt{}$, the square root of that number or quantity is thereby denoted.

But to design or represent the Root of a Power higher than a Square, some Algebraical Writers (whom in this matter I shall follow) are wont to write the Index of the Power within a Circle next after the sign $\sqrt{}$; As for Example, $\sqrt[3]{327}$ signifies the Cubick root of 27, to wit, 3. Likewise, $\sqrt[4]{16}$ denotes the Biquadrate root of 16, to wit, 2; that is, the root from whence 16 considered as the fourth Power is produced. Again, $\sqrt[5]{243}$ signifies the root from whence 243 consider'd as the fifth Power is raised, which Root is 3. And if you please you may write $\sqrt[2]{81}$ to denote the Square root of 81, to wit, 9.

Likewise $\sqrt[3]{a}$ signifies the Cubick root of some number or quantity represented by a . Also $\sqrt[4]{bc}$ signifies the Biquadrate root of the Quantity bc .

Sometimes the Radical Sign belongs to as many of the following Quantities as have a Line drawn over them; as $\sqrt{b+c}$ or $\sqrt{(b+c)}$ signifies the Square root of the sum of the Quantities b and c . Likewise $\sqrt{bb-c}$ imports the Square root of the Remainder when the quantity c is subtracted from the Square of the quantity b . Which Roots, and such like, are called *Universal Roots*.

Again, $d+\sqrt{bb-c}$ signifies that the Quantity c is first to be subtracted from the Square bb , and then the Square root of the Remainder is to be added to the quantity d . But that the Learner may the better perceive my meaning in the three last Examples concerning Universal Roots, let b signify 4; bb , 16; c , 12; and d , 23. Then $\sqrt{b+c}$ signifies $\sqrt{4+12}$: that is, $\sqrt{16}$, to wit, 4. Also $\sqrt{bb-c}$ signifies $\sqrt{16-12}$: that is, $\sqrt{4}$, to wit, 2. And $d+\sqrt{bb-c}$ signifies $23+2$, that is, 25. After the same manner the Universal Square root of $d+\sqrt{bb-c}$ may be express'd thus;

$$\sqrt{d+\sqrt{bb-c}}: \text{that is, } 5.$$

XXIX. Four points set in this form :: are always in the middle of four Geometrical Proportionals, as, for Example, these four numbers $2:4::6:12$ are Geometrical Proportionals, and to be read thus; As 2 is to 4, so is 6 to 12; or, (in the Phrase of The Rule of Three) If 2 give 4, then 6 will give 12.

In like manner these four Quantities, $b, d::c, a$ are to be read thus; As b is to d , so c is to a ; that is, look what proportion b hath to d , the same proportion hath c to a .

Also these four Quantities $b+c, d-a::f, g$ do intimate that the sum of b and c hath such proportion to the Excess of d above a , as f hath to g . The like is to be understood of others.

XXX. This Character $+++$ set at the end of three or more Quantities, imports that they are Continual Proportionals Geometrical; so by $2:4::8:16::32:++$ it is signified that such proportion as 2 hath to 4, the same hath 4 to 8, 8 to 16, and 16 to 32.

Likewise by these $a, b, c+++$ you are to understand that the quantity a hath the same proportion to the quantity b , as b to c .

XXXI. This Character $=$ is the sign of an Equation or Equality, and imports as much as the word *Equal*; as, $8+4=7+5$ signifies that the sum of 8 and 4 is equal to the sum of 7 and 5. Likewise $8=12-4$ that 8 is equal to 12 less 4, to wit, the excess of 12 above 4.

Again, $8 \times 3=4 \times 6$ denotes the Product of 8 multiplied by 3 to be equal to the Product of 4 into 6.

So also, $a+b=c+d$ signifies that the sum of the quantities a and b is equal to the sum of the quantities c and d . This will be further explained in the XI. Chapter.

XXXII. This Character $<$ stands for the word *Greater*, viz. it signifies that the Quantity which stands before, that is, on the left hand of the said Character is greater than the quantity following the same; so $5<4$ must be read thus, 5 is greater than 4. Likewise $a-b<0$ signifies that the Compound quantity $a-b$ is greater than the Simple quantity 0 . And $d<a+c$ signifies that the quantity d is greater than $a+c$.

XXXIII. This Character $>$ signifies that the quantity standing before the Character is less than the quantity following the same; as, $4>5$ must be read thus, 4 is less than 5. Likewise, $a+b>c+d$ signifies that the compound quantity $a+b$ is less than the compound quantity $c+d$.

XXXIV. Quan-

XXXIV. Quantities, whether they be Simple or Compound, which are exprest either wholly by Letters, or partly by Letters and partly by Numbers written upon one Line, are called Algebraical Integers; as these, $a, ab, cd, +ff, a+3, &c.$ But these quantities, $\frac{a+b}{c}, \frac{a+3}{b},$ and others so written, are called Algebraical Fractions, because each of them like a Fraction in Vulgar Arithmetic consists of a Numerator placed above a line; and a Denominator underneath.

CHAP. II.

Addition of Algebraical Integers.

I. **A**lgebraical Addition finds out the Summ or Aggregate of two or more Quantities exprest either wholly by Letters, or partly by Letters and partly by Numbers.

II. The Operations in Algebraical Addition depend principally upon a diligent observation of three things, *viz.*

First. You must observe whether the Quantities to be added be Like or Unlike.

Like Quantities are those which are exprest by the same Letters equally repeated in every one of the Quantities, such are these, $a, 5a, -2a,$ each of which is exprest by the single letter a . Also these are Like quantities, $3aa, aa, -2aa,$ each of which is exprest by a double a , to wit, aa . Likewise these, $2ab, 3ab, -ab$ are called Like quantities because every one of them is exprest by the same letters, to wit, ab .

Unlike Quantities are those which are exprest by different Letters, or else by the same letters unequally repeated; as, for Example, b and c are unlike quantities, because they are exprest by different letters; also $2abc$ and $2ab$ are unlike quantities, because the letter c is in the one, but not in the other. Again, a and aa are unlike quantities, in regard the letter a is not equally repeated in both. The like is to be understood of others.

Secondly. You must observe whether the Signs (to wit, $+$ and $-$) belonging to like quantities given to be added be Like or Unlike: As, for example, these quantities $+1a$ and $+3a$ have like signs, the same sign $+$ being prefixt before each quantity; Also these quantities, $-2a$ and $-3a$ have like signs, the same sign $-$ being prefixt to each quantity; but these quantities $+2a$ and $-3a$ have unlike or different signs prefixt.

Thirdly. The numbers prefixt before the letters must be diligently observed, for their summ or difference will be concern'd in Algebraical Addition; as will be manifest by the following Rules.

III. When two or more simple Algebraical Integers (or whole quantities) propos'd to be added or collected into one Summ, First collect the numbers prefixt into one summ; then to that summ annex the letter or letters by which any one of the quantities propos'd is exprest; lastly, prefix the given sign whether it be $+$ or $-$, so shall this new quantity be the summ desired. As,

$$\begin{array}{r} \text{Add } \left\{ \begin{array}{l} a \\ a \\ 2a \end{array} \right. \begin{array}{l} +1a \\ +1a \\ +2a \end{array} \\ \hline \text{Summ: } 2a \end{array}$$

fixed, so $2a$ or $+2a$ is the summ desired.

In like manner, if to $-2b$ you would add $-b$, the summ will be $-3b$. For the numbers prefixt are 2 and 1 , which added together make 3 , to which annexing b , and prefixing the given sign $-$, there ariseth $-3b$, the summ desired.

$$\begin{array}{r} \text{Add } \left\{ \begin{array}{l} -2b \\ -b \end{array} \right. \\ \hline \text{Summ: } -3b \end{array}$$

More

More Examples of the Rule of Addition in the foregoing Sect. III.

$$\begin{array}{r} \text{To be added, } \left\{ \begin{array}{l} 5a \\ 3a \end{array} \right. \begin{array}{l} -5aa \\ -2aa \end{array} \begin{array}{l} +7ab \\ +13ab \end{array} \\ \hline \text{The Summ, } 8a \quad -7aa \quad +20ab \end{array}$$

$$\begin{array}{r} \text{To be added, } \left\{ \begin{array}{l} ac \\ 2ac \\ 3ac \end{array} \right. \begin{array}{l} -3bcd \\ -bcd \\ -6bcd \end{array} \begin{array}{l} +3a^3 \\ +2a^3 \\ +7a^3 \end{array} \\ \hline \text{The Summ, } 6ac \quad -10bcd \quad +12a^3 \end{array}$$

IV. When two Simple quantities propos'd to be added together be like; and have equal numbers prefixt, but unlike or contrary Signs, the Summ will be 0 , or nothing; for the Affirmative quantity will destroy or extinguish the Negative: As, for example, if it be required to add $1c$, or $+c$, to $-c$, the Summ will be 0 , to wit, nothing. For supposing $-c$, or $-1c$ to be a Debt of one Crown that I owe; and $+c$, or $+1c$ to be one Crown in my purse, it is evident that one Crown in ready money will discharge or strike off a Debt of one Crown; and so that Debt and Credit being added or compared together, the Summ amounts to 0 .

In like manner, if it be desired to add $-6l$ to $+6l$, the Summ will be 0 ; for if my whole Estate be worth but 6 pounds, and I owe a Debt of 6 pounds, it is manifest that my clear Estate is worth or amounts to just nothing.

More Examples of the Rule of Addition in the preceding Sect. IV.

$$\begin{array}{r} \text{To be added, } \left\{ \begin{array}{l} +3a \\ -3a \end{array} \right. \begin{array}{l} -5abc \\ +5abc \end{array} \begin{array}{l} +7add \\ -7add \end{array} \\ \hline \text{The Summ, } 0 \quad 0 \quad 0 \end{array}$$

V. When two Simple quantities propos'd to be added together be like, but their Signs unlike, and the prefixed numbers unequal between themselves, first subtract the lesser number prefixt from the greater, then to the Remainder annex the letter or letters by which either of the Quantities propos'd is exprest; lastly, before the said Remainder set the Sign which stands before the greater number prefixt, so shall this new Quantity be the Summ desired.

As, for Example, if it be desired to add $-2a$ to $+3a$, the summ will be a . For first subtracting 2 from 3 the remainder is 1 , to which annexing a and prefixing $+$ (because $+$ belongs to that Quantity which hath the greater number prefixt) there ariseth $+1a$, or $+a$ for the Summ sought.

Again, to add $+b$ to $-3b$, I subtract 1 the lesser number prefixt, from 3 the greater, and to the Remainder 2 annexing b and prefixing $-$, (because $-$ belongs to $3b$ whose prefixt number 3 is greater than that of $+b$ or $+1b$) I find $-2b$ for the Summ desired.

Thus you see that this last Rule of Addition is performed by Subtraction, and may easily be understood under the notion of discharging or paying off a Debt, or at least part of a Debt by so much ready Money or Credit; and then observing what Debt remains unpaid,

or what Money or Credit remains as an overplus: So in the first of the two last Examples, you may conceive $+3a$ to be three Pounds in ready Cash, and $-2a$ to be a Debt of two Pounds; then comparing the said ready Money and Debt together, you will find by Subtraction that the clear Money remaining after the Debt is paid, will be one Pound, to wit, $+1a$ or a which is the Summ of the quantities $+3a$ and $-2a$. Likewise in the latter Example, if $-3b$ be conceived to represent a Debt of three Pounds, and $+b$ or $+1b$ one Pound in ready Money; 'tis evident that this will strike off one Pound of that Debt; and to the Debt remaining will be two Pounds, to wit, $-2b$, which is the Summ of $-3b$ and $+b$.

More Examples of the Rule of Addition in the preceding Sect. V.

To be added,	$\begin{cases} +5aa \\ -7aa \end{cases}$	$\begin{cases} +6abcd \\ -abcd \end{cases}$	$\begin{cases} -8f^+ \\ +3f^+ \end{cases}$
The Summ,	$-2aa$	$+5abcd$	$-5f^+$

V.I. When three or more Simple Quantities propos'd to be added be like, but have unlike Signs; First, (by the Rule in Sect. III. of this Chap.) collect the Affirmative quantities into one Summ; and the Negative quantities into another; then (by Sect. IV. or V.) add those two Summs into one; so this last Summ shall be that which is sought. As, for example, If the Summ of these four Quantities, $7a$, $2a$, $-3a$, $-5a$ be desired; First, (by Sect. III.) the sum of $7a$ and $2a$ is $+9a$; also the sum of $-3a$ and $-5a$ is $-8a$; lastly (by Sect. V.) $+9a$ added to $-8a$ makes $+a$, that is, a , which is the Summ desired.

More Examples of the Rule of Addition in Sect. VI.

To be added,	$\begin{cases} +5a \\ +3a \\ -8a \end{cases}$	$\begin{cases} -2bc \\ +3bc \\ -4bc \end{cases}$	$\begin{cases} +4d^+ \\ +3d^+ \\ -1d^+ \end{cases}$
The Summ,	0	$-3bc$	$+2d^+$

To be added,	$\begin{cases} +5ccc \\ +2cc \\ -cc \\ -4cc \end{cases}$	$\begin{cases} -4fff \\ -3fff \\ -2fff \\ +8fff \end{cases}$	$\begin{cases} +4ggbb \\ -3ggbb \\ +2ggbb \\ -ggbb \end{cases}$
The Summ,	$+2cc$	$-fff$	$+2ggbb$

VII. When two or more Simple quantities given to be added be unlike, write them down one after another without altering their Signs; as, if the number (or line) a be to be added to the number (or line) b ; I write $a+b$, or, $b+a$ for the Summ. In like manner the Summ of these Quantities a, b, c , may be written thus, $a+b+c$; or thus, $a+b+c$; or thus, $b+a+c$.

More Examples of the Rule of Addition in Sect. VII.

To be added,	$\begin{cases} +3a \\ +2d \end{cases}$	$\begin{cases} +aa \\ -bb \end{cases}$
The Summ,	$3a+2d$	$+aa-bb$

To

	Again,	
To be added,	$\begin{cases} +ab \\ -ac \\ +ad \end{cases}$	$\begin{cases} +5add \\ -3dd \\ -4d \end{cases}$
The Summ,	$+ab-ac+ad$	$+5add-3dd-4d$

Addition of Compound Algebraical Integers.

VIII. The Addition of Compound whole Quantities may easily be dispatched by the help of the Rules in the preceding Sections of this Chap. as will appear by the following Examples.

First then, If this Compound quantity $a+b$ be to be added to $a+2b$, their Summ is $a+b+a+2b$, that is $2a+3b$; for $a+a$ makes $2a$, and $+b+2b$ makes $+3b$. Again, The Summ of these two Compound quantities $3b+5a$ and $2b-2a$ is $3b+5a+2b-2a$, that is, $5b+3a$; for $3b+2b$ makes $5b$, and (by Sect. V.) $+5a-2a$ makes $+3a$.

Likewise, The Summ of these two Compound quantities $5cc+3f-8$ and $3cc-2f+6$ will be found $8cc+f-2$: For $5cc$ added to $3cc$ makes $8cc$; also $+3f$ added to $-2f$ gives $+f$, and -8 added to $+6$ makes -2 .

After the same manner, $3a-8$ added to $10-a$ makes $2a+2$; (for $+3a$ added to $-a$ makes $+2a$, and -8 added to $+10$ gives $+2$.)

Again, The Summ of these two Compound quantities $a+b$ and $c-d$ is $a+b+c-d$; which Summ admits of no Contraction, in regard all the Simple quantities are unlike.

More Examples of the Addition of Compound whole Quantities.

To be added,	$\begin{cases} a+b \\ a-b \end{cases}$	$\begin{cases} aa+2a-3 \\ aa+a-6 \end{cases}$
The Summ,	$2a$	$2aa+3a-9$

To be added,	$\begin{cases} aa-2ab \\ aa+ab \end{cases}$	$\begin{cases} 4c-d+3 \\ -4c+2d-2 \end{cases}$
The Summ,	$2aa-ab$	$d+1$

To be added,	$\begin{cases} 2cc+3ef-ff \\ -3cc+5ef \end{cases}$	$\begin{cases} a^3-abc+6 \\ +3abc-6 \end{cases}$
The Summ,	$-cc+8ef-ff$	a^3+2abc

To be added,	$\begin{cases} -aaa+2bba \\ 8aaa+4bba \\ 6aaa-6bba \end{cases}$	$\begin{cases} aa-5a+24 \\ aa+a-17 \\ -2aa+2a+12 \end{cases}$
The Summ,	$13aaa$	$-2a+19$

To be added,	$\begin{cases} a+b \\ c-d \\ e+f \end{cases}$	$\begin{cases} 5b^3+24 \\ -2b^3+40 \\ 6b^3-64 \end{cases}$
The Summ,	$a+b+c-d+e+f$	$9b^3$, or, $9bbb$

B 2

CHAP.

C H A P. III.

Subtraction in Algebraick Integers.

I. *Algebraical Subtraction* takes one Quantity, whether it be express'd by a letter or letters, or partly by letters and partly by number, out of, or from another, in such manner that if the Remainder be added (according to the Rules of Algebraick Addition) to the Quantity subtracted, the Summ will be always equal to the said other Quantity.

II. A general Rule to find out the Remainder in all cases of Algebraical Subtraction is this; First joyn both the given quantities together, by writing one after the other; but with this caution, that every Sign of the quantity given to be subtracted, be ever changed into the contrary Sign, viz. $+$ into $-$ and $-$ into $+$; then shall the Summ of both quantities so connected be the Remainder sought, which is to be contracted (when it may be done) into the fewest and smallest Terms, by the Rules of Algebraical Addition.

As, for Example, If from $5a$ it be desired to subtract $3a$, first, I write down $5a$, then next after the same I write $-3a$; (where observe, that according to the Rule above given, I change $+$, the Sign belonging to $3a$ the quantity given to be subtracted, into $-$;) so there ariseth $5a - 3a$, which being contracted (by the Rule of Addition in *Self. V. Chap. 11.*) makes $2a$ the Remainder sought.

$$\begin{array}{r} \text{Out of} \\ \text{Subtract} \\ \hline \text{Remainder,} \\ \text{Remainder,} \\ \text{contracted,} \end{array} \begin{array}{r} 5a \\ 3a \\ \hline 5a - 3a \\ 2a \end{array}$$

Likewise, If from $3b$ it be desired to subtract $-2b$, I first write down $3b$, and next after the same I write $+2b$; so $3b + 2b$, that is, $5b$ is the Remainder sought; where observe (as before) that I change the Sign $-$, which belongs to $2b$ the quantity propos'd, to be taken out of $3b$, into the contrary sign $+$. But that the said $5b$ is a true Remainder, we may prove by Addition, for $+5b$ added to $-2b$ the quantity subtracted, makes $+3b$, which is the Quantity out of which the said $-2b$ was subtracted.

$$\begin{array}{r} \text{Out of} \\ \text{Subtract} \\ \hline \text{Remainder,} \\ \text{Remainder,} \\ \text{contracted,} \end{array} \begin{array}{r} 3b \\ -2b \\ \hline 3b + 2b \\ 5b \end{array}$$

Moreover, If a be to be subtracted from a , the Remainder will be $a - a$, that is, 0 or nothing. And if from $2b$ there be subtracted $-4b$, the Remainder will be $2b + 4b$, that is, $6b$.

Likewise, If from $-2m$ it be required to subtract $-m$, the Remainder will be found $-2m + m$, that is, $-m$. In every one of which Examples you may observe that the sign of the Quantity propos'd to be subtracted is changed into the contrary sign.

Again, If from $2bc$, it be desired to subtract $2ab$, the Remainder will be $2bc - 2ab$, which, because it consists of unlike Quantities, cannot be contracted into fewer or lesser Terms, by any of the Rules of Algebraical Addition. But according to the definition of Subtraction, the said $2bc - 2ab$ is a true Remainder; for if it

be added to $2ab$ the quantity subtracted, the Summ is $2bc$, which is the quantity out of which the said $2ab$ was subtracted.

More Examples of Subtraction in Simple Algebraick Integers.

$$\begin{array}{r} \text{Out of} \\ \text{Subtract} \\ \hline \text{Remainder,} \\ \text{Remainder,} \\ \text{contracted,} \end{array} \begin{array}{r} 2b \\ b \\ \hline 2b - b \\ b \end{array} \quad \begin{array}{r} +3c \\ -c \\ \hline +3c + c \\ +4c \end{array} \quad \begin{array}{r} -2n \\ -n \\ \hline -2n + n \\ -n \end{array}$$

Again,

$$\begin{array}{r} \text{Out of} \\ \text{Subtract} \\ \hline \text{Remainder,} \\ \text{Remainder,} \\ \text{contracted,} \end{array} \begin{array}{r} 3a \\ 5a \\ \hline 3a - 5a \\ -2a \end{array} \quad \begin{array}{r} \text{Again,} \\ -8d \\ -10d \\ \hline -8d + 10d \\ +2d \end{array} \quad \begin{array}{r} -a \\ +a \\ \hline -a - a \\ -2a \end{array}$$

$$\begin{array}{r} \text{Out of} \\ \text{Subtract} \\ \hline \text{Remainder,} \\ \text{Remainder,} \\ \text{contracted,} \end{array} \begin{array}{r} -bcd \\ -bcd \\ \hline -bcd + bcd \\ 0 \end{array} \quad \begin{array}{r} -4rs \\ +9rs \\ \hline -4rs - 9rs \\ -13rs \end{array} \quad \begin{array}{r} +4abc \\ -abc \\ \hline +4abc + abc \\ +5abc \end{array}$$

$$\begin{array}{r} \text{From} \\ \text{Subtract} \\ \hline \text{Remainder,} \end{array} \begin{array}{r} d \\ e \\ \hline d - e \end{array} \quad \begin{array}{r} -2b \\ -3a \\ \hline -2b + 3a \end{array} \quad \begin{array}{r} +a \\ -3a \\ \hline +a - 3a \end{array}$$

$$\begin{array}{r} \text{From} \\ \text{Subtract} \\ \hline \text{Remainder,} \end{array} \begin{array}{r} 8bbb \\ 7bbb \\ \hline 8bbb - 7bbb \end{array} \quad \begin{array}{r} +3abd \\ -7aa \\ \hline +3abd + 7aa \end{array}$$

Nor will the Operation be otherwise in the Subtraction of Compound Algebraick Integers; as for Example, if from this Compound quantity $3a + 2b$, it be desired to subtract $a + 3b$. First I write down $3a + 2b$, then next after the same I write $-a - 3b$, where observe, that the sign $+$ which belongs to a , and also to $3b$, in the Quantity propos'd to be subtracted, is changed into the contrary sign $-$ (according to the Rule of Subtraction before given;) so the Remainder sought is $3a + 2b - a - 3b$, that is, $2a - b$, (by *Self. V. Chap. 11.*)

Again, If from $2a + b$, it be desired to subtract $5a - 6b$, the Remainder will be $2a + b - 5a + 6b$, that is, $7b - 3a$.

for (according to the Rule of Algebraical Subtraction) I joyn together the two given Quantities, changing only the Signs of $+5a - 6b$ (the quantity to be subtracted) into the contrary Signs, so there ariseth $2a + b - 5a + 6b$, which contracted (by the Rules of Addition in *Self. 111. and V. of Chap. 11.*) make $7b - 3a$, which is the Remainder sought, as will easily appear by the Proof.

Likewise, to subtract $c - d$ from $a + b$, I change the Signs of $c - d$ into the contrary Signs, viz. instead of $c - d$, I take $-c + d$, which added to $a + b$ makes $a + b - c + d$, which because it consists altogether of unlike Quantities, cannot be contracted into fewer Terms, and therefore the said $a + b - c + d$ is the Remainder sought, to wit, that which ariseth by subtracting $c - d$ from $a + b$.

After the same manner, $cd + 3c$ subtracted from $3aa + bc + 2d$ leaves $3aa + bc + 2d - cd - 3c$, that is, $3aa + bc - cd - 3c$.

And

More Examples of Subtraction in Compound Algebraick Integers.

Out of Subtract	$a+b$ $a-b$	$3c-8$ $c+5$
Remainder, Remainder contracted,	$a+b-a+b$ $+2b$	$3c-8-c-5$ $2c-13$
Out of Subtract	$5a-4b$ $3a-3b$	$29e$ $-3e+7$
Remainder, Remainder contracted,	$5a-4b-3a+3b$ $2a-b$	$29e+3e-7$ $32e-7$
Out of Subtract	$aa+2ba+bb$ $+4ba$	$-2cd+6$ $+cd-2$
Remainder, Remainder contracted,	$aa+2ba+bb-4ba$ $aa-2ba+bb$	$-2cd+6-cd+2$ $-3cd+8$
Out of Subtract	$5a^3+27$ $-8+3a^3$	$3aa+6$ $-3dd$
Remainder, Remainder contracted,	$5a^3+27+8-3a^3$ $2a^3+35$	$3aa+6+3dd$
From Subtract	$a+b$ $c-d$	$aa-bb$ $-cc+dd$
Remainder,	$a+b-c+d$	$aa-bb+cc-dd$

III. The reason of changing the signs of the Quantity to be subtracted into their contraries, to wit $+$ into $-$, and $-$ into $+$ (according to the Rule before given) will be manifest from a serious consideration of the definition of Subtraction, which requires that the Summ of the quantity subtracted and the Remainder be equal to the quantity from which the subtraction is made: For first, (according to the said Rule) the Remainder is always compos'd of both the quantities propos'd for Subtraction, with this caution, that the signs $+$ and $-$ in the quantity to be subtracted be changed into the contrary signs; Secondly, (according to Algebraical Addition) the quantity to be subtracted with its own signs being added to it self with contrary signs, will destroy or extinguish it self, therefore the Summ of the Remainder and the Quantity to be Subtracted will necessarily be equal to the Quantity from which the Subtraction was made: And therefore the certainty of the said Rule of Algebraical Subtraction, and the reason of changing the signs of the quantity to be subtracted into their contraries, to wit, $+$ into $-$, and $-$ into $+$, is manifest: So if from $a+b$ there be subtracted $a-b$, the Remainder (according to the Rule of Algebraical Subtraction before given) will be $a+b-a+b$, to which if $a-b$ (the quantity subtracted) be added, it is evident that $a-b$ will destroy $-a+b$, and so the Summ will be $a+b$, to wit, the quantity from which $a-b$ was subtracted.

CHAP.

CHAP. IV.

Multiplication in Algebraick Integers.

I. *Algebraical Multiplication* doth by two Quantities, whether they be express'd by letters wholly, or partly by letters and partly by numbers, find out a third Quantity, which is called the *Product*, the *Fact*, or the *Rectangle*.

The Quantities given to be multiplied one by the other are called *Factors*, or (as in vulgar Arithmetick) either of them may be called the *Multiplicand*, and the other the *Multiplicator* or *Multiplyer*.

II. When two simple (or single) Quantities express'd by letters, whether like or unlike, be to be multiplied by one another, and have no numbers prefix to them, joyn the letters of both Quantities together, like letters in a word, it matters not in what order they be written; then the new Quantity represented by the letters so set together is the *Product* sought.

As, for example, If the number or line a be to be multiplied by it self, to wit, by a , I write aa for the *Product*: so also to multiply a by b , I write ab or ba for the *Product*; in like manner if I would multiply abc by bc , I write $abcbc$, or $abbcc$, or $acccb$, &c. for the *Product*.

And if a , b , and c be to be multiplied one into another, first a multiplied by b produceth ab , then ab multiplied by c produceth abc , or bac , or cba , to wit, the *Product* made by the continual Multiplication of the three Quantities a , b , and c .

Again, if aa be to be multiplied by ba , the *Product* will be $aaab$; which may also be written thus, a^3b ; where the Learner must diligently note that the figure 3 which stands next after but a little higher than a , must not be taken as a number prefix to b , but as an Index to shew the number of Dimensions in a^3 , or aaa , (as before hath been said in Sect. XVI. and XVII. Chap. I.)

Likewise, if aaa be to be multiplied by aaa , or a^3 by a^3 , the *Product* will be $aaaaaa$, or a^6 , in which latter way of expressing the *Product*, the Index 6 standing at the head of a is the Summ of 3 and 3 the Indices of the Quantities a^3 and a^3 propos'd to be multiplied.

So the *Product* made by the multiplication of $bbbb$ by bbb or b^4 by b^3 will be $bbbbbbb$, or b^7 (7 being the sum of the Indices 4 and 3.)

Likewise if these three Quantities be to be multiplied continually; to wit, $aaaa$, $bbbb$ and ccc , the *Product* may be express'd thus, $aaaabbbccc$; or compendiously thus, $a^4b^4c^3$; and so of others.

More examples of Multiplication in simple Algebraick Integers, according to the preceding Sect. II.

Multiplicand,	b	d	ac	ccc
Multiplicator,	c	d	d	cc
Product,	bc	dd	acd	$ccccc$

Multiplicand,	abc	def	$abbcc$
Multiplicator,	bca	abc	$abbcc$
Product,	$naabccc$	$abacdf$	$a^2b^2c^4$

III. If two simple Quantities, whether like or unlike, having numbers prefix before them, be to be multiplied one by the other, first multiply the numbers prefix, one into the other, then to this *Product* annex the letters of both Quantities, by setting them immediate-

immediately one after another, (as before in *Sett. II.*) so this new Quantity shall be the Product fought.

As, for Example, if it be desired to multiply $2a$ by $3b$, first I multiply 2 by 3 , and the Product is 6 ; to which annexing ab , (to wit, the letters found in both Quantities given to be multiplied) there ariseth $6ab$ the Product fought; which shews that six times the Product of the Multiplication of any two numbers, or right-lines, a and b , is equal to the Product made by the Multiplication of the double of a by the triple of b .

In like manner, if $2b$ be multiplied by c the Product will be $2bc$, or $2cb$; for 2 which is prefix to b in the Multiplicand, being multiplied by 1 which is suppos'd to be prefix to the Multiplier c , makes 2 , to which annexing bc , there is found $2bc$ for the Product fought.

$$\begin{array}{r} \text{Multiply} \\ \text{by} \\ \text{Product,} \end{array} \begin{array}{r} 2a \\ 3b \\ 6ab \end{array}$$

More Examples of Multiplication in Simple Algebraick Integers, according to *Sett. III.*

$$\begin{array}{r} \text{Multiply} \\ \text{by} \\ \text{Product,} \end{array} \begin{array}{r} 4b \\ 2a \\ 8ab \end{array} \quad \begin{array}{r} 12ac \\ 3d \\ 36acd \end{array} \quad \begin{array}{r} 5dfe \\ dgb \\ 5d^2fggb \end{array}$$

$$\begin{array}{r} \text{Multiply} \\ \text{by} \\ \text{Product,} \end{array} \begin{array}{r} aaa \\ 3bbb \\ 3aabb \end{array} \quad \begin{array}{r} 3a^3 \\ b^3 \\ 3a^2b^3 \end{array} \quad \begin{array}{r} 16aab \\ 4 \\ 64aab \end{array}$$

IV. The Multiplication of Compound Quantities depends upon the precedent Rules of multiplying Simple Quantities; for when a Compound quantity is to be multiplied by a Simple (or single) quantity, every member of that must be multiplied by this, also; when two Compound quantities are to be mutually multiplied, every member of the one must be multiplied into every member of the other. It matters not whether you begin to multiply at the right hand or the left, nor in what order the particular Products be set; (for Quantities express'd by Letters retain their peculiar and unaltered values wheresoever they stand;) but due regard must be had to the Signs $+$ and $-$, one of which always belongs to every particular Product, and may be discovered by this Rule, *viz.* $+$ multiplied by $+$, or $-$ by $-$, makes $+$ in the Product; but $+$ multiplied by $-$, or $-$ by $+$, makes $-$ in the Product; lastly, all the particular Products added together (according to the Rules in the preceding *Chap. 2.*) make the total Product fought: All which will be made manifest by the following Examples.

First, if a Compound quantity, as $a+b$, be to be multiplied by a Simple quantity, as c , I begin at the left hand, and multiplying a by $+$ c the Product is $+$ ac , (for $+$ multiplied by $+$ gives $+$;) likewise b multiplied by $+$ c produceth $+$ bc ; which two Products added together make $ac+bc$, which is the Product of the

multiplication of $a+b$ by c . So if $a-b$ be to be multiplied by c , the Product will be $ac-bc$. For $+$ a multiplied by $+$ c produceth $+$ ac ; and $-b$ multiplied by $+$ c produceth $-bc$; (for according to the Rule, $-$ multiplied by $+$ gives $-$;) Therefore $+$ $ac-bc$ or $ac-bc$ is the Product fought.

$$\begin{array}{r} \text{Multiply} \\ \text{by} \\ \text{Product,} \end{array} \begin{array}{r} a+b \\ c \\ ac+bc \end{array}$$

$$\begin{array}{r} \text{Multiply} \\ \text{by} \\ \text{Product,} \end{array} \begin{array}{r} a-b \\ c \\ ac-bc \end{array}$$

After

After the same manner, if it be desired to multiply $a+b$ by $c+d$, the Product will be found $ac+bc+ad+bd$. For, first $a+b$ being multiplied by c , (as in the first Example) produceth $+$ $ac+bc$; likewise $a+b$ again multiplied by d , produceth $+$ $ad+bd$; then adding those Products together, the Summ is $ac+bc+ad+bd$, which is the required Product of $a+b$ multiplied by $c+d$.

Again, if $a-b$ be multiplied by $c-d$ the Product will be $ac-bc-ad+bd$. For first, $a-b$ multiplied by c produceth $ac-bc$, (as in the last Example but one,) then $a-b$ again multiplied by $-d$ produceth $-ad+bd$; (for, according to the Rule, $+$ a multiplied by $-d$ produceth $-ad$, and $-b$ by $-d$ produceth $+$ bd .) Lastly, those particular Products added together make $ac-bc-ad+bd$, which is the Product of $a-b$ multiplied by $c-d$.

Likewise, if $a-b$ be multiplied by $a-b$, the Product will be $aa-bb$. For first, $a+b$ multiplied by a produceth $aa+ba$; then $a+b$ multiplied by $-b$ produceth $-ba-bb$; lastly, the said Products $aa+ba$ and $-ba-bb$ added together make $aa-bb$; (for $+$ ba and $-ba$ by Addition do quite vanish;) Therefore $aa-bb$ is the Product of $a-b$ multiplied by $a-b$.

Moreover, If $aa-ab+bb$ be multiplied by $a+b$, the Product will be only $aaa+bbb$; for the rest of the particular Products will vanish by Addition.

And if $a+b$ be multiplied by it self, to wit, by $a+b$, the Product will be $aa+2ab+bb$, which is the Square of $a+b$.

Likewise the Square of $a-b$ will be found $aa-2ab+bb$. Nor will the Operation be otherwise when Numbers are prefixed to compound Quantities proposed to be multiplied, respect being had to the third *Sett.* of this *Chap.* as, for Example, to multiply $3a-2c$ by $3a-2c$; First $3a-2c$ multiplied by $3a$ produceth $9aa-6ac$, and $3a-2c$ again multiplied by $-2c$ produceth $-6ac+4cc$; which particular Products added together make $9aa-12ac+4cc$, which is the Square of $3a-2c$.

When Absolute numbers are members of Quantities to be multiplied, the Rules of Multiplication in Vulgar Arithmetick and those before given must be exactly observed; as,

If it be desired to multiply
by the Absolute number 3a+6

The Product will be 15a+30

For five times $3a$ makes $15a$, and five times 6 makes 30 . Likewise, if $2aa-3$ be multiplied by $a-6$, the Product will be $2aaa-12aa-3a+18$, and the work will stand as here you see;

$$\begin{array}{r} \text{Multiplicand,} \\ \text{Multiplier,} \\ \text{Product,} \end{array} \begin{array}{r} 2aa-3 \\ a-6 \\ 2aaa-12aa-3a+18 \end{array}$$

For further illustration of the Multiplication of Algebraick Integers, the Learner may peruse the following Examples; in every one of which, as also in those afore-going, I begin to multiply at the left hand, because in Algebraical Multiplication it being a thing indifferent

indifferent to begin the work either at the right hand or the left, it will be easier to write forward than backward. And as to the placing of the particular Products, there is no necessity of observing any Order therein; for whether they be written upon one, two, or more Lines, they retain the same values, and must by Algebraical Addition be collected into one Summ to make the total Product: And therefore you may either write the particular Products all upon one line when there is room, or else upon so many several lines as there be particular Multipliers, setting like Products (when they happen) under one another to facilitate their Addition, or otherwise, as you shall find it most convenient.

More Examples of Multiplication in Compound Algebraick Integers, according to Sect. IV.

Multiplicand, Multiplier,	$a + e$ d	$ab - 3d$ f	$5g - 8$ 6
Product,	$da + de$	$2bf - 3fd$	$30g - 48$

Multiplicand, Multiplier,	$5a + 3c$ $3a - 2c$	$2b + 3$ $4b - 6$
Product, Product contracted,	$+ 15aa + 9ca$ $- 10ca - 6cc$	$8bb + 12b$ $- 12b - 18$
	$15aa + 9ca - 10ca - 6cc$ $15aa - ca - 6cc$	$8bb + 12b - 12b - 18$ $8bb - 18$

Multiplicand, Multiplier,	$3dd + 4de + ee$ $3dd - ee$
Product, Product contracted,	$+ 9dddd + 12ddde + 3ddee$ $- 3ddee - 4deee - eeee$
	$9dddd + 12ddde + 3ddee - 3ddee - 4deee - eeee$ $9dd + 12dde - 4de^2 - e^4$

Multiplicand, Multiplier,	$a + e$ $a + e$	$a + e$ $a - e$
Product,	$aa + ae$ $+ ae + ee$	$aa + ae$ $- ae - ee$
	$aa + 2ae + ee$	$aa - ee$

Multiplicand, Multiplier,	$4aaa + 3aa - 2a + 1$ $aa - 5a + 6$
Product,	$4aaaa + 3aaaa - 2aaa + aa$ $- 20aaaa - 15aaa + 10aa - 5a$ $+ 24aaa + 18aa - 12a + 6$
	$4aaaa - 17aaaa + 7aaa + 29aa - 17a + 6$

Again,

Multiplicand, Multiplier,	$2aa + 3ba - bc$ $3aa - 2ba - cc$
Prod.	$6aaaa + 9baaa - 3bbaa$ $- 4baaa - 6bbba + 2bbca$ $- 2ccaa - 3bccb + bccc$
	$- 3bc^2$ $6aaaa + 5baaa - 6bb^2aa + 2bbcc^2 + bccc$ $- 2cc^2$

V. Sometimes when Compound quantities be to be multiplied one by the other, it will be very commodious to omit the Operation, and to set only the word *into*, or \times (the sign of Multiplication) between the Quantities to be multiplied, to signify the Product of their Multiplication: But in such case, to avoid mistake, it will be convenient to draw a Line over each Compound quantity, to shew that every member of the one is to be multiplied by every member of the other.

As to multiply $4aaa + 3aa - 2a + 1$ by $aa - 5a + 6$, I write

$$\begin{array}{r} 4aaa + 3aa - 2a + 1 \quad \text{into} \quad aa - 5a + 6 \\ \text{Or,} \quad 4aaa + 3aa - 2a + 1 \quad \times \quad aa - 5a + 6 \end{array}$$

But that $+$ multiplied by $-$, or $-$ by $+$ makes $-$; also, that $-$ multiplied by $-$ makes $+$ in the Multiplication of Compound quantities, I shall hereafter make manifest in the last Section of Chap. XI.

CHAP. V.

Division in Algebraick Integers.

I. **Algebraical Division** doth by two Quantities, (whether they be express wholly by letters, or partly by letters and partly by numbers), whereof one is called the Dividend, and the other the Divisor, find out a third called the Quotient, to wit, such a Quantity, that if it be multiplied by the Divisor, the Product will be equal to the Dividend.

II. The nature of Division is to resolve or undo that which is composed or done by Multiplication: For the Dividend always represents the Fact or Product in Multiplication, the Divisor one of the two Factors or Multipliers, and the Quotient the other. As, if 12 be to be divided by 2, the Dividend 12 represents the Fact or Product made by the multiplication of two numbers, one of which is the Divisor 2, and the other is the Quotient sought, to wit, 6.

III. Every Fraction is equal to the Quotient of the Numerator divided by the Denominator: So $\frac{1}{2}$ is the Quotient of 3 divided by 4, for, according to the Proof of Division, If the Quotient $\frac{1}{2}$ be multiplied by the Divisor 4, the Product will be equal to the Dividend 3. Upon this ground, Division in Algebraick Integers, whether Simple or Compound is most commonly performed; viz. by setting the Dividend as the Numerator of a Fraction, and the Divisor as a Denominator, for this Fraction is equal to the Quotient sought.

As, for Example, to divide the Quantity a by b , I write $\frac{a}{b}$, which signifies that that a is divided by b ; or $\frac{a}{b}$ is equal to the Quotient of the quantity a divided by the Quantity b .

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In like manner, if b be propos'd to be divided by ac , I write $\frac{b}{ac}$ to represent the Quotient; also, if ac be to be divided by b , I write $\frac{ac}{b}$ to signify the Quotient.

Again, If $2ab$ be given to be divided by $3cd$, the Quotient will be $\frac{2ab}{3cd}$; and if a be to be divided by 5 , I write for the Quotient $\frac{a}{5}$; also, to divide 1 by a , I write $\frac{1}{a}$ to signify the Quotient.

So also, If $a+b$ be given to be divided by c , the Quotient may be represented by $\frac{a+b}{c}$; and if $3a$ be to be divided by $2b-c$, the Quotient is $\frac{3a}{2b-c}$.

More Examples of Division in Algebraick Integers, according to the foregoing Sect. III.

Dividend, Divisor,	$\frac{bb}{a}$	$\frac{2de}{fg}$	$\frac{3abc}{2dd}$	$\frac{a^2b}{2d^3}$
Quotient,	$\frac{bb}{a}$	$\frac{2de}{fg}$	$\frac{3abc}{2dd}$	$\frac{a^2b}{2d^3}$
Dividend, Divisor,	$\frac{aa+bb}{c}$	$\frac{2ab-3bd}{d+e}$	$\frac{aaa}{a+b-c}$	
Quotient,	$\frac{aa+bb}{c}$	$\frac{2ab-3bd}{d+e}$	$\frac{aaa}{a+b-c}$	
Dividend, Divisor,	$\frac{4aa}{3}$	$\frac{2cc+5dd}{3}$		
Quotient,	$\frac{4aa}{3}$, or $\frac{4}{3}aa$	$\frac{2cc+5dd}{3}$, or, $\frac{2}{3}cc + \frac{5}{3}dd$		

IV. When the Dividend is equal to the Divisor, the Quotient is 1 ; for every Quantity contains it self once, and therefore being divided by it self gives 1 in the Quotient: As to divide 4 by 4 the Quotient is 1 ; likewise, a divided by a gives 1 for the Quotient; also, if $a+b$ be divided by $a+b$ the Quotient is 1 ; and if $3a+2cd$ be divided by $3a+2cd$ the Quotient is 1 . The like is to be understood of others.

V. When the Quotient is expressed Fraction-wise, (according to Sect. III.) if the same letter or letters be found equally repeated in every member of the Numerator and Denominator, cast away those letters, so the remaining Quantities shall signify the Quotient.

As, for Example, If ab be to be divided by a , the Quotient express'd Fraction-wise will be $\frac{ab}{a}$; But because the letter a is found in the Numerator and Denominator, I cast away a out of both, so b only is left, which is the Quotient of ab divided by a .

Likewise, If aa be divided by a the Quotient is $\frac{aa}{a}$, that is, a ; (by casting away a out of the Numerator and Denominator.)

Again, If aaa be to be divided by aa , the Quotient will be $\frac{aaa}{aa}$, that is, a ; by casting away aa out of the Numerator and Denominator. And if abc be to be divided by ab , the Quotient express'd Fraction-wise will be $\frac{abc}{ab}$, that is, c , after ab is cast out of the Numerator and Denominator.

After the same manner, If a^4 be propos'd to be divided by a^3 , (that is, $aaaa$ by aaa) the Quotient will be a^2 , or aa , by expunging a^3 (or aaa) out of the Dividend and Divisor.

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This Contraction of Division is like to the reducing of a Fraction express'd by large numbers to more simple Terms, by dividing the Numerator and also the Denominator by a common Divisor.

Again, If $ab+ac$ be to be divided by $ad-af$, the Quotient express'd Fraction-wise according to the preceding Sect. III. will stand thus, $\frac{ab+ac}{ad-af}$, where because the letter a is found in every member of the Numerator and Denominator, it may be quite struck out, and then the new Quotient will be $\frac{b+c}{d-f}$, which Fraction is equal to the former, and express'd by more simple Terms.

Likewise, If $ab+a$ be divided by a , the Quotient (according to Sect. III.) will be $\frac{ab+a}{a}$, that is, $b+1$; for by casting away a , there will remain $\frac{b+1}{1}$, that is, $b+1$; (for $\frac{b}{1}$ is but b ; and $\frac{1}{1}$ is 1 ;) but that $b+1$ is the true Quotient it will appear by the proof of Division, for $b+1$ multiplied by the Divisor a will produce the Dividend $ab+a$.

So also to divide $3bc-2b$ by $2bb+b$, I write $\frac{3c-2}{2b+1}$ for the Quotient; where observe, that although the letter b be cast out of every member of the given Dividend and Divisor, yet the number prefix to the letter cast out must stand still in the new Quotient.

But note diligently, That in this kind of Division of Compound Algebraick Integers, a letter cannot be cancell'd or cast away, unless it be found in every member of the Dividend and Divisor; and therefore this Quotient $\frac{bc+cd}{c+f}$ cannot be contracted by casting away any letter.

More Examples of Contractions in Algebraick Division, according to the preceding Sect. V.

Dividend, Divisor,	$\frac{aab}{aa}$	$\frac{ddef}{ef}$	$\frac{abc}{b}$	$\frac{a^7}{a^3}$
Quotient,	$\frac{aab}{aa}$	$\frac{ddef}{ef}$	$\frac{abc}{b}$	$\frac{a^7}{a^3}$
Quotient contracted,	b	dd	ac	a^4
Dividend, Divisor,	$\frac{ab+ac-a}{a}$	$\frac{ab-2a}{3a}$		
Quotient,	$\frac{ab+ac-a}{a}$	$\frac{ab-2a}{3a}$		
Quotient contracted,	$b+c-1$	$\frac{b-2}{3}$; or, $\frac{1}{3}b - \frac{2}{3}$		
Dividend, Divisor,	$\frac{2abd+3bd}{3bb-b}$	$\frac{2ba^3+caa-3aa}{baa-daa+aa}$		
Quotient,	$\frac{2ad+3d}{3b-1}$	$\frac{2ba+c-3}{b-d+1}$		

VI. If an Algebraick Integer, whether Simple or Compound, be to be divided by a simple Quantity, and there be such numbers prefix to the letters in the Dividend and Divisor as may all be severally divided by some number as a common Divisor without leaving a Remainder, let the Quotients arising by the Division of those numbers by their common Divisor, before the letters respectively, instead of the numbers that were first prefixt: As, for Example, if $8a$ be to be divided by $6b$; First, the Quotient, express'd Fraction-wise (according to Section III. of this Chap.) will be $\frac{8a}{6b}$, then dividing the prefixed numbers 8 and 6 by their common Divisor 2, I set the Quotients 4 and 3 instead of 8 and 6 before a and b ; so the Quotient sought is $\frac{4a}{3b}$.

In like manner, $6abc - 3abc$ divided by $9fbc$ gives the Quotient $\frac{2a-d}{3f}$; For first, the Dividend and Divisor being set Fraction-wise will stand thus, $\frac{6abc - 3abc}{9fbc}$, then, (according to Sect. V.) bc is to be cast out of the Numerator and Denominator; lastly, the prefixed numbers 6, 3, and 9 being divided by their common Divisor 3, give 2, 1, and 3, which being set before the remaining letters a , d and f respectively, give the contracted Quotient $\frac{2a-d}{3f}$ or $\frac{2a-d}{3f}$.

More Examples of Contractions in Division, according to Sect. V. and VI.

Dividend,	$4cd$	$27ab$	$16gb$
Divisor,	$2c$	$9ad$	$8gb$
Quotient,	$\frac{4cd}{2c}$	$\frac{27ab}{9ad}$	$\frac{16gb}{8gb}$
Quotient contracted,	$2d$	$\frac{3b}{d}$	2

Dividend,	$18aaaa$	$30b^2c^2dd$
Divisor,	$6aa$	$5bbccdd$
Quotient,	$\frac{18aaaa}{6aa}$	$\frac{30b^2c^2dd}{5bbccdd}$
Quotient contracted,	$3aa$	$6b^2ccdd$

Dividend,	$28bbc + 16bbd$
Divisor,	$2obb$
Quotient,	$\frac{28bbc + 16bbd}{2obb}$
Quotient contracted,	$\frac{7c + 4d}{5}$, or $\frac{7c}{5} + \frac{4d}{5}$

VII. If every member of a Compound quantity be multiplied by one and the same Simple quantity, it is evident from the nature of Multiplication and Division, that if the Product of that Multiplication be divided by the said Compound quantity, the Quotient will be the Simple quantity.

As, for Example, if $b+c$ be multiplied by a the Product will be $ba+ca$, and therefore $ba+ca$ divided by the Factor $b+c$ will give the other Factor a . And for

for the same reason, $2bca+a$, that is $2bca+1a$, divided by $2bc+1$ will give the Quotient a .

Likewise, if $6a+5a-a$ (that is, $10a$) be divided by $6+5-1$ (that is, 10), the Quotient will be a .

Again, if $2ba+2ca+2da$ be divided by $b+c+d$, the Quotient will be $2a$; and if $2baa+caa-daa-aa$ be divided by $2b+c-d-1$, the Quotient will be aa .

More Examples of Contractions in Division, according to the preceding Sect. VII.

Dividend,	$2da + 3ca$	$23b + 18b + 1b$
Divisor,	$2d + 3c$	$23 + 18 + 1$
Quotient,	a	b

Dividend,	$2baa - 3caa$	$2af - 2bf + 2cf - 6f$
Divisor,	$2b - 3c$	$a - b + c - 3$
Quotient,	aa	$2f$

VIII. When the Dividend and Divisor are Compound whole Quantities, the precedent Rules of Algebraical Division will not always give the Quotient in the least Terms; but the simplest Quotient may be found out by one of these two ways, viz.

1. When the Dividend and Divisor are Algebraick Integers, and there is a possibility of expressing the Quotient by an Algebraick Integer, it may be found out by the general method of Division handled in the next following Section; which way is like that of dividing whole numbers in vulgar Arithmetick; but if the Learner find it difficult, he may waive it until he hath proceeded as far as the 8. Chapter of the 2. Book.

2. The Quotient, whether it happen to be an Algebraick Integer, or a Fraction, may be found out in its least Terms by the method hereafter delivered in Sect. 7. Chap. 8. of the Second Book, where the manner of finding out all the Aliquot parts or just Divisors, every one of which will divide the Dividend and Divisor propos'd without any Remainder, is exhibited.

IX. In this Section a general method of Division in Algebraick Integers is handled. As to the order of the work, it agrees with that form of Division in whole numbers which I have explained in Mr. Wingate's Arithmetick, but the work it self depends upon the preceding Rules of Algebraical Division, Multiplication, and Subtraction, as also upon this Rule for discovering the due Sign belonging to every particular Quotient, viz. $+$ divided by $+$, or $-$ by $-$, gives $+$ in the Quotient; but $+$ divided by $-$, or $-$ by $+$, gives $-$ in the Quotient. Whether the Operation be begun at the right hand or the left, it matters not; but because 'tis easier to Write forwards than backwards, I shall (as in Vulgar Arithmetick) begin to Divide at the left hand, and proceed towards the right.

Example 1. Let it be required to divide $ac+ad+bc+bd$ by $c+d$.

Having placed the Dividend and Divisor in such order as you see in the next Page, first I divide $+ac$ by $+c$, according to Sect. 5. of this Chap. and there ariseth $+a$, ($+$ \div $+$, because $+$ divided by $+$ gives $+$.) therefore I write $+a$ or a in the Quotient; then multiplying the whole Divisor $c+d$ by the said Quotient a , I write the Product $ac+ad$ under the two first members of the Dividend towards the left hand, to wit, under $ac+ad$; that done, drawing a line under the said Product $ac+ad$, I subtract the same from $ac+ad$, (the two first members of the Dividend) and there remains 0 , which I set under the line, as you may see in the Page following.

Divisor.

$$\begin{array}{r} \text{Divisor.} \quad \text{Dividend.} \quad \text{Quotient.} \\ c + d \quad) \quad ac + ad + bc + bd \quad (a + b \\ \underline{ac + ad} \\ 0 + bc + bd \\ + bc + bd \\ \hline 0 \end{array}$$

Then there remains to be divided $+bc + bd$ which I bring down to the Remainder 0, and renew the work, *viz.* I divide $+bc$ by $+c$, and there ariseth $+b$ which I write in the Quotient next after a ; then multiplying the whole Divisor $c + d$ by the said Quotient b , the Product is $bc + bd$, which being subscribed, and subtracted from that which remained to be divided, there remains 0. So the Division is finished, and the Quotient is found $a + b$; but that it is a true Quotient the Proof will make manifest; for $a + b$ multiplied by the Divisor $c + d$ produceth the Dividend $ac + ad + bc + bd$.

Example 2. In like manner, if $aa - bb$ be to be divided by $a - b$ the Quotient will be found $a + b$; For first, aa divided by a gives a in the Quotient, by which multiplying the whole Divisor $a - b$ the Product is

$$\begin{array}{r} a + b \quad) \quad aa - bb \quad (\quad a - b \\ \underline{aa - ab} \\ bb - ab \\ \underline{ bb - ab} \\ 0 \end{array}$$

ded by \div gives $-$ in the Quotient; then multiplying the whole Divisor $a \div b$ by $-b$ (last \div in the Quotient) the Product is $-ab - bb$, or $-bb - ab$, which subtracted from $-bb - ab$ that remained to be divided, there remains 0 , so the Division is finished and the Quotient is found $a - b$, to wit, such a Quantity that if it be multiplied by the Divisor $a \div b$, it will produce the Dividend $aa - bb$.

Example 3. Again, If it be desired to divide $aaa + bbb$ by $aa - ba + bb$, the Quotient will be found $a + b$, and the work will stand thus:

[illegible]

In which Example, first (as before) I begin at the first Term of the Dividend towards the left hand, and dividing aaa by aa , (not by $-ba$ nor by $+bb$, because each of these will give a Fraction in the Quotient) there aritheth a , which I set in the Quotient; then multiplying the whole Divisor $aa - ba + bb$ by the said Quotient a , the Product is $aaa - baa + bba$, which I subtract from the Dividend $aaa + bbb$; so there remains to be yet divided $+bbb + baa - bba$.

Now I renew the work, and divide $\div bbb$ by its correspondent Divisor $\div bb$, (not by $\div aa$, nor by $\div ba$, becaufe each of these gives a Fraction) and there ariseth $\div b$, which I write next after a in the Quotient; then multiplying the whole Divisor $aa-ba$ $\div bb$ by the said Quotient $\div b$, the Product is $bbb \div baa-bba$, which I set under, and subtrakt from the Quantity that remained to be divided, for there remains 0 , and the Quotient sought is $a+b$: But that it is a true Quotient the proof will discover; for if the Divisor $aa-ba \div bb$ be multiplied by the Quotient $a+b$, it will produce the Dividend $aaa \div bbb$.

Exam-

Example 4. In like manner, if $aaa - bbb$ be divided by $aa + ba + bb$, the Quotient will be $a - b$, and the work will stand thus ;

Divisor.	Dividend.	Quotient.
$aa + ba + bb$	$aaa - bbb \dots \dots \dots$	$(a - b$
	$aaa + baa + bba$	
	<hr/>	
	$- bbb - baa - bba$	
	$- bbb - baa - bba$	
	<hr/>	
	$\begin{array}{ccc} 0 & 0 & 0 \end{array}$	

Example 5. Again, If $9\ dddd + 12\ dde + 4\ dee + eee$ be to be divided by $3\ dd + ee$, the Quotient will be found $3\ dd + 4\ de + ee$, as will be manifest by the subsequent Operation.

$$\begin{array}{r}
 3dd - ee \\
 \hline
 9ddd + 12dde - 4deee - eeee \quad (3dd + 4de + ee) \\
 9ddd - 3ddee \\
 \hline
 12dde + 3ddee - 4deee - 4deee \\
 12dde \qquad \qquad \qquad 4deee \\
 \hline
 3ddee + eeee \\
 3ddee + eeee \\
 \hline
 \end{array}$$

In which Example, first I divide 9ddd by 3dd , and it gives 3dd , which I write in the Quotient; then multiplying the whole Divisor $3\text{dd} - \text{ee}$ by the said Quotient 3dd , the Product is $9\text{ddd} - 3\text{dee}$, which I write under the two first members of the Dividend, and subtract the same from the said two members, so there remains $+12\text{ddd} + 3\text{dee}$; to which I bring down -4eee (the next member of the Dividend) and it makes $+12\text{ddd} + 3\text{dee} - 4\text{eee}$ which comes now to be divided, therefore I renew the work, and dividing $+12\text{ddd}$ by $+3\text{dd}$, it gives $+4\text{de}$, which I set in the Quotient next after 3dd , then multiplying the whole Divisor $3\text{dd} - \text{ee}$ by the said Quotient $+4\text{de}$, the Product is $+12\text{ddd} - 4\text{eee}$, which I write under $+12\text{ddd} + 3\text{dee} - 4\text{eee}$ (the Quantity last set apart to be divided,) and having drawn a line under the said Product I subtract it from the said particular Dividend, so there remains $+3\text{dee}$ which I write underneath the line; that done, to the said Remainder $+3\text{dee}$ I bring down $- \text{eee}$, (the last member of the total Dividend) and it makes $+3\text{dee} - \text{eee}$ which is yet to be divided: Therefore I renew the work, and dividing $+3\text{dee}$ by $+3\text{dd}$, it gives $+ \text{ee}$ which I set in the Quotient next after $+4\text{de}$; (or I might here divide $+3\text{dee}$ by $- \text{ee}$ in regard it will give an Algebraical Integer in the Quotient, as I shall then in the next Example;) then multiplying the Divisor $3\text{dd} - \text{ee}$ by $+ \text{ee}$, (last set in the Quotient), and subtracting the Product $+3\text{dee} - \text{eee}$ from the quantity that remained to be divided, there now remains 0. So the Division is finished without any Quantity remaining, and the entire Quotient is $+3\text{dd} + 4\text{de} + \text{ee}$.

Note. By this General Method of Division the Quotient may oftentimes be found out and exprest various ways, both as to the Order and Multitude of members in the Quotient, but yet the entire Quotient in each form will have one and the same value, as will appear by the following manner of Dividing the two quantities propos'd in the last Example.

Let it therefore be again propos'd to divide $9ddd + 12dde - 4dee - eee$ by $3dd - ee$.

First, I work as before in the last Example to find out the two first members in the Quotient, to wit, $3dd + 4de$, and then there remains to be divided $+ 3dee - eeee$ which you see stands at this mark * in the following Operation: Now because $+ 3dee$ divided by $- ee$ gives an Algebraick Integer for the Quotient, to wit, $- 3dd$, therefore I write $- 3dd$ in the Quotient, then multiplying the whole Divisor $3dd - ee$ by $- 3dd$ (last fer in the Quotient) I subtract the Product $+ 3ddde - qddddd$ from $+ 3dee - eeee$ which remained to be divided; so there remains to be yet divided $- eeee + 3ddde$;

D

$3dd$

$$\begin{array}{r}
 3dd - ee \quad \left\{ \begin{array}{l} 9dddd + 12ddee - 4deee - eeee \\ 9dddd - 3ddee \end{array} \right. \quad \left\{ \begin{array}{l} 3dd + 4de \\ -3dd + ee + 3dd \end{array} \right. \\
 \hline
 + 12ddee + 3ddee - 4deee \\
 + 12ddee - 4deee \\
 \hline
 * \quad + 3ddee - eeee \\
 + 3ddee - 9dddd \\
 \hline
 - eeee + 9dddd \\
 - eeee + 3ddee \\
 \hline
 + 9dddd - 3ddee \\
 + 9dddd - 3ddee \\
 \hline
 0 \quad 0
 \end{array}$$

Then I divide $-eeee$ (which stands immediately under the third black line) by its correspondent Divisor $-ee$, (for it cannot be divided by $3dd$ so as to give an Integer in the Quotient,) and there ariseth $+ee$, which I set in the Quotient; then multiplying the whole Divisor $3dd - ee$ by the said Quotient $+ee$ the Product is $-eeee + 3ddee$, which subtracted from $-eeee + 9dddd$ (to wit, the quantity that remained to be divided) there remains to be yet divided $+9dddd - 3ddee$, (which stands immediately under the last black line but one;) Therefore I divide $+9dddd$ by $+3dd$ and it gives $+3dd$ to be set in the Quotient; then multiplying the whole Divisor $3dd - ee$ by the said $+3dd$, it makes $+9dddd - 3ddee$, which subtracted from $+9dddd - 3ddee$ (the quantity that remained to be divided) leaves 0; so the Division is finished without any quantity remaining, and the Quotient is found $3dd + 4de - 3dd + ee + 3dd$, that is, $3dd + 4de + ee$: So that the Quotient found out by the latter Operation, after it is contracted by Algebraical Addition, is the same found out by the former way of dividing the Quantities given in the fifth Example.

Example 6. Again, If $yyyyy - 8yyy - 124yy - 64$ be divided by $yy - 16$, the Quotient will be found $yyy + 8yy + 4$, and the work will stand thus:

$$\begin{array}{r}
 \text{Divisor.} \quad \text{Dividend.} \quad \text{Quotient.} \\
 yy - 16 \quad \left\{ \begin{array}{l} yyyyy - 8yyy - 124yy - 64 \\ yyyyy - 16yyy \end{array} \right. \quad \left\{ \begin{array}{l} yyy + 8yy + 4 \end{array} \right. \\
 \hline
 + 8yyy - 124yy \\
 + 8yyy - 128yy \\
 \hline
 + 4yy - 64 \\
 + 4yy - 64 \\
 \hline
 0 \quad 0
 \end{array}$$

If the Powers of the Root y in the last Example be expressed according to Cartesius his way, the work will stand thus:

$$\begin{array}{r}
 yy - 16 \quad \left\{ \begin{array}{l} y^5 - 8y^3 - 124yy - 64 \\ y^5 - 16y^4 \end{array} \right. \quad \left\{ \begin{array}{l} y^4 + 8yy + 4 \end{array} \right. \\
 \hline
 + 8y^4 - 124yy \\
 + 8y^4 - 128yy \\
 \hline
 + 4yy - 64 \\
 + 4yy - 64 \\
 \hline
 0 \quad 0
 \end{array}$$

But

But Cartesius in dividing the Quantities propos'd in the last Example works backwards, viz. from the right hand of the Dividend towards the left, as you here see in the following Operation.

$$\begin{array}{r}
 yy - 16 \quad \left\{ \begin{array}{l} y^5 - 8y^4 - 124yy - 64 \\ + 4yy - 64 \end{array} \right. \quad \left\{ \begin{array}{l} 4 + 8yy + y^4 \end{array} \right. \\
 \hline
 - 8y^4 - 128yy \\
 + 8y^4 - 128yy \\
 \hline
 y^5 - 16y^4 \\
 y^5 - 16y^4 \\
 \hline
 0 \quad 0
 \end{array}$$

More Examples are here added for the fuller exercise and illustration of Division in Compound Algebraick Integers, according to the general method in Sect. IX. of this Chapter.

$$\begin{array}{r}
 \text{Divisor.} \quad \text{Dividend.} \quad \text{Quotient.} \\
 2c - 3d \quad \left\{ \begin{array}{l} 6ca - 5da - 8bc + 12ab \\ 6ca - 9da \end{array} \right. \quad \left\{ \begin{array}{l} 3a - 4b \end{array} \right. \\
 \hline
 0 \quad 0 - 8bc + 12ab \\
 - 8bc + 12ab \\
 \hline
 0 \quad 0
 \end{array}$$

$$\begin{array}{r}
 \text{Divisor.} \quad \text{Dividend.} \quad \text{Quotient.} \\
 a - b \quad \left\{ \begin{array}{l} aaa - 3aab + 3abb - bbb \\ aaa - aab \end{array} \right. \quad \left\{ \begin{array}{l} aa - 2ab + bb \end{array} \right. \\
 \hline
 - 2aab + 3abb \\
 - 2aab + 2abb \\
 \hline
 + abb - bbb \\
 + abb - bbb \\
 \hline
 0 \quad 0
 \end{array}$$

$$\begin{array}{r}
 \text{Divisor.} \quad \text{Dividend.} \quad \text{Quotient.} \\
 2aa + 3bb \quad \left\{ \begin{array}{l} 4aaaa + 12aabb + 9bbbb \\ 4aaaa + 6aabb \end{array} \right. \quad \left\{ \begin{array}{l} 2aa + 3bb \end{array} \right. \\
 \hline
 + 6aabb + 9bbbb \\
 + 6aabb + 9bbbb \\
 \hline
 0 \quad 0
 \end{array}$$

$$\begin{array}{r}
 a + b \quad \left\{ \begin{array}{l} aaa - abb \\ aaa + aab \end{array} \right. \quad \left\{ \begin{array}{l} aa - bb - ab + bb, \text{ that is, } aa - ab \end{array} \right. \\
 \hline
 - abb - aab \\
 - abb - bbb \\
 \hline
 - aab + bbb \\
 - aab + abb \\
 \hline
 + bbb + abb \\
 + bbb + abb \\
 \hline
 0 \quad 0
 \end{array}$$

D 2

Again,

Again,

Divisor.	Dividend.	Quotient.
$ab - aa$	$aab^3 + a^3b - 2a^3$	$(abb + a^2 + aab + a^3$
	$aab^3 - a^3bb$	
	$+ a^3b + a^3bb - 2a^3$	
	$- a^3b$	
	$+ a^3bb - a^3$	
	$+ a^3bb - a^3b$	
	$- a^3 + a^3b$	
	$- a^3 + a^3b$	
	0	

Divisor.	Dividend.	Quotient.
$\frac{2}{3}ab - \frac{1}{2}aa$	$\frac{2}{3}aab^3 + \frac{1}{2}a^3b - a^3$	$(\frac{2}{3}abb + \frac{1}{2}a^3 + \frac{1}{2}aab + \frac{1}{2}a^3$
	$\frac{2}{3}aab^3 - \frac{1}{2}a^3bb$	
	$+ \frac{1}{2}a^3b + \frac{1}{2}a^3bb - a^3$	
	$+ \frac{1}{2}a^3b$	
	$+ \frac{1}{2}a^3bb - \frac{1}{2}a^3$	
	$- \frac{1}{2}a^3 + \frac{1}{2}a^3b$	
	$- \frac{1}{2}a^3 + \frac{1}{2}a^3b$	
	0	

If Algebraical Division according to this general Method will not work off just without a Remainder, then you may write the Dividend and Divisor fraction-wise, according to *Sett.* III. of this *Chapt.* Or sometimes the Quotient may be express partly by Integers, and partly by a Fraction; As if $bb + bd + cc$ be to be divided by $b + d$, the Quotient may be express either thus $\frac{bb + bd + cc}{b + d}$; or else thus, $b + \frac{cc}{b + d}$ which latter Quotient is found out by the help of the said general Method; for after you have thereby discovered as many Integers as can arise in the Quotient, you may set the remainder of the Dividend as a Numerator over the Divisor as a Denominator, so this Fraction together with the said Integer or Integers shall be equal to the Quotient sought; as in this following Example.

Divisor.	Dividend.	Quotient.
$a - b$	$2aac + 3aaa - 2abc - 3aab + 2cc$	$(2ac + 3aa + \frac{2cc}{a-b})$
	$2aac$	
	$- 2abc$	
	$+ 3aaa$	
	$- 3aab$	
	$+ 3aaa$	
	$- 3aab$	
	0	
	0	$+ 2cc$

CHAP.

CHAP. VI.

Containing the Arithmetick of Algebraical Fractions.

Of the rise of Algebraical Fractions, and the manner of expressing Integers and mixed quantities Fraction-wise.

I. The Operations about Algebraical Fractions are wrought like those of vulgar Fractions, by the help of the Rules of Algebraical Integers before delivered, as will appear by the following Rules of this Chapter.

II. From the manner of dividing quantities according to *Sett.* 3. of the preceding Chap. 5. Algebraical Fractions arise; as, If a be to be divided by b , the Quotient is represented by the Fraction $\frac{a}{b}$: Likewise $\frac{a+b}{c-d}$, which imports as much as the

Quotient of $a+b$ divided by $c-d$; also $\frac{2aa+3cd}{bb}$, and such like, are called Algebraical Fractions.

III. If the Numerator be equal to the Denominator, that Fraction (or Quotient express fraction-wise) is equal to 1, (to wit, Unity,) as before hath been said in *Sett.* 4. Chap. 5.

So $\frac{aa}{aa} = 1$. And $\frac{abc+dd}{abc+dd} = 1$.

IV. When an Algebraical Integer is to be express fraction-wise, make it a Numerator, and set 1 for the Denominator; as if these quantities ab and $aa-bb$ be to be set in the form of Fractions they will stand thus;

$\frac{ab}{1}$ And $\frac{aa-bb}{1}$.

V. If an Algebraical Integer, as a , be to be set in the form of a Fraction that shall have for its Denominator some Algebraical Integer prescribed, as d , multiply a by the Denominator d , and write the Product ad as a Numerator over the Denominator d , thus, $\frac{ad}{d}$; which Fraction is equal to the Integer a first proposed, and hath for its Denominator the prescribed quantity d .

Likewise the quantity a reduced to the form of a Fraction whose Denominator is prescribed $b+c$ will stand thus, $\frac{ab+ac}{b+c}$.

Moreover, If $a + \frac{aa}{d}$ be to be reduced to the Form of a Fraction that shall have d for a Denominator; let a be multiplied by the Denominator d , and to the Product ad add the Numerator aa ; then set that Summ, to wit, $ad+aa$ over the Denominator d , so there will be $\frac{ad+aa}{d}$ for the Fraction desired. More Examples of this Rule are these following.

$$\frac{bc}{c} = b. \quad \frac{aa+ab}{a+b} = a. \quad \frac{dda}{a} = dd.$$

$$\frac{bc+bb}{c} = b + \frac{bb}{c} \quad \frac{ab-ac+dd}{b-c} = a + \frac{dd}{b-c}$$

How

How to reduce Algebraick Fractions to others of the same value in more simple Terms.

VI. When the same letter or letters be found in the Numerator and Denominator, let them be cast out of both; and if the numbers prefix can be abbreviated by some common Divisor for the Quotients in the places of those numbers prefix, so shall the new Fraction be of the same value with that first proposed: So this Fraction $\frac{abc}{abd}$ will be reduced to $\frac{c}{d}$; and this $\frac{12ab+8ac}{16ad}$ will be reduced to $\frac{3b+2c}{4d}$. This Rule hath already been explained in Sect. 5; and 6. of Chap. 5. and may be further illustrated by the following Examples.

$$\frac{ad}{ac} = \frac{d}{c} \quad \frac{12ad}{4abc} = \frac{3d}{bc}$$

$$a + \frac{bcd}{cd} = a + b \quad \frac{36aa}{4ba+16da} = \frac{9a}{b+4d}$$

VII. The searching out of the greatest common Divisor, for reducing an Algebraick Fraction to the smallest Terms, after the manner used in Vulgar Arithmetick, is for the most part a tedious and intricate work; especially when the Numerator and Denominator are Compound Quantities consisting of many members, and therefore instead of that way of finding out a Common measure (or Divisor,) I shall by a clear Method in Chap. 8. of the Second Book, shew how to find out all such Divisors as will divide the Numerator and Denominator precisely without leaving a Remainder. But in the mean time I shall recommend to the Learners exercise the following Examples of Fractions abbreviated by Division according to the general method in Sect. 9. Chap. 5. of this Book; which Examples, together with the Rule above-delivered in the 6. Sect. will be great helps to reduce Algebraical Fractions to lower terms, when there is a possibility.

Examples of Fractions reduced to their smallest Terms.

$\frac{aa+ab}{a+b} = a$	$\frac{aa-ab}{a-b} = a$
$\frac{aac-aaa}{c-d} = a$	$\frac{aa+2ba+bb}{a+b} = a+b$
$\frac{a^3-2b^2a^2+b^3}{aa+bb} = aa+bb$	$\frac{aa-2ba+bb}{a-b} = a-b$
$\frac{a^3-2b^2a^2+b^3}{aa-bb} = aa-bb$	$\frac{aa-bb}{a+b} = a-b$
$\frac{aaaa-bbbb}{aa+bb} = aa-bb$	$\frac{aa-bb}{a-b} = a+b$
$\frac{aaaa-bbbb}{aa-bb} = aa+bb$	$\frac{aaa-bbb}{aa-ba+bb} = a+b$

aaa+

$$\frac{aaa+bbb}{a+b} = aa-ba+bb \quad \frac{aaa-bbb}{aa+ba+bb} = a-b$$

$$\frac{aaa-bbb}{a-b} = aa+ba+bb \quad \frac{aaa-abb}{aa-ab} = a+b$$

$$\frac{aaa-abb}{aa+ab} = a-b \quad \frac{aaaa-bbbb}{aaa-aab+abb-bbb} = a+b$$

More Examples of Fractions abbreviated.

$$\frac{aa+ab}{ad+bd} = \frac{a}{d} \quad (\text{By the common Divisor } a+b)$$

$$\frac{aa-ab}{ac-bc} = \frac{a}{c} \quad (\text{By the common Divisor } a+b)$$

$$\frac{aac-aad}{cd-dd} = \frac{aa}{d} \quad (\text{By the common Divisor } c-d)$$

$$\frac{aaa-abb}{aa+2ab+bb} = \frac{aa-ab}{a+b} \quad (\text{By } a+b)$$

$$\frac{aaa-bbb}{aa-bb} = \frac{aa+ba+bb}{a+b} \quad (\text{By } a+b)$$

$$\frac{a^3-b^3}{aa+ab} = \frac{aaa-aab+abb-bbb}{a} \quad (\text{By } a+b)$$

How to find out the smallest quantity that can be divided by two or more given quantities severally without a Remainder.

VIII. Two or more Algebraick quantities whether Simple or Compound being proposed, the smallest quantity that can be divided by every one of those given, without a Remainder, may be found out by the following Operation, (which is grounded upon 36. prop. 7. Elem. Euclid.) and the use thereof will hereafter appear.

As, for Example, If it be desired to find the smallest quantity that can be divided by aac and cd , set them in the form of a Fraction

thus, $\frac{aac}{cd}$, and reduce the Fraction to its primitive or equivalent Fraction in the smallest Terms

$\frac{aa}{d}$, which being set near the former, multiply

cross-wise, viz. aac by d , or aa by cd , and there will arise one and the same Product, to wit $aacd$ the Quantity sought, which is the smallest quantity that can be divided

severally by aac and cd without leaving any Remainder.

$$\frac{aac}{cd} \times \frac{aa}{d} = \frac{aacd}{d}$$

In

In like manner to find the smallest quantity that can be divided by $ab + ac$ and $ad - af$ severally, I set them Fraction-wise thus, $\frac{ab+ac}{ad-af}$, this reduced to its lowest

$$\frac{ab+ac}{ad-af} \times \frac{b+c}{d-f}$$

Terms gives $\frac{abd+acd-fab-fac}{add-fff}$; then I multiply cross-wise (as before) viz. $ab+ac$ by $d-f$ or $ad-af$ by $b+c$, and there ariseth $abd+acd-fab-fac$, which is the smallest quantity that can be divided by $ab+ac$ and $ad-af$, so as to leave no Remainder.

IX. But if the given Quantities cannot be reduced to lower Terms, then multiply them one into another, and their Product is the quantity desired: So to find the smallest quantity that can be divided by $bb+cc$ and $dd+ff$ severally

$$\frac{bb+cc}{dd+ff} \times \frac{bb+cc}{dd+ff}$$

$$\frac{bbdd+ccdd+bbff+ccff}{dddd+ffdd+bbff+ccff}$$

without leaving a Remainder, because $\frac{bb+cc}{dd+ff}$ cannot be reduced to more simple Terms, I multiply $bb+cc$ by $dd+ff$, and there is produced $bbdd+ccdd+bbff+ccff$ the Quantity sought.

X. When three or more quantities are given, the smallest quantity that can be divided by them severally without leaving a Remainder may be found out in this manner, viz. To find out the least quantity that can be divided by $aaa-abb$, $aa+2ab+bb$ and $aa-bb$; I first seek (after the manner of the second Example in Sect. 8.) the smallest quantity that can be divided by $aaa-abb$, and $aa+2ab+bb$, so I find $aaaa-aabb$ and $aaaa-aabb$; and because this quantity may be also divided by $aa-bb$ (the third quantity proposed) it is manifest that $aaaa-aabb+aaab-abb$ is the quantity sought.

In like manner if there be given these four quantities, $aaaa-bbbb$, $aa+ab$, $aaaa+aaab$, and $a+b$; First, I find out (as before) the smallest quantity $aaaa-bbbb$ that can be divided by the first and second quantities $aaaa-bbbb$ and $aa+ab$,

$$\frac{aaaa-bbbb}{aa+ab} \times \frac{aaa-aab+abb-bbb}{a}$$

Then because the said $aaaa-bbbb$ cannot be divided by the third quantity $aaaa+aaab$, I seek the smallest quantity that can be divided by $aaaa-bbbb$ and $aaaa+aaab$, so I find (in like manner as before) $aaaaaa-aabbbb$, which, because it is divisible by the fourth quantity proposed, to wit, by $a+b$, shall be the quantity sought; viz. a^6-aab^4 is the smallest quantity that can be divided

$$\frac{aaaaa-aabbbb}{aaaa+aaab} \times \frac{aa-bb}{a}$$

by every one of these four quantities, a^6-b^4 , $aa+ab$, a^4+aaab , and $a+b$. And so of others.

How to reduce Algebraical Fractions which have different Denominators, into other Fractions of the same value that may have a Common Denominator.

XI. When two Fractions having different Denominators are to be reduced into two other Fractions of the same value that shall have a Common Denominator, multiply the Numerator of the first Fraction by the Denominator of the second, and the Product shall be a new Numerator correspondent to the Numerator of that first Fraction; Also,

multiply

multiply the Numerator of the second Fraction by the Denominator of the first, and the Product is a new Numerator correspondent to the Numerator of the second Fraction; lastly, multiply the Denominators one by the other, and the Product shall be a common Denominator to both the new Numerators.

As, for Example, to reduce $\frac{ab}{c}$ and $\frac{bd}{a}$ (whose Denominators c and a are unlike)

into two other Fractions that may be of the same value with those given, and have a common Denominator; First, I multiply cross-wise, viz. the Numerator ab by the Denominator a , and the Product is aab for a new Numerator instead of ab ; likewise I multiply the Numerator bd by the Denominator c , and the Product is bdc , for a new Numerator instead of bd ; lastly, the Denominators c and a multiplied one by the other produce ac for a Denominator to each of those new Numerators aab and bdc : So the Fractions $\frac{aab}{ac}$ and $\frac{bdc}{ac}$

are found out which have a common Denominator ac , and are equal in value to the Fractions first given, viz. $\frac{aab}{ac}$ is equal to $\frac{ab}{c}$, and $\frac{bdc}{ac}$ is equal to $\frac{bd}{a}$. As was required.

In like manner $\frac{aa}{7bc}$ and $\frac{2bb}{5d}$ (which have unlike Denominators) will be reduced into $\frac{4daa}{35bcd}$ and $\frac{4bbbc}{35bcd}$ which have a common Denominator.

Also, $\frac{12}{a}$ and $\frac{b}{5}$ will be reduced into these, $\frac{60}{5a}$ and $\frac{ba}{5a}$.

Again, to reduce $\frac{aa+2bb}{c+d}$ and $\frac{3cc-dd}{ff}$ to a common Denominator, I multiply cross-wise (as before,) viz. $aa+2bb$ by ff , and $3cc-dd$ by $c+d$, so, the Products are $aaff+2bbff$, and $3ccc-cdd+3ccd-ddd$ for new Numerators; then multiplying the Denominators $c+d$ and ff one into the other, the Product is $cff+dff$ for a common Denominator, and the Fractions sought are $\frac{aaff+2bbff}{cff+dff}$ and $\frac{3ccc-cdd+3ccd-ddd}{cff+dff}$.

XII. When three or more Fractions having unlike Denominators are to be reduced into as many other Fractions that may be of the same value, and have a common Denominator, Multiply the Numerator of each Fraction into all the Denominators except its own, so the Products made by that continual Multiplication shall be new Numerators, multiply also all the Denominators one into another, and the Product shall be a Denominator to every one of the new Numerators.

As, for Example, To reduce these three Fractions $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$ into three others that may be of the same value and have a common Denominator, I multiply the Numerator a into the Denominators d and g , so the Product adg is a new Numerator instead of a ; again, I multiply the Numerator c into the Denominators b and g , and the Product cbg is a Numerator instead of c ; likewise, multiplying the Numerator $2ef$ into the Denominators b and d , the Product $2bdef$ is a

Numerator instead of $2ef$; lastly, the Denominators b , d and g multiplied one into another produce bdg for a common Denominator to those three new Numerators, and the three Fractions sought are $\frac{adg}{bdg}$, $\frac{cbg}{bdg}$ and $\frac{2bdef}{bdg}$.

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In

In like manner these three Fractions $\frac{aa+8}{bb}$, $\frac{9}{aa-8}$, and $\frac{dd}{7}$ will be reduced to these three, to wit, $\frac{7aaaa-448}{7aabb-56bb}$, $\frac{63bb}{7aabb-56bb}$, and $\frac{aaddb-8dabb}{7aabb-56bb}$, which have for a common Denominator $7aabb-56bb$.

XIII. But if the Denominators of the given Fractions can be reduced to lower Terms, then those Fractions may oftentimes be reduced more compendiously than by the Rules in the two last preceding Sections, into others in the smallest Terms that have a common Denominator, in this manner, viz. Seek (by the Rules in Sect. 8. and 10. of this Chap.) the smallest quantity that can be divided by every one of the Denominators without a Remainder, which quantity reserve for a common Denominator; then for the Numerators divide the common Denominator by the Denominator of the first Fraction, and multiply the Quotient by the Numerator of the first Fraction, so shall the Product be a new Numerator instead of that first Numerator; work in like manner to find out the other Numerators, and set every one of them over the common Denominator before found out.

As, for Example, to reduce these Fractions $\frac{bbbd}{aac}$ and $\frac{aaa}{cd}$ to a common Denominator; I seek first of all the smallest quantity that can be divided by the Denominators aac and cd , and I find that quantity to be $aacd$, which shall be the common Denominator; then I divide the said $aacd$ by each of the given Denominators aac and cd , and multiply the Quotients d and a by the given Numerators $bbbd$ and aaa , so the Products $bbbdd$ and $aaaa$ shall be the new Numerators, which being severally set over the common Denominator $aacd$, there will arise $\frac{bbbdd}{aacd}$ and $\frac{aaaa}{aacd}$ for the Fractions sought.

Likewise, to reduce $\frac{bbbb}{aac-aad}$ and $\frac{aaa+bbb}{cd-dd}$ to a common Denominator, having first found the common Denominator $aacd-aadd$, to wit, the least quantity that can be divided by the given Denominators $aac-aad$ and $cd-dd$, I divide the said common Denominator by the said given Denominators severally, and the Quotients d and a I multiply by the Numerators $bbbb$ and $aaa+bbb$, and then setting the Products severally over the common Denominator, the Fractions sought will be $\frac{bbbdd}{aacd-aadd}$ and $\frac{aaaa+aabbb}{aacd-aadd}$.

Again, to reduce these three Fractions, to wit, $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$, and $\frac{aa-ab}{aa-bb}$ to a common Denominator; First (as in the first Example in Sect. 10. of this Chap.) I seek the smallest quantity that can be just divided by every one of the three given Denominators, and I find $aaaa+aaab-aabb-aaaa$, for a common Denominator; then dividing this quantity found by every one of the three given Denominators (according to the general Method in Sect. 9. Chap. 5.) the Quotients will be $a+b$, $aa-ab$ and $aa+ab$; that done, I multiply the first of those Quotients by the Numerator of the first Fraction; also the second Quotient by the second Numerator, and the third Quotient by the third Numerator; so the Products $aa-bb$, $aabb-aaaa$ and $aaaa-aabb$ shall be new Numerators, which being severally set over the common Denominator first found, will give the Fractions sought, to wit, these:

$$\begin{array}{r} \frac{aa-bb}{aaaa+aaab-aabb-aaaa} \\ \frac{aabb-aaaa}{aaaa+aaab-aabb-aaaa} \\ \frac{aaaa-aabb}{aaaa+aaab-aabb-aaaa} \end{array}$$

Nor

Nor will the Operation be otherwise to reduce these four Fractions, to wit, $\frac{a^3}{a^2-ab}$, $\frac{a^3-a^2b}{a^2+ab}$, $\frac{a^3-b^3}{a^3+a^2b}$, and $\frac{a^2+ab+b^2}{a+b}$, into these four following Fractions having a common Denominator.

$$\begin{array}{l} 1. \quad \frac{a^3}{a^3-a^2b^2} \\ 2. \quad \frac{a^3-2a^2b+2a^2b^2-2a^2b^3+a^2b^4}{a^3-a^2b^2} \\ 3. \quad \frac{a^3-a^2b^2-a^2b^3+b^3}{a^3-a^2b^2} \\ 4. \quad \frac{a^3+a^2b^2-a^2b^3-a^2b^4}{a^3-a^2b^2} \end{array}$$

For first by the help of the given Denominators, the smallest common Denominator a^3-aab^4 is found out by the operation in the last Example of the preceding Sect. 10. of this Chap.) then the said common Denominator being divided severally by the given Denominators a^3-b^4 , $aa+ab$, a^3+aabb , and $a+b$, the Quotients are aa , $a^3-a^2b+aaab-ab$, $aa-bb$, and $a^3-a^2b+a^2bb-aab^3$; which multiplied respectively by the given Numerators a^3 , a^3-a^2b , a^3-b^3 , and $aa+ab+bb$, will produce those new Numerators which are before set over the common Denominator a^3-aab^4 .

Addition of Algebraical Fractions.

XIV. If two or more Fractions to be added have one common Denominator, add the Numerators together, and set their Summ as a new Numerator over the common Denominator, so shall this new Fraction be the sum of the Fractions given to be added.

As, for Example, to add $\frac{aa}{c}$ to $\frac{bb}{c}$, the Summ will be $\frac{aa+bb}{c}$.

So also, $\frac{2ab}{c+d}$ added to $\frac{3bb}{c+d}$ makes $\frac{2ab+3bb}{c+d}$.

Likewise the Summ of $\frac{5a-3b}{c+d}$ and $\frac{2b-3a}{c+d}$ will be found $\frac{2a-b}{c+d}$; (For the given Numerators $5a-3b$ and $2b-3a$ added together make $2a-b$.)

Again, the Summ of $\frac{a-b+2c}{c+5}$, $\frac{a+b-2c}{c+5}$ and $\frac{4a}{c+5}$ will be found $\frac{6a}{c+5}$.

And if these be added, to wit, $\frac{3ab}{b+c+d}$, $a+\frac{3ac}{b+c+d}$, and $\frac{3ad}{b+c+d}$, the Summ will be $a+\frac{3ab+3ac+3ad}{b+c+d}$; that is, $4a$. (For by Division, $\frac{3ab+3ac+3ad}{b+c+d} = 3a$.)

XV. But if the Fractions propos'd to be added together have unlike Denominators, first reduce them to a common Denominator, and then add them as before; as to add $\frac{ab}{c}$ to $\frac{bd}{a}$, first I reduce them to $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ which have the same Denominator ac ; then setting the sum of the Numerators aab and bdc over the common Denominator ac , there will be $\frac{aab+bdc}{ac}$ for the Summ required.

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So

So also to add $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$, their Summ will be found $\frac{adg + cbg + 2bdef}{bdg}$.

Likewise, to add these three Fractions $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$ and $\frac{aa-ab}{aa-bb}$; first I reduce them to three others of the same value under a common Denominator, (as in the third Example of the preceding 13. Sect.) and then setting the Summ of the three new Numerators over the common Denominator, I find the sum of the given Fractions to be $\frac{aaaa + aa - abbb - bb}{aaaa + aab - aab - abbb}$.

XVI. When Mixed quantities are to be added together, collect the Fractions into one sum, and the Integers into another, then those two summs added together give the sum desired; as, for Example:

To add these mixed quantities . . . $\frac{aa}{b} - a$ and $\frac{dd}{c} + d$.

The sum of the Fractions, after they are reduced to a common Denominator, is $\frac{caa + bdd}{bc}$.

To which sum adding the Integers in the mixed quantities proposed, the sum desired will be $\frac{caa + bdd}{bc} - a + d$.

Or, when mixed quantities are to be added together, you may reduce them to improper Fractions, (by Sect. 5. of this Chap.) and then add these together as in the preceding Examples; as,

To add those mixed quantities in the last Example, to wit, . . . $\frac{aa}{b} - a$ and $\frac{dd}{c} + d$;

I first reduce them to these Fractions . . . $\frac{aa-ba}{b}$ and $\frac{dd+cd}{c}$;

Which reduced to a common Denominator produce these . . . $\frac{caa-cba}{bc}$ and $\frac{bdd+bcd}{bc}$.

Which two last Fractions added together give the sum required, to wit, . . . $\frac{caa-cba+bdd+bcd}{bc}$.

Which is equal to the Summ before found, to wit, . . . $\frac{caa+bdd}{bc} - a + d$.

Subtraction of Algebraical Fractions.

XVII. If the two Fractions given have the same Denominator, subtract the Numerator of the Fraction prescribed to be subtracted, from the other Numerator, and set the Remainder as a new Numerator over the common Denominator, so shall this new Fraction be the Remainder sought.

As, for Example, If from $\frac{aa}{c}$ you desire to subtract $\frac{bb}{c}$, take bb from aa , and set the Remainder $aa-bb$ as a Numerator over the common Denominator c ; so $\frac{aa-bb}{c}$ shall be the Remainder sought.

In like manner, If from $\frac{2ab}{b-c}$ you would subtract $\frac{2ac}{b-c}$, the Remainder will be $\frac{2ab-2ac}{b-c}$, that is, (by Division) $2a$.

Again, If from $\frac{2aa-7b+6}{a+b}$ it be desired to subtract $\frac{2aa+12b-18}{a+b}$, the Remainder

Remainder will be found $\frac{5aa-19b+24}{a+b}$. (For $3aa+12b-18$ subtracted from $2aa-7b+6$, leaves $5aa-19b+24$.)

So also, from $d + \frac{bb}{b+d}$ subtracting $\frac{bd}{b+d}$, there remains $\frac{dd+bb}{b+d}$; For, (by Sect. 5. of this Chap.) $d + \frac{bb}{b+d}$ will be reduced to $\frac{db+dd+bb}{b+d}$; from which subtracting $\frac{bd}{b+d}$, the Remainder is $\frac{dd+bb}{b+d}$.

XVIII. But if the two Fractions given have different Denominators, first reduce them to a common Denominator, and then subtract as before; so if from $\frac{dd}{c}$ it be desired to subtract $\frac{aa}{b}$, I reduce them to $\frac{ddb}{cb}$ and $\frac{aac}{cb}$, which have the same Denominator cb ; then from $\frac{ddb}{cb}$ subtracting $\frac{aac}{cb}$, there remains $\frac{ddb-aac}{cb}$, which is the Remainder sought.

After the same manner, If from $\frac{aa+d}{b-c}$ you would take away $\frac{aa}{b}$, there will remain $\frac{db-aac}{bb-bc}$.

Likewise from $\frac{aaa+bbb}{cd-ad}$ to take away $\frac{bbbb}{aac-aad}$, I first reduce these given Fractions to a common Denominator, (as in the second Example of Sect. 13. of this Chap.) and so I find $\frac{aaaaa+aaabb}{aacd-aadd}$ and $\frac{bbbbb}{aacd-aadd}$, which latter Fraction subtracted from the former there remains $\frac{aaaaa+aaabb-bbbbb}{aacd-aadd}$.

Again, If from a it be desired to subtract $\frac{aa-ab}{a+b}$, I reduce a into the form of a Fraction whose Denominator shall be $a+b$, and so instead of a , I find $\frac{aa+ab}{a+b}$, from which subtracting $\frac{aa-ab}{a+b}$, there remains $\frac{2ab}{a+b}$.

Multiplication of Algebraical Fractions.

XIX. When two Algebraical Fractions are given to be multiplied one by the other, multiply their Numerators one into the other, and take the Product for a new Numerator; likewise multiplying the Denominators one into the other, this Product shall be a new Denominator, and the new Fraction is the Product sought.

As, for Example, to multiply $\frac{2a}{c}$ by $\frac{b}{3d}$, I multiply (as in vulgar Fractions) the Numerator $2a$ by the Numerator b , and the Product $2ab$ is a new Numerator; likewise I multiply the Denominators $3d$ and c one into the other, and the Product $3dc$ shall be a new Denominator; so $\frac{2ab}{3dc}$ is the Product sought.

In like manner, $\frac{aa-bb}{c}$ multiplied by $\frac{2ab}{b+c}$ gives the Product $\frac{2aaab-2abbb}{bc+cc}$.

XX. When either or both the given Terms are mixed Quantities, reduce the mixt Quantity to the form of a Fraction (by the Rule in Sect. 5. of this Chap.) and then multiply as before: So to multiply $c + \frac{bb}{d}$ by $a + \frac{ad}{c-d}$, I first reduce those

those mixt quantities to these Fractions, $\frac{cd+bb}{d}$ and $\frac{ac}{c-d}$, then multiplying the Numerator $cd+bb$ by the Numerator ac , the Product is $accd+acbb$ for a new Numerator; also multiplying the Denominators d and $c-d$ one by the other, the Product is $dc-dd$ for a new Denominator, and the Product sought is $\frac{accd+acbb}{dc-dd}$.

XXI. When an Integer is to be multiplied by a Fraction, expresse the Integer fraction-wise by giving it unity, (to wit, 1) for a Denominator, (according to Sect. 4. of this Chap.) and then multiply as in the preceding Examples.

As, to multiply a by $\frac{b}{c}$, that is, $\frac{a}{1}$ by $\frac{b}{c}$, the Product will be $\frac{ab}{c}$. Likewise to multiply $aa-bb$ by $\frac{aa+bb}{cd+fg}$, I reduce $aa-bb$ to $\frac{aa-bb}{1}$, then multiplying the Numerator $aa+bb$ by the Numerator $aa-bb$, the Product $aaaa-bbbb$ shall be a new Numerator; Likewise the Denominator $cd+fg$ multiplied by the Denominator 1 gives $cd+fg$ for a new Denominator, and the new Fraction $\frac{aaaa-bbbb}{cd+fg}$ is the Product sought.

XXII. But oftentimes there may be this useful Contraction in the Multiplication of Fractions, viz. When the Numerator of the one and the Denominator of the other may be severally divided by some common Divisor without a Remainder, take the Quotients instead of the said Numerator and Denominator, and then multiply as in the preceding Examples.

As, for Example, to multiply $\frac{aa+2ab+bb}{cd-dd}$ by $\frac{dd}{a+b}$:

Forasmuch as the Numerator of the first Fraction and the Denominator of the latter may be divided severally by $a+b$ without a Remainder, I set the Quotients $a+b$ and 1 in the places of $aa+2ab+bb$ and $a+b$, and by that exchange these Fractions will arise, to wit;

$$\frac{a+b}{cd-dd} \text{ and } \frac{dd}{1}.$$

In like manner, because $cd-dd$ the Denominator of the first of the two Fractions left above-written, and dd the Numerator of the latter Fraction, may be severally divided by d without a Remainder, I set the Quotients $c-d$ and d in the places of $cd-dd$, and dd , and to these new Fractions arise, to wit;

$$\frac{a+b}{c-d} \text{ and } \frac{d}{1}.$$

Then I multiply (as before) the Numerators $a+b$ and d ; one by the other, and the Product $da+db$ is a new Numerator: Also multiplying the Denominator $c-d$ by the Denominator 1, the Product $c-d$ is a new Denominator, and the new Fraction $\frac{da+db}{c-d}$ is the Product sought; being equal to that which would be made by the mutual multiplication of $\frac{aa+2ab+bb}{cd-dd}$ and $\frac{dd}{a+b}$ the Fractions first proposed to be multiplied.

So also, If it be desired to multiply $a+\frac{bb}{a-b}$ by $a-2b+\frac{bb}{a}$, that is, $\frac{aa-ab+bb}{a-b}$ by $\frac{aa-2ab+bb}{a}$; Forasmuch as the Numerator $aa-2ab+bb$ of the latter Fraction, and the Denominator $a-b$ of the former, being severally divided by their common Divisor $a-b$ will give the Quotients $a-b$ and 1; therefore I set these in the places of $aa-2ab+bb$ and $a-b$, whence these Fractions will arise, to wit;

$$\frac{aa-ab+bb}{1} \text{ and } \frac{a-b}{a}.$$

Which

Which being multiplied one by the other will give $\frac{aaa-2aab+2abb-bbb}{a}$, or $aa-2ab+2bb-\frac{bbb}{a}$, the Product sought.

Again, this Fraction $\frac{aac-aad-bbc-bbd}{aa+2ab+bb}$ multiplied by $\frac{aaa-abb}{cd-ad}$, will produce $\frac{aaaa-aaab-aabb+abbb}{ad+bd}$; For the Numerator of the first Fraction and the Denominator of the latter being severally divided by their common Divisor $c-d$, the Quotients will be $aa-bb$ and d ; Also, the Denominator of the first Fraction and the Numerator of the second being severally divided by their common Divisor $a+b$, the Quotients will be $a+b$ and $aa-ab$; then setting the two former Quotients in the places of the two first Dividends, and the two latter Quotients in the places of the two latter Dividends, these two Fractions will arise, to wit;

$$\frac{aa-bb}{a+b} \text{ and } \frac{aa-ab}{d}.$$

Lastly, multiplying the Numerators $aa-bb$ and $aa-ab$ one into the other; as also the Denominators $a+b$ and d , (as in former Examples,) you will find the Product sought, to wit;

$$\frac{aaaa-aaab-aabb+abbb}{ad+bd}.$$

XXIII. When a Fraction is to be multiplied by some Integer that happens to be the same with the Denominator of the Fraction, take the Numerator for the Product required. As, for Example, to multiply $\frac{aa+ab+bb}{a+d}$ by $a+d$; I write $aa+ab+bb$ for the Product of their multiplication.

Likewise, if $\frac{b}{c}$ be to be multiplied by the Denominator c ; I write the Numerator b for the Product. The reason of this Contraction is evident; for if $\frac{b}{c}$ be multiplied by c , or $\frac{c}{1}$, in the ordinary way, the Product will stand thus, $\frac{bc}{c}$, which, by casting away the common Factor c out of the Numerator and Denominator, gives b for the Product, to wit, the Numerator of the given Fraction $\frac{b}{c}$.

Hence also, if an Algebraical Fraction be to be multiplied by some letter or letters that are found among others in every member of the Denominator, that multiplication needs no other work but the casting away such letter or letters out of the Denominator: As to multiply $\frac{ab}{cd}$ by c , the Product is $\frac{ab}{d}$; where observe, that because the multiplier c is found in the given Denominator cd , I strike it quite out.

Likewise, to multiply $\frac{ab}{cd}$ by d , I write $\frac{ab}{c}$ for the Product: And to multiply $\frac{bbb-ccc}{3faa-3gaa}$ by $3aa$, I cancel $3aa$ in the Denominator, and write $\frac{bbb-ccc}{f-g}$ for the Product required.

Note. The taking of $\frac{2}{3}$ parts of the Quantity a , imports the same thing with the multiplying of a by $\frac{2}{3}$, and the Product may be exprest either thus, $\frac{2a}{3}$, or thus, $\frac{2}{3}a$.

Likewise $\frac{2}{3}$ of $b+c$, or the Product of $b+c$ multiplied by $\frac{2}{3}$, may be exprest either thus, $\frac{2b+2c}{3}$, or thus, $\frac{2}{3}b+\frac{2}{3}c$. And so of others.

Division

Division in Algebraical Fractions.

XXIV. When the two given Fractions, to wit, the Dividend and Divisor, have a common Denominator, cast away the Denominator, and divide the Numerator of the Dividend by the Numerator of the Divisor; so that which ariseth shall be the Quotient sought. As, to divide $\frac{aab}{c}$ by $\frac{bb}{c}$; I cast away the common Denominator c , and divide aab by bb , so the Quotient sought is $\frac{aab}{bb}$, that is, $\frac{aa}{b}$.

In like manner, $\frac{aabb}{d}$ divided by $\frac{ab}{d}$ gives $\frac{aabb}{ab}$, that is, ab for the Quotient.

Again, if $\frac{aaa-abb}{c-d}$ be divided by $\frac{aa+2ab+bb}{c-d}$, there will arise $\frac{aaa-abb}{aa+2ab+bb}$, which abbreviated (by dividing the Numerator and Denominator severally by their common Divisor $a+b$) gives $\frac{aa-ab}{a+b}$ the Quotient sought.

XXV. If the given Fractions have not a common Denominator, then (as in Division of vulgar Fractions) multiply the Numerator of the Dividend by the Denominator of the Divisor, and the Product shall be a new Numerator; also, multiply the Denominator of the Dividend by the Numerator of the Divisor, and the Product shall be a new Denominator; so the new Fraction is the Quotient sought.

As, for Example, to divide $\frac{ab}{c}$ by $\frac{dd}{a}$, I multiply ab by a , and the Product $\frac{a}{a}$) $\frac{ab}{c}$ ($\frac{aab}{ddc}$ is aab for a new Numerator; also, multiplying c by dd , the Product is ddc for a new Denominator; so the Quotient sought is $\frac{aab}{ddc}$.

Likewise, if $\frac{aa-bb}{c+d}$ be divided by $\frac{c-d}{aa+bb}$, the Quotient will be $\frac{aaaa-bbbb}{cc-dd}$; For $aa-bb$ the Numerator of the Dividend being multiplied by $aa+bb$ the Denominator of the Divisor, the Product $aaaa-bbbb$ is the new Numerator; and $c+d$ the Denominator of the Dividend being multiplied by $c-d$ the Numerator of the Divisor produceth $cc-dd$ for a new Denominator; whence the Quotient sought is $\frac{aaaa-bbbb}{cc-dd}$.

XXVI. But, oftentimes there may be this useful Contraction in the Division of Fractions, viz. when either the two Numerators, or the two Denominators may be divided by some common Divisor without a Remainder, set the Quotients arising out of such Division (or imagine them to be set) in the places of the said Numerators or Denominators that were divided, and then divide as in the former Examples.

As, to divide $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$; Forasmuch as the Numerators $aa-ab$ and $a-b$ may be reduced to more simple Terms, to wit, a and 1 , (for $aa-ab$ and $a-b$ being severally divided by their common Measure $a-b$ give a and 1 . And, because the Denominators cc and cd may likewise be reduced to more simple Terms c and d , (by dividing the said cc and cd by their common Divisor c), therefore in the places of the two given Numerators $aa-ab$ and $a-b$ I set the two former Quotients a and 1 , and in the places of the two given Denominators cc and cd I set the two

latter Quotients c and d ; so there will be $\frac{a}{c}$) $\frac{a}{c}$ ($\frac{da}{c}$ and $\frac{1}{d}$ for a new Dividend and Divisor; then (as

before) I multiply a by d , and the Product is ad or da for a new Numerator; Also, c multiplied by 1 gives c for a new Denominator, and the new Fraction $\frac{da}{c}$ is the

the Quotient sought; which is equal to that which would arise by dividing $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$, to wit, the Fractions first proposed.

Again, if it be desired to divide $\frac{aaaa-bbbb}{aa-2ab+bb}$ by $\frac{aa+ab}{a-b}$; Forasmuch as the Numerators $aaaa-bbbb$ and $aa+ab$ may be reduced to $aaa-aab+abb-bbb$ and a by their common Divisor $a+b$; and the Denominators $aa-2ab+bb$ and $a-b$ may be reduced to $a-b$ and 1 , by the common Divisor $a-b$; therefore instead of multiplying $aaaa-bbbb$ by $a-b$, I multiply the said $aaa-aab+abb-bbb$ by 1 , and the Product is $aaa-aab+abb-bbb$ for a new Numerator; and instead of multiplying $aa-2ab+bb$ by $aa+ab$, I multiply $a-b$ by a , so the Product $aa-ab$ shall be a new Denominator, whence the Quotient sought is $\frac{aaa-aab+abb-bbb}{aa-ab}$.

In like manner, if $\frac{aaaa-625}{aa-10a+25}$ be divided by $\frac{aa+5a}{a-5}$, the Quotient will be $\frac{aaa-5aa+25a-125}{aa-5a}$; For $aaaa-625$ and $aa+5a$ may be reduced to $aaa-5aa+25a-125$, and a by the common Divisor $a-5$; Also, $aa-10a+25$ and $a-5$ may be reduced to $a-5$ and 1 by the common Divisor $a-5$ and 1 ; whence instead of the Fractions given we may divide $\frac{aaa-5aa+25a-125}{a-5}$ by $\frac{a}{1}$,

and the Quotient sought will be $\frac{aaa-5aa+25a-125}{aa-5a}$.

Again, to divide $\frac{aaa-2aab+abb}{aa+ab}$ by $\frac{aa-ab}{a+b}$, I set 1 for a Denominator under the Dividend $aaa-2aab+abb$, and it stands thus $\frac{aaa-2aab+abb}{1}$; then forasmuch as the Numerators $aaa-2aab+abb$ and $aa-ab$ may be reduced to $a-b$ and 1 , (by the common Divisor $aa-ab$), therefore instead of the given Dividend and Divisor we may take $\frac{a-b}{1}$ and $\frac{1}{a+b}$, whence the Quotient sought will be found $aa-bb$.

So also, if $\frac{aa+3ab}{a+4b}$ be to be divided by $\frac{a+b}{a+4b}$, that is, $\frac{aa+3ab+3abb}{a+4b}$ by $\frac{a+b}{1}$, the Quotient will be found $\frac{aa+3ab}{a+4b}$; And $\frac{xx+5x}{x-5}$ divided by $\frac{x+5x}{x-5}$, gives the Quotient $\frac{x}{x-5}$; Lastly, $\frac{xx+5x}{x-5}$ divided by $x+5$ gives the Quotient $\frac{x}{x-5}$.

C H A P. VII.

The Rule of Three in Quantities represented by Letters.

I. **A**S in Vulgar Arithmetick so here in Algebraical, if three Quantities be given to find out a fourth in a direct Proportion, that is, when the nature of the Question is such, that as the first Term is in proportion to the second, so is the third to the fourth sought; then (respect being had to the preceding Rules of Algebraical Multiplication and Division) multiply the second and third Terms one into another, and divide the Product by the first Term; so the Quotient shall be the fourth Proportional sought.

As, for example, If the Quantity a give b , what shall c give, in a direct Proportion? Or, to the same effect, find out a quantity which shall have the same proportion to c , as b hath to a ; here I multiply b by c , and then dividing the Product bc by a , the Quotient $\frac{bc}{a}$ is the fourth Proportional sought; as will appear by the

$$\frac{abc}{a} = bc.$$

Proof of the Rule of Three direct: For if the fourth Term $\frac{bc}{a}$ be multiplied by the first Term a , the Product will be $\frac{abc}{a}$, which (by *Sett. 5. Chap. 5.*) is equal to bc , to wit, the Product of the second Term multiplied by the third.

In like manner, If $a+b$ give d , what shall $c+d$ give in a Direct proportion? Answer, $\frac{dc+dd}{a+b}$.

Again, If 4 give 3, what shall $8aa$ give? Ans. $\frac{24aa}{4}$, that is, $6aa$.

Moreover, If $aaa - aab + abb - bbb$ give $aa + bb$, what shall $aa - bb$ give? Ans. $a + b$: For the second and third Terms being multiplied one by the other will produce $aaaa - abbb$, which divided by the first Term $aaa - aab + abb - bbb$ (according to the general method of Division in *Sett. 9. Chap. 5.*) gives $a + b$ the fourth Proportional sought.

II. When any one of the three given Quantities is an Algebraick Fraction, let the other two if they be Integers, in the form of Fractions, by placing 1 as a Denominator under each Integer.

Also, when any one of the three given Quantities is compos'd of an Integer and a Fraction, let it be reduced into the form of a Fraction, (by *Sett. 5. Chap. 6.*) then if the Proportion be Direct, multiply and divide as before.

As, for example, If $a + \frac{bb}{c}$ give cd , what shall $\frac{ab}{f}$ give in a direct proportion? Ans. $\frac{abcd}{acf + bbf}$: For first, $a + \frac{bb}{c}$ being reduced to the form of a fraction will stand thus $\frac{ac + bb}{c}$; also cd set fraction-wisely is $\frac{cd}{1}$; then multiplying the third Term $\frac{ab}{f}$ by the second Term $\frac{cd}{1}$, the Product is $\frac{abcd}{f}$, which divided by the first Term $\frac{ac + bb}{c}$ gives $\frac{abcd}{acf + bbf}$ for the fourth Proportional sought.

In like manner, If $\frac{ab}{c}$ give d , then $\frac{bd}{ab}$ will give $\frac{cbb}{abd}$, that is, $\frac{cb}{a}$, (for $\frac{cbb}{abd}$ being abbreviated according to *Sett. 5. Chap. 5.* gives $\frac{cb}{a}$.)

Also,

Also, If $\frac{a+c}{d}$ give $\frac{aa}{bb}$; then $\frac{bb}{a-c}$ will give $\frac{daa}{aa-cb}$.

III. If after the three given Quantities are ordered or set in the Rule according to the usual manner in Vulgar Arithmetick, the Proportion flows backwards, *viz.* If the nature of the Question be such, that as the third Term is in proportion to the second, so is the first to the fourth Term sought; then (as in the Inverse or backward Rule of Three in Vulgar Arithmetick) multiply the first and second Terms one by the other, and divide the Product by the third, so the Quotient shall be the fourth Proportional sought. But I shall not need to give Examples of this Rule, nor to make application of Algebraical Arithmetick to the Double Rule of Three, Rules of Fellowship and Alligation; since he that understands the manner of working those Rules in Vulgar Arithmetick, as also the Rules of Algebraical Arithmetick before delivered, cannot miss of performing the like work Algebraically when there is occasion.

C H A P. VIII.

An Introduction to the Extraction of ROOTS out of Algebraical Quantities.

I. **I**T is not my design in this Chapter to treat of the Extraction of Roots in general; (that Doctrine being hereafter handled in the third and fourth Chapter of the second Book) but chiefly to shew how to extract the Roots or sides of Simple Powers express'd by Letters, as also of Squares formed from Rational Binomial Roots, in order to the explication of divers Equations in the following Chapters: For I would not willingly affright the Learner with tedious and intricate Operations until he hath had a considerable taste of the practice of Algebra in the solving of Arithmetical Questions.

II. As in Vulgar Arithmetick, the extraction of the Square root of a given number imports nothing else but the finding out such a number that being multiplied by itself will produce the given number; so the extracting of the Square root of the quantity aa implies onely the finding out such a quantity, which if it be multiplied by itself will produce aa ; and since a multiplied by a produceth aa , therefore a is the Root or side of the Square aa .

Likewise the square Root of $4bb$ is $2b$; for $2b$ multiplied by $2b$ produceth $4bb$: And for the same reason, the square Root of $\frac{aa}{4}$ (or $\frac{aa}{4}$) is $\frac{a}{2}$; (or $\frac{a}{2}$) Also, the square Root of $bbaa$ is ba ; and the square Root of $aaaa$ is aa .

Moreover, Forasmuch as aa , or the Square of the Root a , being multiplied by the Root a produceth aaa , or the Cube of a , therefore the cubick Root of aaa being extracted there will come forth again the Root a . In like manner, the cubick Root of $8aaa$ is $2a$; for $2a$ multiplied cubically, (that is, first by itself and then again by the Product) produceth $8aaa$.

III. The like is to be understood in the extraction of the Root of a Compound Power; For, as the Binomial Root $a+b$, which may represent the Summ of the two parts into which some Number or Right-line is divided, being squared or multiplied by itself, produceth the Square $aa + 2ab + bb$; So the square Root of $aa + 2ab + bb$ being extracted, there will arise the Root $a + b$. Here the Learner may observe, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa + 2ab + bb$) which is made of $a+b$ the Summ of the parts, is compos'd of (aa and bb) the Squares of the parts, and of ($2ab$) the double Product made by the multiplication of the parts (a and b) one into the other.

F 2

So

So the Square of 8, or of $5+3$, is equal to $25+9+30$, that is, 64.
Again, As the Binomial, or (as some call it) the Residual Root $a-b$, or $b-a$ being multiplied by it self produceth the Square $aa-2ab+bb$; So the square Root of

$$\begin{array}{r} a-b \quad \text{The Root.} \\ a-b \\ \hline aa-2ab+bb \quad \text{The Square.} \end{array}$$

the double Product of the Multiplication of the parts one into the other: So the Square of $5-3$, that is, of 2, is equal to $25+9-30$, that is, 4.

IV. From what hath been said in the last Section, this Theorem may be infer'd, viz. If a Compound quantity consists of three such members or Simple quantities, that two of them are Squares, each of them having the sign $+$ prefix to it, and the third is the double Product made by the mutual multiplication of the Roots of those simple Squares, the said double Product also having the sign $+$ prefix to it; that Compound quantity shall be a Square whose Root is the sum of the two Roots of the said two simple Squares: But if the said double Product hath the sign $-$ prefix to it, then the difference of the said Roots shall be the Root of the said compound Square.

Hence $aa+6a+9$ will be found a Square, whose Root is $a+3$; for it is evident that aa and 9 are Squares, whose Roots are a and 3 ; and $6a$ is the double Product of the multiplication of those Roots a and 3 one by the other.

Likewise, $9bb+6bc+cc$ is a Square, whose Root is $3b+c$; for $9bb$ and cc are Squares whose Roots are $3b$ and c , and $6bc$ is the double Product of the multiplication of the Roots $3b$ and c one into the other. Also, $aaaa-baa+bbb$ will be found a Square, whose Root is $aa+bb$.

Moreover, (agreeable to the last Case in the Theorem) This Compound quantity $aa-2ca+25$ will be discovered to be a Square whose Root is $a-5$, or $5-a$. And $bbaa-2bca+cc$ is a Square whose Root is $ba-c$, or $c-ba$; For from either of these Roots, the same Square $bbaa-2bca+cc$ will be produced by Algebraical Multiplication.

If the Learner be well vers'd in this Theorem, he may oftentimes discern at first sight whether a Compound quantity that consists of three members or Single quantities be a Square or not; and if a Square, what its Root is.

V. If a quantity out of which a Root is to be extracted be such, that the Root cannot any manner of way be exactly extracted, that Root is usually design'd or represented by prefixing the Radical sign before the Quantity proposed. So to extract the square Root of the quantity a , (whether it represents a Plane number or a Superficies) I write \sqrt{a} , or $\sqrt{(2)a}$, which signifies that the square Root of a is extracted or to be extracted.

So also, $\sqrt{aa+bb}$; or, $\sqrt{(2)aa+bb}$ denotes the Square Root of the sum of the Squares aa and bb .

Likewise, to extract the Cubick Root of b , I write $\sqrt[3]{b}$; as also $\sqrt[3]{(3)abb}$, to signify the Cubick Root of abb ; which kind of Roots are called Surd or Irrational Quantities. (As hereafter in Chap. 9. of the II. Book will be more fully declared.)

VI. When it is required to extract the Root of a Fraction, the Root of the Numerator and the Root of the Denominator shall give a new Fraction which is the Root sought. As, for example, If the Square root of $\frac{aa}{bb}$ be desired; so far as the

square Root of aa is a , and the square Root of bb is b , I write $\frac{a}{b}$ for the Root sought.

1a

In like manner, the square Root of $\frac{aabb}{dd}$ is $\frac{ab}{d}$; (for the square Root of $aabb$ is ab , and the Root of dd is d .)

Again, the square Root of $\frac{aa+6a+9}{bb}$ is $\frac{a+3}{b}$; For (by the foregoing Sect. 4.) the square Root of the Numerator $aa+6a+9$ is $a+3$; and the square Root of the Denominator bb is b . Also, the square Root of $\frac{9bb+6bc+cc}{dd}$ is $\frac{3b+c}{d}$; and the cubick Root of $\frac{27ddd}{64}$ is $\frac{3d}{4}$, or $\frac{3}{4}d$.

VII. But if the Root sought cannot be extracted out of the Numerator and Denominator as before, the Radical sign is to be set before the given Fraction; as to extract the square Root of $\frac{aa}{b}$, I write $\sqrt{\frac{aa}{b}}$; or because the square Root of the Numerator is a , the square Root of $\frac{aa}{b}$ may be express'd thus $\frac{a}{\sqrt{b}}$; likewise the square Root of $\frac{aa+bb}{aabb}$ may be written either thus, $\sqrt{\frac{aa+bb}{aabb}}$; or thus, $\frac{\sqrt{aa+bb}}{ab}$.

CHAPTER IX.

Which teacheth how by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.

FROM Sect. 3. of the precedent 8. Chap. it is evident that every Square formed from a Binomial Root, that is, a Root of two Names or Parts, consists of three Members or distinct Quantities, to wit, two Affirmative Squares, and the double of the Product made by the mutual multiplication of the two Roots of those Squares; which double Product is sometimes Affirmative, and sometimes Negative: So each of these compound Squares $9aa+12a+4$, and $9aa-12a+4$, whose Roots are $3a+2$, and $3a-2$, (or $2-3a$) consists of two Squares, to wit, $9aa$ and 4 , together with $12a$, the double Product of $3a$ multiplied by 2 ; which $3a$ and 2 are the Roots of the said Squares $9aa$ and 4 : Now if any two of the three members of a Square formed from a Binomial root be given; we may find out the third member by one of these two following Rules.

II. When two Affirmative Squares are given as two of the three members or parts of a compound Square formed from a Binomial root to find out the third or mean member, extract the Square root out of each of those given Squares, then the double of the Product made by the multiplication of those Roots one into the other shall be the mean or middle member sought, which if it be annexed to the two given Squares either by $+$ or $-$, will make a compleat Compound Square having a Binomial Root.

As, for example, If the Squares $9aa$ and 4 be given, first I extract their Roots which are $3a$ and 2 , then multiplying these Roots one by the other the Product is $6a$, which doubled makes $12a$; the middle member sought; this joined by $+$ to the sum of the given Squares $9aa$ and 4 makes the compound Square $9aa+12a+4$, or $9aa+12a+4$, whose Root is $3a+2$: But if the said double Product $12a$ be joined to the sum of the Squares by $-$ there will arise the compound Square $9aa+12a+4$, or $9aa-12a+4$, whose Root is $3a-2$; or, $2-3a$.

In like manner, If $4aa$ and $9bb$ be propos'd as two of the three members of a compound Square that hath a Binomial Root, the third member will be found $12ab$, and the Square sought will be either $4aa+12ab+9bb$, whose Root is $2a+3b$; or else $4aa-12ab+9bb$, whose Root is $2a-3b$, or $3b-2a$.

III. When

III. When the double Product and either of the two Affirmative Squares aforesaid are given as two of the three members of a compound Square having a Binomial Root, to find out the other Square or third member; divide half the said double Product by the Root of the given Square, and the Square of the Quotient shall be the third member sought, which added by \pm to the two given members will compleat the Compound Square.

As, for example, If $9aa \pm 12a$ be proposed; the half of $12a$ is $6a$; this divided by $3a$ (the square Root of $9aa$) gives 2 whose Square is 4 , which added by \pm to $9aa \pm 12a$ makes $9aa \pm 12a \pm 4$, which is a compleat Compound Square, whose Root is $3a \pm 2$.

In like manner, If $12a \pm 4$ be given; the half of $12a$ is $6a$, which divided by 2 (the square Root of 4) gives $3a$, whose Square is $9aa$, which added by \pm to $12a \pm 4$, makes the compound Square $12a \pm 4 \pm 9aa$, that is, $9aa \pm 12a \pm 4$, whose Root is $3a \pm 2$.

Again, If $aa \pm 2ba$ be given; the half of $2ba$ is ba , which divided by a (the square Root of aa) gives the Quotient b , whose Square is bb ; which added to $aa \pm 2ba$ makes the Square $aa \pm 2ba \pm bb$, whose Root, because $-$ is prefix to $2ba$, shall be $a - b$, or $b - a$; But if \pm had been prefix to $2ba$, then the Root would have been $a + b$, or $b + a$.

Note. If the said Affirmative Square given be express by letters, and hath only x (to wit, Unity) prefix to it, then instead of the Rule above delivered in this Sect. 3. there may be this *Compendium*, viz. The Square of half that quantity which in the double Product given is drawn into the Root of the given Square shall be the third Member sought to compleat the compound Square: As in the last Example, where $aa \pm 2ba$ was given, because x is prefix (or must be imagined to be prefix) to aa ; I take the half of $2b$ because x is prefix (or must be imagined to be prefix) to aa ; I take the half of $2b$ to wit, b , which multiplied by it self gives bb , which added by \pm to $aa \pm 2ba$, will make (as before) the compleat Compound Square $aa \pm 2ba \pm bb$. So also to make $aa \pm 6da$ a Compleat Square, I take the half of $6d$ which is $3d$, whose Square $9dd$ added by \pm to $aa \pm 6da$ makes the compound Square $aa \pm 6da \pm 9dd$, whose Root is $a \pm 3d$. This will be further illustrated in the next Section.

IV. If a Compound quantity consists of two such quantities that one of them is an Affirmative Square express by letters, before which 1 is prefix, (or suppos'd to be prefix) and the other is the Product made by the multiplication of the Root of that Square by some quantity, which is usually called the Coefficient; that Compound quantity may be made a compleat Square thus, viz. Add by the sign \pm the Square of half the Coefficient to the Compound quantity given, so shall the sum be a Square, whose Root, when \pm is prefix to the said Product, is the sum of the Roots of the Square given and the Square added: But when $-$ is prefix to the said Product, then the Root of the Compound Square found shall be the difference of those two Roots.

As, for example, If the Compound quantity $aa \pm ca$ be proposed, I take the half of the Coefficient c , to wit, $\frac{1}{2}c$; then the Square of $\frac{1}{2}c$ is $\frac{1}{4}cc$, which added to $aa \pm ca$ makes $aa \pm ca \pm \frac{1}{4}cc$, which is a Square whose Root or Side is $a \pm \frac{1}{2}c$, to wit, the sum of the Roots of the Squares aa and $\frac{1}{4}cc$; But if the said $\frac{1}{4}cc$ be added to $aa - ca$, then there will arise the Square $aa - ca \pm \frac{1}{4}cc$, whose Root is $a - \frac{1}{2}c$, or $\frac{1}{2}c - a$.

In like manner, To make $aa \pm 5ba$ a compleat Square, and to discover its Root; I take the half of $5b$, to wit, $\frac{5}{2}b$, the Square whereof is $\frac{25}{4}bb$, which added to the given Compound quantity $aa \pm 5ba$ makes $aa \pm 5ba \pm \frac{25}{4}bb$, which is a Square whose Root is $a \pm \frac{5}{2}b$, as will easily appear by multiplying the said Root into it self.

So also, To make $aa - 12a$ a perfect Square, I add 36 (the Square of half the Coefficient 12) to $aa - 12a$, and it makes the compound Square $aa - 12a \pm 36$, whose Root is $a - 6$, or $6 - a$.

Again, To find what Quantity must be added to $aaaa \pm aa$, or $aaaa \pm 1aa$, to make a compleat Square; I take $\frac{1}{2}$, to wit, half the Coefficient 1 which is prefix to aa , (the Square root of $aaaa$) and then the Square of the said $\frac{1}{2}$ is $\frac{1}{4}$; this added to $aaaa \pm 1aa$ makes the Square $aaaa \pm 1aa \pm \frac{1}{4}$, or, $aaaa \pm aa \pm \frac{1}{4}$, whose Root is $aa \pm \frac{1}{2}$, to wit, the sum of the Roots of the Squares $aaaa$ and $\frac{1}{4}$.

After

After the same manner, To make this Compound Quantity a compleat Square,

I take the half of the Coefficient $\frac{2b \pm 3c}{d}$, to wit,

Then the Square of that half Coefficient is

Which Square added to the Compound quantity proposed, makes

Which last Compound quantity is a Square, whose Root is

Likewise, If it be desired to make this Compound quantity a compleat Square, to wit, $aaaaa \pm baaa$, I add to it the Square of half the Coefficient b , to wit, $\frac{1}{4}bb$; so there will be $aaaaa \pm baaa \pm \frac{1}{4}bb$ the Square desired, whose Root is $aaa \pm \frac{1}{2}b$.

CHAP. X.

A Collection of easie Questions to exercise the Rules hitherto delivered.

I. There are two Quantities whereof the greater is a (or, 3), the lesser is b , (or 2); What is their Summ? What is their Difference? What is the Product of their Multiplication? What is the Quotient of the greater divided by the lesser? What is the Quotient of the lesser divided by the greater? What is the Summ of their Squares? What is the Difference of their Squares? What is the sum of the Summ and Difference of the two Quantities first proposed? What is the difference of their Summ and Difference? What is the Product made by the multiplication of the Summ by the Difference? What is the Square of the Summ? What is the Square of the Difference? What is the Summ of the Squares of the Summ and Difference? What is the Difference between the Square of the Summ, and the Square of the Difference? What is the Square of the Product of the multiplication of the said two Quantities?

Answers by Letters, by Numbers.

1. The Summ of the two Quantities proposed is . . .	$a + b$	5
2. Their Difference, or the excess of the greater above the less, is	$a - b$	1
3. The Product of their Multiplication is	ab	6
4. The Quotient of the greater divided by the less is . .	$\frac{a}{b}$	$\frac{1}{2}$
5. The Quotient of the lesser divided by the greater is	$\frac{b}{a}$	$\frac{2}{3}$
6. The Summ of their Squares is	$aa + bb$	13
7. The Difference of their Squares is	$aa - bb$	5
8. The sum of the Summ and Difference of the two Quantities first proposed is	$2a$	6
9. The difference of their Summ and Difference is . .	$2b$	4
10. The Product of the Multiplication of the Summ by the Difference is	$aa - bb$	5
11. The Square of the Summ is	$aa \pm 2ab \pm bb$	25
12. The Square of the Difference is	$aa \pm 2ab \pm bb$	1

13. The

13. The Summ of the Squares of the Summ and Difference is	$2aa + 12cc$	26
14. The difference between the Square of the Summ and the Square of the Difference is	$4ac$	24
15. The Square of the Product of the multiplication of the two Quantities is	$aacc$	36

In like manner, If the greater of two Quantities be c , (or 4,) and the lesser be $\frac{b-d}{c}$, (which we may suppose to represent $\frac{20-12}{4}$, that is, 2; by putting b for 20, and d for 12;) then

1. The Summ of those two Quantities will be	$c + \frac{b-d}{c}$	6
2. Their Difference is	$c - \frac{b-d}{c}$	2
3. The Product of their Multiplication is	$b - d$	8
4. The Quotient of the greater divided by the less is	$\frac{cc}{b-d}$	2
5. The Quotient of the lesser divided by the greater is	$\frac{b-d}{cc}$	$\frac{1}{2}$
6. The summ of their Squares is	$cc + \frac{bb-2bd+dd}{cc}$	20
7. The difference of their Squares is	$cc - \frac{bb-2bd+dd}{cc}$	12
8. The summ of the Summ and Difference of the two quantities is	$2c$	8
9. The difference between the Summ and Difference is	$\frac{2b-2d}{c}$	4
10. The Product of the Summ multiplied by the Difference is	$cc - \frac{bb-2bd+dd}{cc}$	12

11. There are two Quantities whose Summ is b , (or 20,) and the greater of them is put a , (or 12;) What is the Lesser? What is their Difference? What is the Product of their multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. If from the Summ of two quantities the greater be subtracted, the Remainder shall be the lesser; therefore the lesser quantity sought is	$b - a$	8
2. If from the greater quantity a , the lesser $b - a$ be subtracted, the Remainder or Difference will be	$2a - b$	4
3. The Product of the multiplication of the two quantities is	$ba - aa$	96
4. The Summ of their Squares is	$2aa + bb - 2ba$	208
5. The Difference of their Squares is	$2ba - bb$	80

1. But if the Summ of two quantities be represented by	b	20
2. And for the lesser of them there be put	c	8
3. The Greater quantity shall be	$b - c$	12
4. Their Difference shall be	$b - 2c$	4
5. The Product of their Multiplication	$bc - cc$	96
6. The Summ of their Squares	$2cc + bb - 2bc$	208
7. The Difference of their Squares	$bb - 2bc$	80

III. There

III. There are two Quantities whose Difference is d , (or 4,) and if for the Greater quantity there be put a , (or 12;) What is the Lesser? What is their Summ? What is the Product of their Multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. By subtracting the Difference from the Greater quantity, the Lesser will be	$a - d$	8
2. The Summ of the two quantities is	$2a - d$	20
3. The Product of their Multiplication is	$aa - da$	96
4. The Summ of their Squares is	$2aa + dd - 2da$	208
5. The Difference of their Squares is	$2da - dd$	80

1. But if the Difference of two quantities be	d	4
2. And for the Lesser quantity you put	c	8
3. The Greater shall be the summ of the Difference and the Lesser, to wit,	$d + c$	12
4. The Summ of the two Quantities is	$d + 2c$	20
5. The Product of their Multiplication is	$de + ce$	96
6. The Summ of their Squares is	$dd + 2de + 12cc$	208
7. The Difference of their Squares is	$dd + 2de$	80

IV. There are two Quantities, whereof the Greater hath such proportion to the Lesser as r (3) to s , (2,) now if for the Greater quantity there be put a , (15,) What is the Lesser? What is their Summ? What is their Difference? What is the Product of their Multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. First, say by the Rule of Three, If r give s ; what will a give? <i>Ans.</i> $\frac{sa}{r}$, which is the Lesser quantity sought	$\frac{sa}{r}$	10
2. Then the Summ of the two quantities will be	$a + \frac{sa}{r}$	25
3. Their Difference is	$a - \frac{sa}{r}$	5
4. The Product of their Multiplication is	$\frac{sa^2}{r}$	150
5. The Summ of their Squares is	$aa + \frac{51aa}{rr}$	325
6. The Difference of their Squares is	$aa - \frac{51aa}{rr}$	125

But if the Lesser of two quantities be e (10,) and hath such proportion to the Greater as s (2,) to r (3,) Then

1. The Greater quantity will by the Rule of Three be found	$\frac{re}{s}$	15
2. And the Summ of the two quantities will be	$\frac{re}{s} + e$	25
3. Their Difference is	$\frac{re}{s} - e$	5
4. The Product of their Multiplication is	$\frac{ree}{s}$	150
5. The Summ of their Squares is	$\frac{rree}{ss} + ee$	325
6. The Difference of their Squares is	$\frac{rree}{ss} - ee$	125

G

V. There

V. There are two Quantities, the Product of whose multiplication is $b(20)$ and if for the Greater quantity there be put $a(5)$ What is the Lesser? What is their Summ? What is their Difference? What is the Summ of their Squares? What is the Difference of their Squares?

1. The Product b divided by the Greater quantity a , gives the Lesser, to wit,	$\frac{b}{a}$	4
2. Then the Summ of the two quantities is	$a + \frac{b}{a}$	9
3. Their Difference is	$a - \frac{b}{a}$	1
4. The Summ of their Squares is	$aa + \frac{bb}{aa}$	41
5. The Difference of their Squares is	$aa - \frac{bb}{aa}$	9

But if the Product of the multiplication of two quantities be $b(20)$, and for the Lesser there be put $e(4)$

1. The greater quantity will be	$\frac{b}{e}$	5
2. The Summ of the two quantities is	$\frac{b}{e} + e$	9
3. The Difference is	$\frac{b}{e} - e$	1
4. The Summ of their Squares is	$\frac{bb}{ee} + ee$	41
5. The difference of their Squares is	$\frac{bb}{ee} - ee$	9

VI. The extraction of Roots may be exercised by these following Questions, respect being had to *Self*. 28. *Chap.* 1. as also *Chap.* 8.

1. What is the Square Root of $144aa$? *Ans.* $12a$.
2. What is the Square Root of $16aabb$? *Ans.* $4ab$.
3. What is the Square Root of $9aa - 6ab + bb$? *Ans.* $3a - b$, or $b - 3a$.
4. What is the Square Root of $\frac{4aa + 16ab + 16bb}{9cc}$? *Ans.* $\frac{2a + 4b}{3c}$.
5. What is the Cubick Root of $125aaabbb$? *Ans.* $5ab$.
6. If b be put for 65, and e for 8, what number is signified by $\sqrt[3]{b + \frac{1}{4}cc} = \frac{1}{2}c$? *Ans.* 5.
7. The same things being put as in the last Question, what number is signified by $\sqrt[3]{b + \frac{1}{4}cc} + \frac{1}{2}c$? *Ans.* 13.
8. If d be put for 8, and f for 48, what number is signified by $\sqrt[3]{f + \frac{1}{2}dd} = \frac{1}{2}d$? *Ans.* 2.
9. But the same things being put as in the last Question, this quantity $\sqrt[3]{f + \frac{1}{2}dd} + \frac{1}{2}d$ signifies $\sqrt[3]{12}$, or, 3.464, &c. that is, $3\frac{1}{2}\frac{1}{1000}$, &c.
10. If g be put for 4, and b for 837, what number is signified by $\sqrt[3]{(3)\sqrt{b + \frac{1}{2}gg} - \frac{1}{2}g}$? *Ans.* 3.
11. But the same things being put as in the last Question, this quantity $\sqrt[3]{(3)\sqrt{b + \frac{1}{2}gg} + \frac{1}{2}g}$ signifies $\sqrt[3]{(3)31}$, or, 3.141, &c.

VII. The Rules of the ninth *Chap.* may be exercised by these following Questions.

1. What Quantity is that which if it be added to $aa + 25$, will make the sum a Square? *Ans.* The Quantity to be added may be either $+10a$, or $-10a$; and

and the Square sought is either $aa + 10a + 25$, whose Root or side is $a + 5$; or else the Square is $aa - 10a + 25$, whose Root is $a - 5$, or $5 - a$.

2. What Quantity is that which if it be added to $\frac{1}{4}aa + \frac{1}{2}bb$, will make the sum a Square? *Ans.* The Quantity to be added may be either $+ab$, or $-ab$; and the Square is either $\frac{1}{4}aa + ab + \frac{1}{4}bb$, whose Root is $\frac{1}{2}a + \frac{1}{2}b$; Or else the Square is $\frac{1}{4}aa - ab + \frac{1}{4}bb$, whose Root is $\frac{1}{2}a - \frac{1}{2}b$; or $\frac{1}{2}b - \frac{1}{2}a$.

3. What Quantity is that which if it be added to $aa + 3a + \frac{1}{4}$ will make the sum a Square? *Ans.* The Quantity to be added is $\frac{1}{2}a$; and the Square is $aa + 3a + \frac{1}{4}$, whose Root is $a + \frac{1}{2}$.

4. What Quantity is that which together with $aaaa - 2bbaa$ will make a perfect Square? *Ans.* The Quantity to be added is $bbbb$; and the Square is $aaaa - 2bbaa + bbbb$, whose Root is $aa - bb$, or $bb - aa$.

5. What Quantity is that which if it be added to $aa + \frac{bb}{c}a$, will make the sum a Square? *Ans.* The Quantity to be added is $\frac{bbbb}{4cc}$; and the Square is $aa + \frac{bb}{c}a + \frac{bbbb}{4cc}$, whose Root is $a + \frac{bb}{2c}$.

6. What Quantity is that which together with $aaaaa - ada$ will make a complete Square? *Ans.* The Quantity to be added is $\frac{1}{4}a$; and the Square sought is $aaaaa - ada + \frac{1}{4}a$, whose Root is $aaa - \frac{1}{2}$, or $\frac{1}{2} - aaa$.

CHAP. XI.

Concerning an Equation, and the Reduction of Equations.

I. **AN** Equation in the Algebraical Art is a mutual Comparing of two Equal quantities or things of different Denominations: as, If the value of three shillings be compared to thirty six pence of *English* money, that comparison imports an Equation, which may be Symbolically exprest thus, $3s = 36d$, that is, three shillings are equal to thirty six pence. Likewise, forasmuch as nine Crowns are of equal value with the sum of two Pounds and five Shillings of *English* money, the comparing of these two sums to one another is nothing else but an Equation which may be briefly exprest thus, $9c = 2l + 5s$. In each of which Equations the Moneys compared are of different kinds; for Equations between equal things of one and the same name, as $2s = 2s$, or $5 = 5$, and such like, are fruitless.

After the same manner, this Equation $a = b + c$ may signify that some number or line represented by a is equal to two other numbers or lines b and c taken together as one; or, if the number or line a be divided into two parts b and c , then also $a = b + c$; for the whole is equal to all its parts.

II. Every Equation consists of two Parts, which are usually separated one from another by this Character $=$, so in the first Equation in the precedent *Self*. $3s$ is the first Part, and $36d$ the latter; also in the second Equation, $9c$ is the first Part, and $2l + 5s$ the latter; likewise in the last Equation of the same *Section*, a is the first Part, and $b + c$ the latter.

III. The single Quantities or things, whereof each part of an Equation is composed, are called the Terms of an Equation; as in this Equation, $a = b + c$, the Terms are a , b and c .

IV. How Equations are found out, the Resolution of Questions will hereafter shew; but when known quantities are intermingled with unknown in an Equation, the first scope is to clear the Equation from all superfluous quantities, and to separate the known quantities from the unknown, that at length an Equation may remain in the fewest and simplest

Terms, so disposed, that the unknown quantity or quantities may possess one part of the Equation, and the known the other; this work is called *Reduction*, and how 'tis perform'd the Examples in the following *Sections* will make manifest.

Reduction by Addition.

V. Reduction by Addition is grounded upon this Axiom, (or common Notion) *viz.* If equal quantities, or one and the same quantity, be added to equal quantities, the whole shall be equal. As, for Example;

If the letter a represent some number unknown, and it be granted or found out that $a - 3 = 12$
Then by adding $+3$ to each part of that Equation, this ariseth, to wit, $a - 3 + 3 = 12 + 3$
That is, (because -3 and $+3$ added together make 0,) $a = 15$

In like manner, to reduce this Equation $3a - 4 = 6 - a$
I add $+4$ to each part, and there ariseth $3a - 4 + 4 = 6 - a + 4$
Which Equation contracted makes $3a = 10 - a$
Then by adding $+a$ to each part of the last Equation, this ariseth, $3a + a = 10 - a + a$
That is, alter each part is contracted, $4a = 10$

Again, If this Equation be propos'd to be reduced $aa - b = d + b$
By adding $+b$ to each part, this Equation ariseth, $aa - b + b = d + b + b$
Which last Equation, after due contraction gives $aa = d + 2b$

So also, If $a - b = 0$
By adding $+b$ to each part, there ariseth $a = b$

Likewise, If $b - a = 0$
By adding a to each part there ariseth $b = a$

Moreover, If $aa - bb - cc = dd$
Then by adding $bb + cc$ to each part this Equation comes forth, $aa = dd + bb + cc$

Lastly, If $aa - bb = cc - da$
By adding $+bb$ to each part, this Equation ariseth, $aa = cc - da + bb$
And by adding $+da$ to each part of the last Equation, this ariseth, to wit, $aa + da = cc + bb$

From the premises it is evident, That if in any Equation any Quantity which hath the sign $-$ prefixed to it, be transfer'd to the other part of the Equation with the sign $+$, that work effects the same thing as the adding of that Quantity to each part of the Equation, and is called *Transposition*.

Reduction

Reduction by Subtraction.

VI. If from equal Quantities you take away equal Quantities, or one and the same Quantity, the Quantities remaining will be equal; therefore,

If it be taken for granted that $a - 3 = 12$
Then by subtracting -3 from each part, there ariseth $a = 15$

In like manner, If $b + a = 4b$
I subtract $+b$ from each part, and there ariseth $a = 3b$

Again, If $bb + 2aa = ad + cc$
First, I subtract bb from each part, and there remains $2aa = ad + cc - bb$
Then aa subtracted from each part of the last Equation leaves this, to wit, $aa = cc + bb$

So also, If $aa + b + c = 2ca + df$
By subtracting $+b + c$ from each part, there ariseth $aa = 2ca + df - b - c$
And by subtracting $2ca$ from each part of the last Equation, this ariseth, to wit, $aa - 2ca = df - b - c$

Hence it is evident, That if in any Equation any Quantity which hath the sign $+$ prefixed to it be transfer'd to the other part of the Equation with the sign $-$, that work effects the same thing as the subtracting of that Quantity from each part of the Equation; and is also called *Transposition*.

Reduction by Multiplication.

VII. If equal Quantities be multiplied by equal Quantities, or by one and the same Quantity, the Products shall be equal. Hence Equations exprest by Algebraical Fractions are reduced to other Equations consisting altogether of Integers.

As, for Example, If $\frac{a}{5} = 6$

Then by multiplying each part by 5, this Equation is produced $a = 30$

Again, to reduce this Equation to another in Integers, *viz.* $a = \frac{dd}{a - b}$

I multiply each part by $a - b$ and there comes forth $aa - ab = dd$

Likewise, to reduce this Equation to another in Integers, $\frac{3aa}{c} = \frac{dd}{b}$

First, I multiply each part by the Denominator b , and there will be produced $\frac{3aab}{c} = dd$

Then multiplying each part of the last Equation by the Denominator c , I find this Equation $3aab = cdd$

Hence it is manifest, That an Equation whereof each part is a Fraction, may be reduced to another Equation in Integers, by multiplying cross-wise, as in the reduction of

of Fractions to a common Denominator, and then omitting the common Denominator, a new Equation may be instituted between the new Numerators only.

When either part of an Equation is compos'd of Integers and Fractions, first reduce that part into a Fraction, (after the manner of the latter Example in *Self. 16. Chap. 6.*) and then multiply as in the preceding Examples: as,

If this Equation be propos'd, $\frac{aa}{b} + c + d = \frac{bc}{a} + \frac{dd}{a}$
 First, I reduce that Equation to this, $\frac{aa}{b} + \frac{bc}{a} + \frac{bd}{a} = \frac{bc}{a} + \frac{dd}{a}$
 Which last Equation reduced by Multipli-
 cation as in the preceding Examples, gives $aaa + abc + abd = bba + bdd$

But here is to be noted, that in reducing Equations which consist of Fractions into other Equations in Integers, the Operation may oftentimes be facilitated by the same compendium that hath before been shewn in the Division of Fractions (in *Self. 26. Chap. 6.*) viz. When either the Numerators or Denominators can be reduced to more simple Terms by some common Divisor, set the Quotients in the places of those Numerators or Denominators; and then reduce these new Fractions into an Equation in Integers, by multiplying cross-wise as before: As, for example,

To reduce this Equation to another in Integers, $\frac{aaa}{aa-bb} = \frac{ba-bb}{a+b}$
 First, after the Denominators $aa-bb$ and $a+b$ are reduced to $a-b$ and 1 , by the common Divisor $a+b$, this new Equation ariseth, $\frac{aaa}{a-b} = \frac{ba-bb}{1}$
 Whence, by multiplying cross-wise, (as in the preceding Examples) this Equation in Integers is produced, $aaa = baa - 2bba + bbb$

Again, to reduce this Equation to another in Integers, $\frac{bba-cca}{a+b} = \frac{bbb-bcc}{a}$
 First, the Numerators reduced to a and b by the common Divisor, $bb-cc$ will give
 Whence by multiplying cross-wise, this Equation is produced $aa = ba + bb$

In like manner, to reduce this Equation, $\frac{baa-caa}{cc-ca} = \frac{bb-bc}{c}$
 First, I reduce the Numerators to aa and b , by the common Divisor $b-c$; also, the Denominators to $c-a$ and 1 , by the common Divisor c ; which new Numerators and Denominators constitute this Equation, $\frac{aa}{c-a} = \frac{b}{1}$
 Whence by multiplying cross-wise, this Equation is produced $aa = bc - ba$

So also to reduce this Equation $\frac{ba^3-ca^3}{aa-ba+bb} = \frac{bc-cc}{1}$
 First, I set 1 for a Denominator under the Integer $bc-cc$, so the Equation proposed will stand thus, $\frac{ba^3-ca^3}{aa-ba+bb} = \frac{bc-cc}{1}$

Then

Then, after the Numerators ba^3-ca^3 and $bc-cc$ are reduced to a^3 and c , by the common Divisor $b-c$, this Equation ariseth, $\frac{a^3}{aa-ba+bb} = \frac{c}{1}$
 Which last Equation, by multiplying cross-wise, gives this in Integers, $aaa = caa - cba + cbb$

When one part of an Equation is a Surd quantity, (that is, such which hath a Radical sign prefix to it, as, $\sqrt{}$, or $\sqrt{3}$, &c.) and the other part is a Rational quantity; that Equation may be reduced to another which shall be free from any Surd quantity, by casting away the Radical sign, and multiplying the rational part of the given Equation either quadratically or cubically, &c. according to the import of the Radical sign; as,

If there be propos'd $\sqrt{a} = 6$
 Forasmuch as the Squares of equal Roots or Sides are also equal, therefore by squaring each part of that Equation, this is produced, to wit, $a = 36$
 Likewise, if $\sqrt{a} = bc$
 By multiplying each part into it self, this Equation is produced, $a = bbcc$
 Again, if $\sqrt{a} = \sqrt{5}$
 By squaring each part, there comes forth $a = 5$
 And, if $\sqrt{a} = \sqrt{bcc-b}$
 By squaring each part, which is done by casting away $\sqrt{}$, there will arise $a = bcc - b$
 So also if this Equation be propos'd, $\sqrt{ca} = b-d$
 By multiplying each part into it self, this Equation is produced, $ca = bb - 2bd + dd$
 And, if $\sqrt{(3)a} = 8$
 By multiplying each part into it self cubically, there ariseth $a = 512$
 Also, if $\sqrt{(3)a} = \sqrt{(3)b+c}$
 By casting away $\sqrt{(3)}$ from each part it gives $a = b+c$

Reduction by Division

VIII. If equal Quantities be divided by equal Quantities, or by one and the same Quantity, there will come forth equal Quotients. Hence Equations are reduced to others of lower Degrees: As, for example;

If it be granted or found out that $aa = 5a$
 Then by dividing each part by a , you will find $a = 5$
 Again, if $aaa + baa = bba$
 By dividing each part by a , this Equation ariseth, $aa + ba = bb$
 Also, if $5a = 15$
 By dividing each part by 5 , there ariseth $a = 3$
 Likewise, if $ba = bc$
 By dividing each part by b , this Equation ariseth, $a = c$
 Again, if $ba-ca = cc$
 By dividing each part by $b-c$, there ariseth $a = \frac{cc}{b-c}$
 Also, if $baa + caa = ba + ca$
 By dividing each part by $b+c$, there ariseth $aa = d$

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Moreover, If $3aa + 4a = 39$
 By dividing each part by 3, there ariseth $aa + \frac{4}{3}a = 13$
 Likewise, If $caa - ba = cda$
 By dividing each part by c , there ariseth $aa - \frac{b}{c}a = dd$

Reduction by Extraction of ROOTS.

IX. Forasmuch as the Sides or Roots of equal Squares and Cubes, &c. are also equal between themselves; therefore,

If there be proposed $aa = 36$
 By extracting the square Root of each part, there ariseth $a = 6$
 In like manner, If $aa = bb + 2bc + cc$
 By extracting the square Root of each part, there comes forth $a = b + c$
 Again, If $aa = 29$
 By extracting the square Root of each part, there will arise $a = \sqrt{29}$
 Likewise, If $aa = bb - dd$
 Then, by extracting the square Root out of each part, there ariseth $a = \sqrt{bb - dd}$
 Again, If $aaa = 27$
 Then, the cubick Root being extracted out of each part, there comes forth $a = 3$
 Also, If $aaa = 12$
 By extracting the cubick Root out of each part, this Equation will arise, $a = \sqrt[3]{12}$
 Likewise, If $aaa = bbc + ccd$
 Then, the cubick Root extracted out of each part, gives $a = \sqrt[3]{(3): bbc + ccd}$

X. By the help of some of the foregoing Reductions, I shall here shew, (after the manner of *Fran. van Schooten* in his *Principia Mathes. universal.*) the certainty of the Rule before given concerning $+$ and $-$ in the Algebraical Multiplication of Compound quantities: *viz.* That $+$ multiplied by $-$, or $-$ by $+$ makes $-$; also, That $-$ multiplied by $-$ makes $+$.

First, let $a - b$ be to be multiplied by c , then the Product according to Algebraical Multiplication is $ac - bc$: now it must be proved that $-b$ multiplied by $+$ makes $-bc$; to which end, let f be put equal to $a - b$, and then if it be proved that $ac - bc = fc$, it is evident that $ac - bc$ is the true Product sought; and consequently, $-b$ multiplied by $+$ makes $-bc$: But that $ac - bc = fc$ may be proved thus,

Forasmuch as by supposition, $a - b = f$
 Therefore by adding b to each part, it makes $a = f + b$
 And by multiplying each part of the last Equation by c , there will be produced $ac = fc + bc$
 Wherefore, by subtracting bc from each part of the last Equation there remains $ac - bc = fc$
 Which was to be proved.

After the same manner it may be proved that $-$ multiplied by $-$ makes $+$: For, if $a - b$ be to be multiplied by $c - d$, and there be put (as before) $f = a - b$, it may be shewn that $ac - bc - ad + bd$ is equal to $a - b \times c - d$ the Product sought; and therefore $-b$ multiplied by $-d$ produceth $+bd$. For,

By

By supposition $f = a - b$
 Therefore, by multiplying each part into $c - d$ $f \times c - d = a - b \times c - d$
 That is, $fc - fd = a - b \times c - d$
 But it hath been proved in the former Example, that $ac - bc = fc$
 Therefore instead of fc in the third Equation of this latter Example, taking $ac - bc$ (equal to fc) there ariseth $ac - bc - fd = a - b \times c - d$
 Again, If each part of the first Equation be multiplied by d , this will be produced, $fd = ad - bd$
 Wherefore, If from $ac - bc$ in the fifth Equation there be subtracted $ad - bd$ instead of fd equal to $ad - bd$, there will remain according to the Rule of Algebraical Subtraction. $ac - bc - ad + bd = a - b \times c - d$
 Which was to be proved.

CHAP. XII.

Which shews in what Order the Reductions in the foregoing Chap. 11. are to be used to resolve Equations, or at least to prepare them for Resolution.

I. BY the help of the precedent Reductions, either the value of the unknown Root or Quantity sought in an Equation will be found equal to some known Quantity or Quantities, and consequently the Quantity sought is then known also, or else a new Equation will be discovered, from whence the same Quantity sought may be made known by some other Rule or Rules hereafter delivered: But in the use of those Reductions, the work may oftentimes be facilitated by an orderly process, which is the scope of the five following Sections, where I assume the Vowel a to stand for the unknown Root or Quantity sought, and Consonants for known Quantities.

II. If in any Equation the Quantity sought, or any Power or Degree of it, be found in a Fraction, reduce that Equation to another that may be exprest altogether by Integers, (by *Self. 7. Chap. 11.*) As, for Example,

If this Equation be proposed, $\frac{b-a}{c} = d + f$
 By multiplying each part thereof by the Denominator c , this Equation ariseth $b - a = cd + cf = cg$
 in Integers, $b - a = cd + cf = cg$

After the same manner, this Equation multiplied by 4, $\frac{aa + 6}{4} = 15$
 Will be reduced into $aa + 6 = 60$

Likewise this Equation $\frac{aa + bb}{d} + \frac{c}{d} = e$
 Will be reduced to $aa + bb + db + dc = da - dc$

III. When Quantities given or known be intermingled with those that are sought in an Equation, let Quantities be transferr'd from one part of the Equation to the other under a contrary Sign, (according to *Self. 5. and 6. of Chap. 11.*) until at length the

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unknown Quantity may make one part of an Equation, and all the known Quantities the other: As, for example;

If there be proposed $2a - 26 = 8$
By transposition of -26 to the other part
of the Equation, under the contrary
sign $+$, there will arise $2a = 8 + 26 = 34$.

In like manner, If $aa + 24 = 60$
By transposition of $+24$, under the con-
trary sign $-$ it gives $aa = 60 - 24$
That is, $aa = 36$

Again, If $6a - 4 = 20 - a$
First, by transposition of -4 , this E-
quation aritheth $6a = 20 - a + 4$
Then by transposition of $-a$, I find $6a + a = 20 + 4$
Which last Equation being contracted by
Addition, gives $7a = 24$

Likewise, If $b - a = cd - cf$
After due transposition, this Equation
will arise, $b + cf - cd = a$
Or, $a = b + cf - cd$

IV. When some Power or Degree of the Quantity sought happens to be multiplied into every Term or Member of an Equation, divide every Term by that Degree, so will that Degree or Power quite vanish, and consequently the Equation will be depressed, that is, reduced to lower Degrees or more simple Terms: As, for example,

If there be proposed $aa + 3a = 20a$
Forasmuch as a is drawn into every Term
of that Equation, I divide every Term
by a , and there aritheth $a + 3 = 20$
Whence by equal subtraction of 3 I find $a = 17$

In like manner, If $aaa = 3aa$
By casting away aa , that is, by dividing each
part by aa , there will arise $a = 3$

Again, If $aaaa + baaa = ddaa$
By expunging aa out of every Term, there
aritheth $aa + ba = dd$

V. When some known Quantity is multiplied into the highest Power or Degree of the Quantity unknown or sought in an Equation; divide each part of the Equation by that known Quantity, to the end the said highest unknown Power may have no Co-efficient or fellow-multiplyer but 1, (or unity;) As, for example,

If there be proposed $5a = 60$
Because the unknown quantity a is multi-
plied by 5, I divide each part of the
Equation by 5, and there aritheth $a = 12$

Again, If $ca = cc + dd$
Because c is drawn into a the Root sought,
I divide every Term of the Equation
by c , and there aritheth $a = c + \frac{dd}{c}$

Like-

Likewise, If $2ba + 3ca = 2add + 3cdd$
Because $2b + 3c$ is drawn into the un-
known Root a , I divide each part by
 $2b + 3c$, and there aritheth $a = dd$

So also, If $4aa = 60$
By dividing each part by 4 which is drawn
into aa , there aritheth $aa = 15$

Again, If $3aa - 5a = 24$
Because 3 is drawn into aa which is the
highest unknown Power in the Equation,
I divide every Term by 3, and there aritheth $aa - \frac{5}{3}a = 8$

Likewise, If $2ccaa - 4dda = 5bbcc$
Because $2cc$ is drawn into aa which is the
highest unknown Power in the Equation,
I divide every Term by $2cc$, and there
aritheth $aa - \frac{4dd}{cc}a = \frac{5bb}{2}$

Again, If $2bbaa + 3cdaa - dda = ccdd$
Because $2bb + 3cd$ is drawn into aa the
highest unknown Degree in the Equa-
tion, I divide each part by $2bb + 3cd$,
and there aritheth $aa = \frac{ccdd}{2bb + 3cd}$

Also, If $3aaa + 24aa - 6a = 1200$
Because 3 is drawn into aaa the highest un-
known Power in the Equation, I divide
each part by 3, and there aritheth $aaa + 8aa - 2a = 400$

VI. If there be a Surd quantity in an Equation, that is, if a Radical sign as $\sqrt{\quad}$, or $\sqrt[3]{\quad}$ be prefixed before some Quantity; first by Transposition (according to Sect. 4. or 6. of Chap. 11.) make the Surd quantity sole possessor of one part of an Equation, then cast away the Radical sign, and exalt the other part of the Equation to the same degree or Power which is denoted by the Radical sign, by multiplying Quadratically or Cubically, &c. so at length an Equation will be found express'd altogether by Rational quantities: As, for example;

If this Equation be proposed $\sqrt{a} = \frac{3}{2}$
By squaring each part, there will be produced $a = \frac{9}{4}$

In like manner, If $\sqrt{ba} = 3bc$
By multiplying each part into it self qua-
dratically, there comes forth $ba = 9bbcc$
Then dividing each part of the last Equation
by b , there aritheth $a = 9bcc$

Again, If $b + \sqrt{ba} = c$
First by transposition of b there aritheth $\sqrt{ba} = c - b$
Then by squaring each part of the last
Equation, there will be produced $ba = cc - 2cb + bb$
Whence, by dividing each part by b ,
there aritheth $a = \frac{cc}{b} - 2c + b$

H 2

Like-

Likewise, If $-d$, this Equation } $-d + \sqrt{ba + da} = b$
 First by transposition of $-d$, this Equation } $\sqrt{ba + da} = b + d$
 ariseth.
 Then by squaring each part, there will be } $ba + da = bb + 2bd + dd$
 produced.
 Lastly, by dividing each part of the last } $a = b + d$
 Equation by $b + d$, there ariseth . . . }

Again, If . . . $\sqrt{(3)9a} = 3$
 By multiplying each part Cubically, there } $9a = 27$
 will be produced.
 And, by dividing each part of the last Equa- } $a = 3$
 tion by 9 there ariseth . . . }

Likewise, If . . . $\sqrt{(3)ba - ca + c} = b$
 First, by transposition of $+c$ this E- } $\sqrt{(3)ba - ca} = b - c$
 quation ariseth, . . .
 Then multiplying each part of the last Equation cubically, this Equation will be pro-
 duced, to wit,

$$ba - ca = bbb - 3bbc + 3bcc - ccc:$$

Whence, by dividing each part by $b - c$, the value of a will be discovered, viz.
 $a = bb - 2bc + cc.$

VII. When after the using of all or any of the foregoing Rules of this Chapter an Equation ariseth between a perfect Square, Cube or other higher Power of the quantity fought, and some known quantity; then extract such a Root out of each part of the said Equation as the Index of the said unknown Power denoteth; so will the value of the unknown Root or Quantity fought be made known: As, for example;

If this Equation be proposed, to wit, $\frac{6aa}{5} + 8 = 128$
 First by subtracting 8 from each part, this } $\frac{6aa}{5} = 120$
 Equation ariseth, . . .
 Then each part of the last Equation being } $6aa = 600$
 multiplied by 5, gives
 And by dividing each part of the last E- } $aa = 100$
 quation by 6, this ariseth, . . .
 Lastly, the Square root of each part of the } $a = 10$
 last Equation being extracted, the value
 of a will be discovered, to wit, . . . }

Again, If . . . $\frac{3aaaa}{4} - 8a = 1544$
 Then by transposition of $-8a$ there ariseth $\frac{3aaaa}{4} = 1624$
 And by multiplying each part of the last } $3aaaa = 6484$
 Equation by 4, this will be produced,
 And by dividing each part of the last Equa- } $3aaa = 648$
 tion by 3 this ariseth, to wit, . . .
 Likewise each part of the last Equation di- } $aaa = 216$
 vided by 3 gives
 Lastly, by extracting the Cubick root } $a = 6$
 out of each part of the last Equation, the
 value of a will be discovered, to wit, . . . }

Like-

Likewise, If . . . $na + 2ba + bb = cc.$
 The Square root extracted out of each part, } $a + b = c.$
 gives
 And then by transposition of b , the value } $a = c - b$
 of a is discovered, to wit, . . . }

CHAPTER XIII.

Which shews how to convert Analogies into Equations,
 and Equations into Analogies.

I. IF four right-lines or numbers be Proportionals, the Product made by the multipli-
 cation of the two extremes is equal to the Product of the two means. And if three
 right-lines or numbers be Proportionals, the Product of the extremes is equal to the
 Square of the mean, (by Prop. 16. and 17. of 6. Elem. and by 19. and 20. of 7. Elem.
 Euclid.) Hence Analogies may be converted into Equations, as in the following Examples;
 where for the greater evidence let a represent 2; b 6; c 12; and d 3; Then

1. Let there be four Proportionals, } $d : b :: d - a : a$
 suppose these, . . . } $3 : 6 :: 1 : 2$

Then by the Theorem above express, this } $da = bd - ba$
 Equation will follow,

Now to find the value of a in that Equation, } $da + ba = bd$
 first by transposition of $-ba$ this E-
 quation ariseth, . . . }

Then each part divided by $d + b$ gives . . . $a = \frac{bd}{d+b}$

2. If there be three Continual proportion- } $\frac{4a}{8} : \frac{c}{12} : \frac{9a}{18}$
 nals, suppose these, . . . } $\frac{4a}{8} : \frac{c}{12} : \frac{9a}{18}$
 That is, If . . .

Then, by the latter part of the said Theo- } $36aa = cc$
 rem, this Equation will follow, . . .

Now to find the value of a in that Equation, } $6a = c$
 extract the Square root out of each part,
 and there ariseth . . .

Lastly, each part of the last Equation di- } $a = \frac{c}{6}, \text{ or } \frac{1}{2}c.$
 vided by 6 gives . . .

II. If the Product of the multiplication of two Quantities be found equal to the
 Product of two other Quantities, that Equation may be resolved into Proportionals, for
 as either of the Factors in either of the two equal Products is to a Factor of the same
 kind in the other Product, so is the remaining Factor in this latter Product to the other
 Factor in the former. Hence Equations may oftentimes be resolved into Proportionals, as;

If there be proposed . . . $3ba = cd$
 From that Equation this Analogy may be } $3b : c :: d : a$
 infer'd, viz. As . . .

Again, If . . . $bd = da + ba$
 That Equation may be resolved into these } $d + b : b :: d : a$
 Proportionals, viz. As . . .

Likewise, If . . . $6da = bb$
 Then it shall be, As . . . $6d : b :: b : a$

III. When

III. When there happens to be an Equation between an Algebraical Fraction and an Integer, and the Numerator of the Fraction can be resolved into two such quantities that being mutually multiplied will produce the said Numerator, then that Equation may be resolved into Proportionals in this manner, viz. Let the Denominator of the Fraction, and the Integer to which the Fraction is equal, be made the extreme Terms of an Analogy; and let the two quantities which being mutually multiplied will constitute the Numerator be made the mean Terms, but with this caution in Geometrical Questions, that the first and second Terms be of one and the same kind, that is, either both Lines, or both Planes, or both Solids. As, for example;

If this Equation be proposed, $\frac{cd}{3b} = a$

It may be resolved into these Proportionals, $3b : c :: d : a$

But that they are Proportionals, I prove thus;

First, It is evident that these are Proportionals, (because the Product of the extremes is equal to the Product of the means)

And by the Equation proposed, $a = \frac{cd}{3b}$

Therefore $3b : c :: d : a$ ($\frac{cd}{3b}$)

Again, If $\frac{bb}{b+c} = a$

That Equation may be resolved into these Proportionals, $b+c : b :: b : a$

Likewise this Equation $\frac{cc-bb}{5b+2c} = a$

may be resolved into this Analogy, $5b+2c : c+b :: c-b : a$

And this Equation $\frac{bb+2bc+cc}{54d} = a$

may be converted into these Proportionals, $54d : b+c :: b+c : a$

Also, this Equation $\frac{bbc}{36d} = aa$

may be resolved into these Proportionals, $36d : b :: bc : aa$

Or into these, $36d : c :: bb : aa$

But this Equation $\frac{b}{c} = a$

cannot be resolved into Proportionals any otherwife than thus, $c : \sqrt{b} :: \sqrt{b} : a$

Nor can this Equation $\frac{bb+cd}{f} = a$

be converted into Proportionals, unless thus, $g : \sqrt{bb+cd} :: \sqrt{bb+cd} : a$

CHAP.

CHAP. XIV.

Various Arithmetical Questions Algebraically resolved; whereby most of the Rules hitherto delivered are exercis'd, in the Invention and Resolution of pure or simple Equations.

I. Equations may be divided into two kinds, viz. 1. Pure or Simple, 2. Affected or Compound.

II. A pure or simple Equation is of two kinds, viz. First, when the quantity sought is express'd by a simple Root only, as a , as in this Equation, $6a = 12$; secondly, when the quantity sought is express'd by a simple Power only, as aa , or aaa , &c. as in this Equation, $3aaa = 24$; likewise in this, $2aaaa = 32$, and such like.

III. An affected or compounded Equation is that, wherein there are two or more different Degrees or Powers of the quantity sought, as in this Equation, $aa+b = 27$, where aa and a express two different Degrees or Powers of the quantity sought, the one signifying a Square, and the other its Root or side; also in this Equation, $aaa+6aa-2a = 28$, there are three unlike Powers or Degrees of the quantity sought, to wit, aaa , aa , and a .

IV. The Invention and Resolution of Pure or Simple Equations is copiously illustrated by Arithmetical Questions in this Chapter, as also in the second and third Books of my *Algebraical Elements*; and the Resolution of Affected or Compound Equations in Numbers is handled in the 15, 16, and 17, Chapters of this Book, as also in the 12, and 11, Chapters of the second Book. But how Algebraical operations are applicable to the solving of Geometrical Problems, I shall shew in my fourth Book of *Algebraical Elements*.

V. When an Arithmetical Question is proposed, the number sought must first of all be assumed or supposed to be known, and you may represent it by the letter a , or any other Vowel: you may likewise represent the given numbers by Consonants, as b, c, d , &c. but for Quantities sought the latter letters x, y, z , &c. Then with the letters representing the numbers given and sought, in orderly process must be made, by adding, subtracting, multiplying or dividing, &c. according to the import of the Question, until at length an Equation be found out between the number sought, or some Power or Powers of it, and some number or numbers given: Lastly, when the Equation is found out, if it is a Pure or Simple Equation the number sought may be discovered by some of the Resolutions in the foregoing 12, and 13, Chapters; but when the Equation is Affected or Compound, the Resolution thereof belongs either to the 15, Chapter of this first Book, or to the 10, and 11, Chapters of the second Book.

VI. In the Resolution of every Question, I proceed from the beginning to the end by steps numbred in the Margin, by 1, 2, 3, 4, 5, &c. And because *Numerical Algebra* is more ealie for Learners than the *Literal*, (though not so useful for the reasons before given in *Sett. 8. Chap. 1.*) I have in many Questions express'd the Operation belonging to every step in both kinds of *Algebra*, that the one may explain the other: So in the second step of the Resolution of the following first Question, the letter number sought is express'd by *Numerical Algebra* thus, $26 - a$; but by *Literal Algebra* thus, $61 - a$. Also, in the fourth step, the Equation by *Numerical Algebra* is $2a - b = c$; but by *literal Algebra* it is $2a - b = c$.

VII. When an Equation is found out, in any of the following Questions, I take it for granted that the Reader knows how to reduce it, if need be, according to the Rules in the foregoing 11, 12, and 13, Chapters, that I may avoid tedious repetitions of what hath been already explain'd. These things premis'd, I proceed to the Questions themselves.

QUEST.

QUEST. 1.

There are two numbers whose Summ is 26, (or b ;) and their difference, (to wit, the excess of the greater above the lesser) is 8, (or c ;) What are the Numbers?

RESOLUTION:

	Numerical,	Literall.
1. For the greater number put	a	a
2. Then subtracting that number a from the given Summ, the Remainder will be the lesser number, to wit,	$26 - a$	$b - a$
3. And by subtracting the lesser number from the greater, the Remainder will be their difference, to wit,	$2a - 26$	$2a - b$
4. Which difference found out in the last step must be equal to the given difference 8, (or c ;) whence this Equation ariseth	$2a - 26 = 8$	$2a - b = c$
5. From which Equation, after it is duly reduced according to Sect. 3. and 5. of Chap. 12. the greater number sought will be discovered, to wit,	$a = 17$	$a = \frac{1}{2}b + \frac{1}{2}c$
6. And consequently from the fifth and second steps the lesser number is also discovered, to wit,	9, that is,	$\frac{1}{2}b - \frac{1}{2}c$

So the numbers sought are found 17 and 9, whose summ is 26, and their difference is 8, as was prescribed.

Moreover, If the two last steps of the literal Resolution be exprest by words, they will give this

THEOREM.

Half the difference of any two numbers added to half their Summ, gives the greater number: But half the difference of any two numbers subtracted from half their Summ, leaves the lesser number.

Therefore the Summ and difference of any two numbers being given severally, the numbers themselves are also given by the said Theorem; but it presupposeth that the number given for the Difference must not exceed the number given for the Summ.

Note here once for all, That the numbers given in a Question cannot always be chosen at pleasure, but sometimes they must be subject to one or more Determinations, which for the most part (though not always) are discoverable by the Theorem or Canon that resulteth from the Resolution. But how limits or Determinations are discovered, I shall have occasion to shew hereafter in my second, third, and fourth Books of *Algebraical Elements*.

QUEST. 2.

There are two numbers whose Summ is 40, (or b ;) and the greater number hath such proportion to the lesser as 3 to 2, or as r to s ;) What are the Numbers?

1. For the greater number sought put	a	a
2. Then to find the lesser number, say by the Rule of Three,		
If 3 . . 2 :: a . . $\frac{2a}{3}$	$\frac{2a}{3}$	$\frac{2a}{3}$
Or, r . . s :: a . . $\frac{sa}{r}$	$\frac{sa}{r}$	$\frac{sa}{r}$
whence the lesser number is		

3. There-

3. Therefore the Summ of the two numbers fought is	$\frac{5a}{3}$	$a + \frac{sa}{r}$
4. Which Summ found out in the last step must be equal to the given Summ 40, (or b ;) whence this Equation	$\frac{5a}{3} = 40$	$a + \frac{sa}{r} = b$
5. Which Equation, after due Reduction according to Sect. 2. and 5. of Chap. 12. gives the greater number	$a = 24$	$a = \frac{rb}{r+s}$
6. And from the fifth, first, and second steps, the lesser number is also discovered, to wit,	16, or	$\frac{sb}{r+s}$

So the numbers fought are found 24 and 16, which will satisfy the conditions in the Question: for their summ is 40, and the greater hath such proportion to the less as 3 to 2, as was prescribed.

Moreover, If the two last steps of the literal Resolution be resolved into Proportionals, according to Sect. 3. Chap. 13. there will arise this

THEOREM.

As the Summ of both the Terms which exprest the Reason (or Proportion) of two numbers, is to the Summ of the same two numbers; so is the greater Term to the greater number; and so is the lesser Term to the lesser number.

Therefore the Summ of two numbers being given, as also their Reason, or Proportion; the numbers shall also be given severally by the said Theorem.

QUEST. 3.

There are two numbers whose difference is 8, (or d ;) and the greater number hath such proportion to the lesser as 3 to 2, (or as r to s ;) what are the Numbers?

1. For the greater number put	a	a
2. Then to find the lesser number say by the Rule of Three,		
If 3 . . 2 :: a . . $\frac{2a}{3}$	$\frac{2a}{3}$	$\frac{2a}{3}$
Or if r . . s :: a . . $\frac{sa}{r}$	$\frac{sa}{r}$	$\frac{sa}{r}$
whence the lesser number is		
3. Therefore by subtracting the lesser number from the greater, the Remainder shall be their difference, to wit,	$\frac{a}{3}$	$a - \frac{sa}{r}$
4. Which difference must be equal to the given difference 8 (or d ;) hence this Equation ariseth	$\frac{a}{3} = 8$	$a - \frac{sa}{r} = d$
5. Which Equation, after due Reduction, discovers the greater number sought, to wit,	$a = 24$	$a = \frac{rd}{r-s}$
6. And from the fifth, first, and second steps the lesser number will be also made known, to wit,	$= 16$	$= \frac{sd}{r-s}$

So the Numbers sought are found 24 and 16, which will solve the Question: for their difference is 8, and they are in the proportion of 3 to 2, as was prescribed.

Moreover, If the two last steps of the literal Resolution be converted into Proportionals (according to Sect. 3. Chap. 13.) there will arise this

THEOREM.

As the difference of the two Terms which exprest the Reason or Proportion of two numbers is to the difference of the same two numbers, so is the greater Term to the greater Number; and so is the lesser Term to the lesser Number.

Therefore the Difference and Reason of two numbers being severally given, the numbers themselves shall be also given by the said Theorem.

I

QUEST.

QUEST. 4.

There are two numbers whose Summ is 7, (or b), and the difference of their Squares is 21, (or d ;) what are the numbers?

- | | | |
|---|-----------------|-------------------------|
| 1. For the greater number sought put | a | a |
| 2. Then subtracting the greater number from the given Summ, the Remainder is the lesser number, to wit, | $7 - a$ | $b - a$ |
| 3. Therefore from the first step the Square of the greater number is | aa | aa |
| 4. And from the second step the Square of the lesser number is | $aa - 14a + 49$ | $aa - 2ba + bb$ |
| 5. Therefore the difference of the Squares of the two numbers sought shall be | $14a - 49$ | $2ba - bb$ |
| 6. Which difference must be equal to the given difference 21 (or d ;) whence this Equation ariseth | $14a - 49 = 21$ | $2ba - bb = d$ |
| 7. Which Equation, after due Reduction according to <i>Self. 3</i> , and 5. of <i>Chap. 12</i> , discovers the greater number sought, to wit, | $a = 5$ | $a = \frac{bb + d}{2b}$ |
| 8. And from the seventh and second steps, the lesser number will be also made known, to wit, | $= 2$ | $= \frac{bb - d}{2b}$ |

So the numbers sought are found 5 and 2, which will solve the Question; for their Summ is 7, and the difference of their Squares is 21, (to wit, $25 - 4$;) as was prescribed.

Moreover, If the two last steps of the literal Resolution be exprest by words, they will give this

THEOREM.

If to the Square of the summ of any two numbers the difference of their Squares be added, and the summ of that addition be divided by the double summ of the two numbers; the Quotient will be the greater number: But if from the Square of the summ of two numbers the difference of their Squares be subtracted, and the Remainder be divided by the double summ of the two numbers, the Quotient will give the lesser number.

Therefore the Summ of two numbers being given, as also the difference of their Squares, the numbers themselves shall be given severally; but it presupposeth the square of the given summ to exceed the given difference.

QUEST. 5.

There are two numbers whose difference is 3, (or c ;) and the difference of their Squares is 21, (or d ;) what are the numbers?

- | | | |
|---|---------------|-------------------------|
| 1. For the lesser number sought put | a | a |
| 2. To which adding the given difference 3, (or c ;) the summ will make the greater number, to wit, | $a + 3$ | $a + c$ |
| 3. Therefore the Square of the greater number is | $aa + 6a + 9$ | $aa + 2ca + cc$ |
| 4. And the Square of the lesser number is | aa | aa |
| 5. Therefore the difference of those Squares is | $6a + 9$ | $2ca + cc$ |
| 6. Which difference must be equal to the given difference of the Squares; whence this Equation ariseth, to wit, | $6a + 9 = 21$ | $2ca + cc = d$ |
| 7. Which Equation, after due Reduction (according to <i>Self. 3</i> , and 5. of <i>Chap. 12</i> ;) discovers the lesser number, to wit, | $a = 2$ | $a = \frac{d - cc}{2c}$ |
| 8. And from the seventh and second Equations, the greater number will be found | $= 5$ | $= \frac{d + cc}{2c}$ |

So

So the numbers sought are 5 and 2, which will solve the Question; for their difference is 3, and the difference of their Squares is 21; as was prescribed. Moreover, the two last steps of the literal Resolution afford this

THEOREM.

If to the difference of the Squares of any two numbers the Square of their difference be added, and the summ of that addition be divided by the double of the difference of those two numbers, the Quotient will give the greater number: But if from the difference of the Squares of two numbers the Square of their difference be subtracted, and the Remainder be divided by the double of the difference of those two numbers, the Quotient shall be the lesser number.

Therefore the difference of any two numbers being given, as also the difference of their Squares, the numbers themselves shall also be given severally by this Theorem; but it presupposeth the given difference of the Squares of the two numbers to exceed the Square of the given difference of the same two numbers.

QUEST. 6.

A certain person being asked what was the age of every one of his four Sons, answered, the eldest was four years (or b) elder than the second, the second was four years elder than the third, the third was four years elder than the fourth or youngest, and the double of the youngest Sons age was equal to the age of the eldest; what was the age of each Son?

- | | | |
|---|---------------|---------------|
| 1. For the age of the eldest Son put | a | a |
| 2. Then from the age of the eldest Son subtracting 4 (or b) there will remain the second Sons age, to wit, | $a - 4$ | $a - b$ |
| 3. Likewise from the second Sons age subtracting 4 (or b) the Remainder will be the third Sons age, to wit, | $a - 8$ | $a - 2b$ |
| 4. Again, from the third Sons age subtracting 4 (or b) there will remain the fourth or youngest Sons age, to wit, | $a - 12$ | $a - 3b$ |
| 5. But according to the Question, the double of the age in the fourth step must be equal to the age in the first step, whence this Equation will arise, | $2a - 24 = a$ | $2a - 6b = a$ |
| 6. Which Equation duly reduced discovers the age of the eldest Son, to wit, | $a = 24$ | $a = 6b$ |

Wherefore the ages of the four Sons were 24, 20, 16, and 12; for the first exceeds the second by 4, which is also the excess of the second above the third, the third above the fourth, and the double of the fourth is equal to the first, as was prescribed in the Question.

Moreover the last step of the literal Resolution shews, that if instead of 4, any other number be given for the common difference of the four Sons ages, then six times that common difference will give the eldest Sons age, which shall be equal to the double of the age of the youngest.

QUEST. 7.

A Merchant began to Trade with a certain number of pounds: By his first Voyage he doubled that Stock, by his second he lost 1200. pounds (or b ;) by his third he doubled his remaining Stock, by his fourth he lost again 1200. pounds; and then had no money left. The question is, to find how many pounds the Merchant began to Trade with?

12

1. For

- | | | |
|--|-----------------|--------------------|
| 1. For the number of pounds which the Merchant began to trade with put . . . | a | a |
| 2. Then the double of that number gives the number of pounds he had at the end of his first voyage, to wit, . . . | $2a$ | $2a$ |
| 3. From which last number subtracting 1200 (or b) the Remainder shews the number of pounds that remained to the Merchant at the end of his second voyage, to wit, . . . | $2a - 1200$ | $2a - b$ |
| 4. Which remaining number being doubled gives the number of pounds which the Merchant had at the end of his third voyage, to wit, . . . | $4a - 2400$ | $4a - 2b$ |
| 5. From which last number subtracting again 1200 (or b) pounds lost by the fourth voyage, the Remainder must be equal to nothing; hence this Equation, . . . | $4a - 3600 = 0$ | $4a - 3b = 0$ |
| 6. Which Equation, after due Reduction, gives | $a = 900$ | $a = \frac{3}{2}b$ |

Whence it is found that the Merchant began to trade with 900. pounds; which number will satisfy the conditions in the Question.

Moreover the last step of the literal Resolution shews, that if instead of 1200. any other number were given, the Merchants stock at first would be three quarters of that given number.

QUEST. 8.

A Gentleman hired a Servant for a year, for 120. shillings (or c), together with a livery Cloak valued at a certain number of shillings: But when $\frac{7}{12}$ (or d) parts of the year were expired, the Master falling at variance with his Servant puts him away, and gives him the Cloak with 50. shillings, (or f), and so the Servant received full satisfaction for the time of his service. The question is, to find How many shillings the Cloak was valued at?

- | | | |
|--|-------------------------------|-------------------|
| 1. For the number of shillings which the Cloak was valued at put . . . | a | a |
| 2. Then to find what part of the value of the Cloak was due to the Servant when $\frac{7}{12}$ (or d) parts of the year were expired, say by the Rule of Three, . . . | $\frac{7a}{12}$ | da |
| Or, if $1 : a :: \frac{7}{12} : d$ whence the desired part of the value of the Cloak is found . . . | $\frac{7a}{12}$ | da |
| 3. Find likewise what part of the 120 (or c) shillings was due to the Servant when $\frac{7}{12}$ (or d) parts of the year were expired, and say, . . . | 70 | cd |
| Or, if $1 : 120 :: \frac{7}{12} : 70$ whence the part desired is found . . . | 70 | cd |
| 4. Now forasmuch as the Cloak together with the 50. shillings the Servant received, ought to be equal to the part of the Cloak, together with the part of the 120. shillings that was due to him at the time he left his service, therefore from the premises there ariseth this Equation: . . . | $a + 50 = \frac{7a}{12} + 70$ | $a + f = da + cd$ |

$$a + 50 = \frac{7a}{12} + 70 \quad \text{Or,} \quad a + f = da + cd$$

5. Which

5. Which Equation after due Reduction according to *Self. 2, 3, and 5. of Chap. 12.* will give the desired value of the Cloak, to wit,

$$a = 48 = \frac{cd \propto f}{1 \propto d}$$

Whence it is evident that the Cloak was valued at 48. shillings; and the last Equation discovers this

CANON

Multiply the money which the Servant was to receive besides the Cloak for a years wages, by the time he served, then divide the difference between that Product and the money he received when he left his service by the difference between 1. (or unity) and the same time he served, so the Quotient gives the value of the Cloak.

By which Canon the value of the Cloak will be found to be 48. s : as above.

The Proof.

$$48 + 50 = 98$$

$$\frac{7}{12} \text{ of } 48, + \frac{1}{12} \text{ of } 120 = 98$$

QUEST. 9.

A certain man finding divers poor persons at his door, gave every one of them three pence (or b), and had six pence (or e) left, but if he would have given them four pence (or f) a piece, he should have wanted two pence (or g). How many poor persons were there?

- For the number of poor persons put . . .
- Then forasmuch as, that number multiplied by 3 (or b) and the Product increased with 6 (or e) makes the whole number of pence that the giver had: And, because if the same number of poor persons be multiplied by 4 (or f), the Product less by 2 (or g) must also make the same number of pence: hence this Equation, . . .

$$3a + 6 = 4a - 2$$

$$\text{Or, } ba + e = fa - g$$

- Which Equation after due Reduction according to *Self. 3, and 5. of Chap. 12.* discovers the number of poor persons to be 8: viz.

$$8 = \frac{e + g}{f - b} = a$$

QUEST. 10.

One being asked what a Clock it was, answered, That the time then past from noon was equal to $\frac{1}{10}$ (or b) parts of the time remaining until midnight: What was the present Hour? supposing the time between noon and midnight to be divided into 12 (or d) equal Hours.

- | | | |
|--|-------------------------|-------------------------|
| 1. For the Hour sought after noon put . . . | a | a |
| 2. Which subtracted from 12 (or d) leaves the time remaining until midnight, to wit, . . . | $12 - a$ | $d - a$ |
| 3. Then $\frac{1}{10}$ (or b) parts of the said remaining time will be . . . | $\frac{12 - a}{10}$ | $\frac{bd - a}{10}$ |
| 4. Therefore from the first and third steps (according to the Question) this Equation ariseth, to wit, . . . | $a = \frac{12 - a}{10}$ | $a = \frac{bd - a}{10}$ |
| 5. Which Equation after due Reduction according to <i>Self. 2, 3, and 5. of Chap. 12.</i> gives the Hour sought, to wit, . . . | $a = \frac{12}{9}$ | $a = \frac{bd}{9}$ |

So the time sought was $5\frac{11}{9}$ Hours after noon, and consequently the remaining time until midnight was $6\frac{2}{9}$ Hours, whereof $\frac{1}{10}$ is equal to the said $5\frac{11}{9}$; as was prescribed in the Question.

QUEST.

QUEST. 11.

A General of an Army having set his Souldiers in a Square Battel, there happened to be 500 (or b) Souldiers to spare; but to increase the Square so as that its side might consist of 1 (or c) Souldier more than the side of the former Square, there would be 29 (or d) Souldiers wanting. The question is, to find How many Souldiers the General had in his Army.

1. For the number of Souldiers that made the side of the first Square, put a
2. Then that side multiplied by it self gives the number of Souldiers in the first Square Battel, to wit, aa
3. Therefore the number of Souldiers in the whole Army was $aa + 500$
4. Then to the end the side of another Square may exceed the side of the former by 1 (or c), let it be $a + 1$
5. Which latter side multiplied by it self gives the number of Souldiers in the latter square Battel, to wit, $aa + 2a + 1$
6. But the number of Souldiers in the last step exceeded the number of Souldiers in the Generals Army by 29 (or d); therefore subtracting 29 (or d) from the number in the last step, the Remainder must be equal to the number in the third step: hence this Equation ariseth, to wit,

$$aa + 2a + 1 - 29 = aa + 500;$$

$$\text{Or, } aa + 2a + 1 - 29 = aa + 500;$$

7. Which Equation after due Reduction (according to Sect. 3, and 5. of Chap. 12.) makes known the side of the first Square, viz.

$$a = 264 = \frac{b+d}{2c}.$$

8. Lastly, if the side or number found out in the last step be multiplied by it self, and the Product be increased with 500 (or b), there will come forth the number of Souldiers that were in the Generals Army; to wit,

$$70196 = \frac{bb + 2bd + dd}{4cc} + \frac{1}{2}c + \frac{1}{2}b - \frac{1}{2}d.$$

Whence it is manifest that the General had 70196 Souldiers in his Army: Also, the Side of the first Square Battel consisted of 264 Souldiers; and the Side of the latter 265; this multiplied by it self produceth 70225, which exceeds the said 70196 by 29: Moreover, the said 70196 exceeds the Square of 264 by 500; as the question requires.

QUEST. 12.

Two persons, A and B , discourse of their Money in this manner; viz. A saith, if B would give him a Crown (or c), then A should have as many Crowns as B had left; but B saith, if A would give him a Crown, then B should have twice as many Crowns as A had left. How many Crowns had each person?

1. For the number of Crowns which A had, put a
2. Then, according to the question, if that number be increased with 1 Crown (or c), the sum will be the number of Crowns that remained to B after he had given 1 Crown to A , to wit, $a + c$
3. And consequently, by adding 1 Crown (or c) to the said number of Crowns that remained to B after he had given 1 Crown to A , the sum will be the number of Crowns which B had at first; to wit, $a + 2c$

4. Again,

4. Again, according to the question, if 1 Crown (or c) be added to the said $a + 2c$ in the last step, and subtracted from a in the first step, the sum must be equal to the double of the Remainder; hence this Equation, $a + 3c = 2a - 2b$
 5. Which Equation, after due Reduction, discovers the number of Crowns that A had at first, to wit, $a = 5c$
 6. And from the fifth and third steps, the number of Crowns which B had at first will also be made known, to wit, $a + 2c = 7b$
- So it is found that A had 5 Crowns, and B 7 Crowns, as will be evident by

The Proof.

$$\begin{array}{r} 5 + 1 = 7 - 1 = 6 \\ 7 + 1 = 4 + 4 = 8 \end{array}$$

QUEST. 13.

A Vintner having two sorts of French Wines, to wit, one sort worth 10. $d.$ (or b) the quart, and the other 6. $d.$ (or c) per quart, would have a mixed quantity of both sorts to consist of 100. quarts (or m) that might be worth 7. $d.$ (or f) per quart. The question is, to find What quantity of each sort of Wine must be taken to make that mixture?

1. For the number of quarts that must be taken of the better sort of wine to make the mixture put a
2. Which number subtracted from 100 (or m) leaves the number of quarts of the worser sort of wine in the mixture, to wit, $100 - a$
3. Then find the worth of the better sort of wine in the mixture at 10. $d.$ (or b) per quart, and say by the Rule of Three, If 1. 10 :: a . (10 a , Or, if 1. b :: a . (ba .) So the quantity of the better sort of wine in the mixture is found worth $10a$
4. Find likewise the worth of the worser sort of wine in the mixture at 6. $d.$ (or c) per quart, and say, If 1. 6 :: $100 - a$. (600 - $6a$, Or, 1. c :: $m - a$. ($cm - ca$.) So the quantity of the worser sort of wine in the mixture is found worth $600 - 6a$
5. Therefore the Summ of the values of both the quantities mentioned in the two last steps is $10a + 600 - 6a$
6. Which Summ must be equal to the Product made by the multiplication of 100 (or m) the total mixed quantity, by 7 (or f) the prescribed mean price; hence this Equation ariseth, to wit, $4a + 600 = 700$

$$\text{Or, } ba + cm - ca = fm.$$

7. Which Equation, after due Reduction, discovers the value of a , to wit, the number of quarts that must be taken of the better sort of wine to make the mixture, viz. $a = 25 = \frac{fm - cm}{b - c}$
8. And from the seventh and second steps the number of quarts that ought to be taken of the worser sort of wine to make the mixture will also be made known, viz. $75 = \frac{bm - fm}{b - c}$
9. From the two last steps it is evident, That 25 quarts of the better sort of wine, and 75 quarts of the worser sort, must be taken to make the prescribed mixture, for those quantities

quantities at their respective prices will be worth in the whole 700 pence, which is also the just worth of 100 quarts at 7 pence per quart.
Moreover, If the latter parts of the two last Equations be resolved into Proportionals, (according to *Self. 3. Chap. 13.*) and be express'd by words, they will give this following

THEOREM.

As the difference between the given prices of two sorts of Wines or other things whereof a mixture is desired, is to the total quantity required to be in the mixture, So is the excess by which some mean price prescribed for the total quantity mixed exceeds the lesser of the two given prices, to the quantity to be taken of the better sort of Wine: And so is the excess of the greater of the two given prices above the mean price, to the quantity that is to be taken of the worse sort of Wine.

This Theorem contains the substance of the Rule of Alligation-alternate in Vulgar Arithmetick. - But how Questions of this nature, when three or more things are to be mixed, may be solved more generally than by that Rule, I shall hereafter shew in *Chap. 13.* of my second Book of *Algebraical Elements.*

QUEST. 14.

A Cistern in a certain Conduit is supplied with water by two Pipes, of such capacities, that by both their Cocks *A* and *B* set open at once the Cistern will be filled in 12 (or *b*) hours; but by the Cock *A* alone in 20 (or *c*) hours: the question is, to find in what time the Cistern will be filled by the Cock *B* alone?

1. Suppose the time sought to be a
2. Then find what part of the Cistern will be filled by the Cock *B* alone in 12 (or *b*) hours, and say by the Rule of Three,

$$\text{If } a : 1 :: 12 : \left(\frac{12}{a}\right)$$

$$\text{Or, if } a : 1 :: b : \left(\frac{b}{a}\right)$$
 whence the said part is found
3. Find likewise what part of the Cistern will be filled by the Cock *A* alone in 12 (or *b*) hours, and say,

$$\text{If } 20 : 1 :: 12 : \left(\frac{3}{5}\right)$$

$$\text{Or, if } c : 1 :: b : \left(\frac{b}{c}\right)$$
 whence the said part is found
4. But those parts found out in the second and third steps must be equal to the whole Cistern; to wit, 1; hence this Equation ariseth,

$$\frac{12}{a} + \frac{1}{5} = 1.$$

$$\frac{b}{a} + \frac{b}{c} = 1.$$
5. Which Equation, after due Reduction according to *Self. 2, 3, and 5. of Chap. 12.* discovers the value of *a*, to wit, the time sought, viz.

$$a = 30$$

$$a = \frac{bc}{c-b}$$

Whence it appears, that by the Cock *B* set open alone the Cistern would be filled in 30 hours: And, if the last Equation of the literal Resolution be resolved into Proportionals according to *Self. 3. Chap. 13.* there will arise this following

CANON.

As the difference of the two numbers or spaces of Time given in the Question is to either of them, so is the other to the Time sought, viz.

$$\text{As } 8 \text{ (} 20 - 12 \text{)} : 12 :: 20 : 30,$$

$$\text{Or, as } \dots c - b : b :: c : \frac{bc}{c-b}.$$

The

The Proof may be made by solving this Question, viz.

If a Cistern will be filled with water by a Cock *A* in 20 hours; and by another Cock *B* in 30 hours; in what time will the Cistern be filled by both Cocks set open at once? *Ans.* 12 hours.

First find what part or parts of the Cistern will be filled by each Cock in one and the same time; then it shall be, As the sum of those parts is to that common time, so is the whole Cistern (to wit, 1,) to the time wherein the whole Cistern will be filled by both Cocks set open at once; viz.

$$\begin{array}{rcl} \text{ho.} & \text{Cist.} & \text{ho.} \\ \text{If, if } & 30 & 1 :: 20 & \left(\frac{1}{5} \text{ Cistern.} \right. \\ & & & \text{add } 1 \text{ Cistern.} \end{array}$$

Summ. $\frac{1}{5}$ Cist.

So it is found that $\frac{1}{5}$ Cistern will be filled in 20 hours by both Cocks *A* and *B* set open at once; then say again by the Rule of Three,

$$\begin{array}{rcl} \text{Cist.} & \text{ho.} & \text{Cist.} \\ 1 \frac{1}{5} & 20 & :: 1 & (12 \text{ hours.}) \end{array}$$

If the Operation of this latter Question be formed Algebraically by letters, it will afford this

CANON.

As the Summ of the two given numbers expressing spaces of time in the latter Question, is to either of them; So is the other to the Time sought.

QUEST. 15.

A Shepherd in the time of war driving a flock of Sheep, fell into the hands of three Companies of plundering Souldiers, who compell'd him to deliver the half of his flock with half a Sheep over and above to the first Company; also half of his remaining flock with half a Sheep to the second Company; likewise the half of the rest of his flock with half a Sheep to the third Company: All which Divisions the Shepherd exactly perform'd without killing a Sheep, and then there remained only 20 (or *b*) Sheep for himself. The question is, to find How many Sheep the Shepherd had in his Flock at first?

1. Let the number of Sheep which the Shepherd had in his flock at first be represented by a
2. Then the half of that number is $\frac{1}{2}a$, to which adding $\frac{1}{2}$, (that is, half a Sheep,) the sum will be the number of Sheep delivered to the first Company of Souldiers; to wit, $\frac{1}{2}a + \frac{1}{2}$
3. And by subtracting the said $\frac{1}{2}a + \frac{1}{2}$ from a , the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied the first Company of Souldiers, to wit, $\frac{1}{2}a - \frac{1}{2}$
4. Then the half of that remaining flock is $\frac{1}{4}a - \frac{1}{4}$, to which adding $\frac{1}{4}$, (that is, $\frac{1}{4}$ Sheep,) the sum will be the number of Sheep delivered to the second Company of Souldiers, to wit, $\frac{1}{4}a + \frac{1}{4}$
5. Which $\frac{1}{4}a + \frac{1}{4}$ being subtracted from $\frac{1}{2}a - \frac{1}{2}$ in the third step, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied the second Company of Souldiers, to wit, $\frac{1}{4}a - \frac{3}{4}$
6. Then the half of the remaining flock in the last step is $\frac{1}{8}a - \frac{3}{8}$, to which adding $\frac{1}{8}$, (to wit, $\frac{1}{8}$ Sheep,) the sum will be the number of Sheep delivered to the third Company, to wit, $\frac{1}{8}a + \frac{1}{8}$
7. Which $\frac{1}{8}a + \frac{1}{8}$ being subtracted from $\frac{1}{4}a - \frac{3}{4}$ in the fifth step, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied all the three Companies, to wit, $\frac{1}{8}a - \frac{7}{8}$
8. But the remainder in the last step must be equal to 20 (or *b*) the number given in the Question; hence this Equation, $\frac{1}{8}a - \frac{7}{8} = b$
9. Which Equation, after due Reduction, discovers the number sought, to wit, $a = 8b + 7 = 167$

So it appears that the Shepherd had 167 Sheep in his Flock at first.

K

The

The Proof.

1. The half of 167 is $83\frac{1}{2}$, to which adding $\frac{1}{2}$, the sum is 84, which was the number of Sheep delivered to the first Company of Souldiers; and then there remained 83 Sheep to the Shepherd.

2. Again, the half of 83 is $41\frac{1}{2}$, which increased with $\frac{1}{2}$ makes 42, the number of Sheep delivered to the second Company; and then there remained 41 Sheep to the Shepherd.

3. Lastly, the half of 41 is $20\frac{1}{2}$, which increased with $\frac{1}{2}$ makes 21, which was the number of Sheep delivered to the third Company; and so there remained 20 Sheep to the Shepherd, as the Question declareth.

Moreover, the Equation in the last step of the Resolution shews, That if any whole number instead of 20 be prescribed in the Question, that number multiplied by 8, and the Product increased with 7 will give a number capable of the like Division as 167 that answered the Question: So if there had been but one Sheep left for the Shepherd, then his Flock at first was 15 Sheep; if 2 Sheep had been left, his Flock at first was 23; if 3 Sheep had been left, then he had 31 when he first met with the Souldiers; and so by a continual addition of 8, all the odd numbers capable of that Division the Question requires may be orderly found out. But to have nothing left after such Division is made, the number first to be divided is 7.

It is also evident, that by continuing the Resolution an odd number may be found out, that shall be capable of being divided according to the import of the Question, as many times as shall be desired.

QUEST. 16.

Two Merchants, *A* and *B*, were Copartners in traffick: the sum of their Stocks was 300 *l.* (or *b*;) the Stock of *A* continued in company 9 (or *c*) months, and the Stock of *B* 11 (or *d*) months; they gained a certain sum of Money which they divided equally. The question is, to find What each Merchants Stock was at first?

- For the Stock of *A* when he entered Partnership, put
- Then subtracting that stock from the joyn't stock 300 *l.* (or *b*;) the Remainder will be the stock of *B*, to wit,
- The first stock multiplied by the time it continued in Company produceth
- And the other stock multiplied by its time produceth
- Now forasmuch as the Merchants divided the gain equally, therefore the Products in the third and fourth steps must be equal to one another, (according to the nature of the Rule of Fellowship with Time.) Hence this Equation arithet;

$$9a = 300 - 11a, \\ \text{Or, } \dots \quad ca = db - da.$$

- Which Equation, after due Reduction, according to *Self. 3.* and 5. of *Chap. 12.* will discover the Stock which *A* put in, viz.

$$a = 165 = \frac{db}{c+d}.$$

- And from the 6. and 2. steps the Stock which *B* put in will also be made known, to wit,

$$135 = \frac{cb}{c+d}.$$

So it is found that the Stock of *A* was 165 *l.* and that of *B*, 135 *l.* For, $165 \times 9 = 135 \times 11$.

Moreover, If the latter parts of the two Equations in the sixth and seventh steps be resolved into Proportionals, according to *Self. Chap. 13.* there will arise this

CANON.

As the sum of both spaces of time given in the Question, is to the given sum of the two particular Stocks fought; so is the greater time to the particular Stock belonging to the lesser time: and so is the lesser time to the Stock belonging to the greater time.

QUEST.

QUEST. 17.

A certain man being asked how many years old he was, answered, If $\frac{1}{20}$ (or *b*) part of the number of years he had lived, were multiplied by $\frac{1}{2}$ (or *c*) parts of the said number, the Product would give his Age. What was his Age?

- For the number of years fought put
- Then according to the Question, multiplying $\frac{1}{20}a$ by $\frac{1}{2}$ (or *ba* by *ca*) the Product will be
- Which Product must be equal to the number of years fought, viz.
- Then, by reducing that Equation according to *Self. 4.* and 5. of *Chap. 12.* the number of years fought will be discovered, viz.

Whence it is manifest that the Respondent was 32 years of age; for if $\frac{1}{20}$, that is, $\frac{1}{20}$ of 32, be multiplied by $\frac{1}{2}$, that is, $\frac{1}{2}$ of 32; the Product will be 32, to wit, the number of years fought. It is also evident by the last Equation in the literal Resolution, that if 1 (to wit Unity) be divided by the Product made by the multiplication of the two numbers given in the question, the Quotient will be the number fought.

QUEST. 18.

There are two numbers, the greater of which hath such proportion to the lesser as 3 to 2, (or as *r* to *s*;) and the sum of the said numbers hath such proportion to the sum of their Squares, as 1 to 13, (or as *b* to *c*;) What are the numbers?

- For the greater number fought put
- Then (according to *Quest. 2.* in *Self. 4.* *Chap. 10.*) the sum of the two numbers will be found
- And (according to *Quest. 5.* in the said *Self. 4.* *Chap. 10.*) the sum of the Squares of the two numbers fought will be
- Again, by the help of the latter Proportion given in the Question, and of the sum found in the second step search out the sum of the Squares of the two numbers fought; viz. say by the Rule of Three,

$$\text{If } 1 : 13 :: \frac{sa}{3} : \left(\frac{6sa}{3} \right)$$

$$\text{Or, if } b : c :: a + \frac{sa}{r} : \frac{cra + csa}{br}$$

whence the sum of the said Squares is found

- But the sum of the Squares found out in the third step must be equal to the sum in the fourth, hence this Equation, viz.

$$\frac{13aa}{9} = \frac{6sa}{3} \\ \text{Or, } \dots \quad aa + \frac{11aa}{rr} = \frac{cra + csa}{br}$$

- Which Equation, after due Reduction, will discover the greater of the two numbers fought, viz.

$$a = 15 = \frac{crr + crs}{brr + bss}$$

- Whence, by the help of the first proportion given in the Question, the lesser number fought will also be made known, viz.

$$10 = \frac{css + crs}{brr + bss}$$

K 2

So

So the numbers fought are 15 and 10; for they are in the given Reason of 3 to 2; and their sum 25 is to 325 the sum of their Squares, as 1 to 13; as was prescribed.

Moreover, the letters in the latter parts of the two last Equations give a Canon to find out the numbers required.

QUEST. 19.

There are two numbers, the greater of which hath such proportion to the lesser, as 3 to 2, (or as 7 to 5;) and the sum of the said numbers hath such proportion to the Product of their multiplication, as 1 to 6, (or as 6 to 1.) What are the numbers?

1. For the greater number fought put a
2. Then (according to *Quest. 1. in Sect. 4. Chap. 10.*) the sum of the two numbers will be $a + \frac{2a}{3}$
3. And (by *Quest. 4. in Sect. 4. Chap. 10.*) the Product of their multiplication is $\frac{2aa}{3}$
4. Again, by the help of the latter proportion given in the Question, and of the sum found in the second step, search out the Product of the multiplication of the two numbers fought; viz. say by the Rule of Three,

$$\text{If } 1 : 6 :: \frac{2a}{3} : 10a,$$

$$\text{Or, if } b : c :: a + \frac{2a}{3} : \frac{2aa}{3},$$

whence the Product is found

5. But the Products found out in the two last steps must be equal to one another; hence this Equation, viz.

$$\frac{2aa}{3} = 10a,$$

$$\text{Or, } \frac{2aa}{3} = \frac{2aa + 4aa}{3}.$$

6. Which Equation, after due Reduction, discovers the greater of the two numbers fought, viz.

$$a = 15 = \frac{2a + 4a}{3}.$$

7. Whence, by the help of the first proportion given in the Question, the lesser number fought will also be made known, viz.

$$10 = \frac{2a + 4a}{3}.$$

So the numbers fought are found 15 and 10; but that they will solve the Question the Proof will make manifest: For the greater is to the lesser as 3 to 2; and their sum 25, is to 150 the Product of their Multiplication, as 1 to 6; as was prescribed.

Moreover, the two last Equations give a Canon to find out the number fought.

QUEST. 20.

There are two numbers, the greater of which hath such proportion to the lesser as 2 to 1 (or as 7 to 5,) and the sum of the Squares of the said numbers is 125 (or 6;) What are the numbers?

1. For the greater number fought put a
2. Then (according to *Quest. 1. in Sect. 4. Chap. 10.*) the lesser number will be found $\frac{a}{2}$
3. Therefore the sum of their Squares shall be $\frac{5aa}{4}$
4. Which

Chap. 14.

which produce simple Equations.

4. Which sum must be equal to 125 (or 6) the given sum of the Squares; hence this Equation, $\frac{5aa}{4} = 125$
5. Which Equation, after due Reduction (according to *Sect. 2, 5, and 7. of Chap. 12.*) will discover the greater number fought, viz. $a = 10$
6. But if a had been put for the lesser number, it would by the like process have been found $= 5$

From the two last steps the numbers fought are found 10 and 5, which will solve the Question: For the greater is to the lesser as 2 to 1, and the sum of their Squares is 125; as was prescribed.

Moreover, to find out the numbers fought, the two last steps of the literal Resolution give this

CANON.

Multiply severally the Squares of the Terms of the given Reason; by the given sum of the Squares of the number fought; then divide the Products severally by the sum of the Squares of the said Terms; lastly, extract the square Root out of each Quotient, so shall these square Roots be the numbers fought.

QUEST. 21.

There are two numbers, the greater of which hath such proportion to the lesser as 2 to 1, (or as 7 to 5,) and the difference of their Squares is 75, (or 4;) What are the numbers?

1. For the greater number fought put a
2. Then (according to *Quest. 1. in Sect. 4. Chap. 10.*) the lesser number will be $\frac{a}{2}$
3. Therefore the difference of their Squares is $\frac{3aa}{4}$
4. Which Difference must be equal to the given Difference 75 (or 4) hence this Equation, viz. $\frac{3aa}{4} = 75$
5. Which Equation, after due Reduction, discovers the greater number, viz. $a = 10$
6. But if a had been put for the lesser number, it would have been found by the like process $= 5$

So the numbers fought are 10 and 5, which will solve the Question: For the greater is to the lesser as 2 to 1, and the Difference of their Squares is 75; as was prescribed.

Moreover, to find out the numbers fought, the two last steps of the literal Resolution give this

CANON.

Multiply severally the Squares of the Terms of the given Reason by the given Difference of the Squares, then divide the Products severally by the Difference of the Squares of the said Terms; lastly, extract the square Root of each Quotient, so shall these square Roots be the numbers fought.

QUEST. 22.

There are two numbers, the sum of whose Squares is 125 (or 6;) and the difference of their Squares is 75 (or 4;) what are the numbers?

1. For the greater number put a
2. Then its Square will be aa
3. Which subtracted from 125 (or 6) the given sum, leaves the Square of the lesser number, to wit, $125 - aa$
4. And

4. And from the second and third steps by subtracting the lesser Square from the greater, their difference is $2aa - 125$ $2aa - b$
5. Which Difference must be equal to the given Difference 75 (or d), whence this Equation $2aa - 125 = 75$ $2aa - b = d$
6. From which Equation after due Reduction, according to *Self*. 3, 5, and 7. of *Chap.* 12, the greater number fought will be made known, *viz.* $a = 10$ $a = \sqrt{\frac{b+d}{2}}$
7. But if a had been put for the lesser number fought, it would by the like process have been found $= 5$ $= \sqrt{\frac{b-d}{2}}$

So the numbers fought are found 10 and 5, which will solve the Question; for the sum of their Squares is 125, and the difference of their Squares is 75, as was prescribed. Moreover, to find out the numbers fought, the two last steps of the literal Resolution give this

CANON.

The square Root of half the sum of the given sum and difference of the Squares of the two numbers fought, is equal to the greater number; and the square Root of half the difference of the said given sum and difference gives the lesser number.

QUEST. 23.

There are two numbers, the sum of whose Squares is 340 (or b), and the Product made by the multiplication of the two numbers is equal to $\frac{5}{2}$ (or c) parts of the Square of the greater number; what are the numbers?

1. For the greater number put a a
2. Then its Square is aa aa
3. And $\frac{5}{2}$ (or c) parts of that Square is $\frac{5aa}{2}$ caa
4. Therefore also (according to the condition in the Question) the Product of the multiplication of the two numbers fought, shall be $\frac{6aa}{2}$ caa
5. Which Product divided by the greater number a will give the lesser number, to wit, $\frac{6a}{2}$ ca
6. Therefore from the last step the Square of the lesser number is $\frac{36aa}{49}$ $ccaa$
7. And by adding together the Squares in the second and sixth steps, their sum will be $\frac{85aa}{49}$ $ccaa + aa$
8. Whose sum must be equal to the given sum 340 (or b), whence this Equation aritheth, $\frac{85aa}{49} = 340$ $ccaa + aa = b$
9. From which Equation, after it is duly reduced according to *Self*. 2, 5, and 7. of *Chap.* 12, the greater number fought will be made known, *viz.* $a = 14$ $a = \sqrt{\frac{b}{cc+1}}$
10. And from the ninth and fifth steps the lesser number will also be discovered, $= 12$ $= \sqrt{\frac{bcc}{cc+1}}$

So the two numbers fought are found 14 and 12, which will solve the Question; for the sum of their Squares 196 and 144 is 340; also, 14 multiplied by 12 makes 168, which is equal to $\frac{5}{2}$ of the greater Square 196.

QUEST. 24.

A Merchant bought a certain number of Yards of linnen Cloth at 12 pence (or b) per Yard; and if the number of pence paid for all the Cloth be multiplied by the number of

of Yards bought, the Product will be 30000, (or c). The Question is, to find the number of Yards bought.

1. For the number of yards bought put a a
2. Then the number of pence paid for the whole Cloth will be $12a$ ba
3. Which number multiplied by a (the number of yards bought,) produceth $12aa$ baa
4. Which Product must, according to the Question, be equal to 30000 (or c); therefore $12aa = 30000$ $baa = c$
5. From which Equation, after due Reduction, the number of yards fought will be discovered, *viz.* $a = 50$ $a = \sqrt{\frac{c}{b}}$

So it is found that the Merchant bought 50 yards of Cloth, which at 12. d. per yard makes 600. d. this 600 multiplied by 50 (the number of yards bought,) produceth 30000; as was prescribed in the Question.

QUEST. 25.

Two Merchants, A and B , were Copartners in traffick; A brought in a certain number of pounds, which continued in Company 4 (or c) months, B brought in 100 (or b) pounds, which continued in Company such a time, that if it be multiplied by the Stock of A it makes 50 (or d). At the end of their Partnership they had gained 60 pounds, whereof A had 40 (or r) pounds for his share, and B the rest, to wit, 20 (or s) pounds. What was the Stock which A put in at first, and how many months did the Stock of B continue in Company?

1. For the Stock of A put a a
2. Then multiplying that Stock by the time it continued in company, to wit, by 4 (or c), it maketh $4a$ ca
3. Then divide 50 (or d) the Product given in the Question, by a the (Stock of A) and the Quotient will give the time that the Stock of B continued in Company, to wit, $\frac{50}{a}$ $\frac{d}{a}$
4. The Stock of B , to wit, 100 l. (or b) multiplied by its time $\frac{50}{a}$ (or $\frac{d}{a}$) produceth $\frac{5000}{a}$ $\frac{bd}{a}$
5. Then according to the nature of the Rule of Fellowship with Time, this Analogy will arise, *viz.* As the Product made by the mutual multiplication of the Stock and Time of A , is to the Product of the Stock and Time of B , so is the gain of A to the gain of B : *viz.*

$$\text{As, } 4a \cdot \frac{5000}{a} :: 40 \cdot 20,$$

$$\text{Or, } ca \cdot \frac{bd}{a} :: r \cdot s.$$

6. Which Analogy (according to *Self*. 1. *Chap.* 13.) may be converted into this Equation, *viz.* $80a = \frac{200000}{a}$

$$\text{Or, } 8ca = \frac{rbd}{a}.$$

7. From which Equation, (after due Reduction according to *Self*. 2, 5, and 7. of *Chap.* 12.) the Stock of A will be discovered, *viz.*

$$a = 50 = \sqrt{\frac{rbd}{8c}}.$$

8. And

8. And from the seventh and third steps, the Time that the Stock of *B* continued in Company will also be made known, viz.

$$\frac{50}{50} = 1 = \sqrt{\frac{scd}{rb}}$$

9. So it is found that the Stock which *A* put in at first was 50 *l.* and the Time during which the Stock of *B* continued in Company was one month; as will appear by

The Proof.

$$\begin{array}{r} 50 \times 4 = 200 \\ 100 \times 1 = 100 \end{array}$$

$$\text{Then if } \dots 300 \dots 60 :: \left\{ \begin{array}{l} 200 \dots 40 \\ 100 \dots 20 \end{array} \right.$$

QUEST. 26.

Certain Noble-men made a Progress for their pleasure; every Noble-man carried along with him the same sum of pounds; the number of the Noble-men was equal to the number of Servants which attended upon each Noble-man; the number of pounds that each Noble-man had was the double of the number of all their Servants; and the sum of all their money was 3456 pounds: the Question is, to find out the number of Noble-men; also, how many pounds and Servants each Noble-man had?

1. For the number of Noble-men put *a*
2. Then (according to the Question) the number of Servants that attended upon each Noble-man was also *a*
3. Therefore the number of all the Servants was *aa*
4. Which last number doubled gives the number of pounds that each Nobleman had, to wit, *2aa*
5. And if the said number of pounds be multiplied by the number of Noble-men, it produceth the sum of all their money, to wit, *2AAA*
6. Which sum must be equal to the given sum 3456, therefore *2AAA = 3456*
7. Therefore by taking the half of that Equation, there ariseth *AAA = 1728*
8. Lastly, by extracting the Cubick root of each part of the last Equation, the number of Noble-men is discovered, to wit, *a = 12*

So it is found that there were 12 Noble-men; also, every one of them had 12 Servants and 288 pounds, as will appear by

The Proof.

$$\begin{array}{r} 12 \times 12 = 144 \\ 144 \times 2 = 288 \\ 288 \times 12 = 3456 \end{array}$$

QUEST. 27.

A Merchant bought as many pounds of Pepper for one Crown as was half the number of Crowns he laid out, then in selling the Pepper he received for every 25 lb of Pepper as many Crowns as he paid for all the Pepper; and in conclusion he had 20 Crowns, The question is, to find how many Crowns he laid out.

1. For the number of Crowns which the Merchant laid out, let there be put *a*
2. Then the number of pounds of Pepper which he bought for one Crown was $\frac{a}{2}$
3. Whence the whole quantity of Pepper bought will be found $\frac{aa}{2}$, for, If $1 \dots \frac{a}{2} :: a \dots \left(\frac{aa}{2} \right)$

4. Then

Chap. 14.

which produce simple Equations.

4. Then find how many Crowns the Merchant received for the total quantity of Pepper sold, saying by the Rule of Three,

$$\text{If } 25 \dots a :: \frac{aa}{2} \dots \left(\frac{aaa}{50} \right)$$

whence the number of Crowns for which all the Pepper was sold is found

5. Which number of Crowns found out in the last Step, must be equal to 20 the number of Crowns given in the Question; hence this Equation, $\frac{aaa}{50} = 20$

6. From which Equation, after it is reduced according to *Self. 2.* and 7. of *Chap. 12.* there will come forth the first cost of the Pepper, to wit, $a = 10$

So the number of Crowns which the Merchant laid out was 10, as will appear by the Proof; for first, the half of 10, to wit, 5, will be the number of pounds of Pepper which he bought for 1 Crown; then say,

$$\begin{array}{l} \text{If } 1 \dots 5 :: 10 \dots 50 \parallel \text{pounds of Pepper bought,} \\ \text{If } 25 \dots 10 :: 50 \dots 20 \parallel \text{Crowns received for Pepper sold.} \end{array}$$

QUEST. 28.

There are two numbers, the greater of which hath such proportion to the lesser as 3 to 2, (or as *r* to *s*) and the sum of the Cubes of the two numbers is 4375, (or *b*); what are the numbers?

1. For the greater number put *a*
2. Then (according to *Quest. 1.* in *Self. 4.* of *Chap. 10.*) the lesser number will be found $\frac{2a}{3}$
3. Therefore from the first step, the Cube of the greater number is *aaa*
4. And from the second step the Cube of the lesser number is $\frac{8aaa}{27}$
5. Therefore from the third and fourth steps, the sum of the Cubes of both numbers is $\frac{35aaa}{27}$
6. Which sum must be equal to the given sum 4375, (or *b*); whence this Equation ariseth, viz., $\frac{35aaa}{27} = 4375$

$$\text{Or, } \frac{35aaa}{27} + \frac{8aaa}{27} = b$$

7. From which Equation, after due Reduction, (according to *Self. 2, 5, and 7.* of *Chap. 12.*) the greater number sought will be made known, viz., $a = 15 = \sqrt[3]{(3) \frac{rrrb}{ssb + rrr}}$

8. And from the seventh and second steps, the lesser number will also be discovered, to wit, $10 = \sqrt[3]{(3) \frac{ssb}{ssb + rrr}}$

So the numbers sought are found 15 and 10, which will solve the Question; for they are in the given Reason of 3 to 2; and the sum of the Cubes of the said 15 and 10, to wit, of 3375 and 1000 makes 4375; as was prescribed.

Moreover, to find the numbers sought, the latter parts of the Equations in the seventh and eighth steps give this

CANON.

Multiply severally the Cubes of the Terms of the given Reason (or Proportion) by the given sum of the Cubes of the numbers sought; divide the Products severally by the sum of the Cubes of the said Terms; lastly, extract the Cubick Root of each of the Quotients; so these Roots shall be the numbers sought

L

CHAP.

C H A P. XV.

Concerning the Resolution of such affected or compounded Equations wherein there are two different Powers of the quantity sought, and those Powers such, that the higher of them is a Square whose Side or Square Root is the lower Power.

I. The Equations treated of in this Chapter fall under three heads or forms here-under specified, which I shall first explain, and then shew how they may be Arithmetically resolved.

Equations of the first form.

$$\begin{array}{l} aa + 6a = 55. \\ aaaa + 8aa = 48. \\ aaaaa + 4aaa = 837. \end{array} \quad \left| \right. \quad \begin{array}{l} aa + ca = b. \\ aaaa + daa = f. \\ aaaaa + gaaa = h. \end{array}$$

Equations of the second form.

$$\begin{array}{l} aa - 10a = 24. \\ aaaa - 6aa = 27. \\ aaaaa - 2aaa = 48. \end{array} \quad \left| \right. \quad \begin{array}{l} aa - ba = k. \\ aaaa - paa = d. \\ aaaaa - maaa = g. \end{array}$$

Equations of the third form.

$$\begin{array}{l} 10a - aa = 24. \\ 5aa - aaaa = 4. \\ 9aaa - aaaaa = 8. \end{array} \quad \left| \right. \quad \begin{array}{l} ca - aa = n. \\ yaa - aaaa = s. \\ daaa - aaaaa = t. \end{array}$$

II. Every Equation which falleth under any of the said three forms, consists of three distinct Terms or Members, whereof two are unknown and the third is known; of the two unknown terms, one is a Square, (by which in this place I mean a square number) which is called the highest term in the Equation; and the other unknown term is the Product made by the multiplication of the square Root of the said square number by some known number, which Product is called the middle term; and the third or lowest term is a number purely known: So in this Equation $aa + 6a = 55$, the highest term is aa , which may represent an unknown square number whose Root is a ; the middle term is $6a$, which is the Product of the multiplication of the said unknown Root a by the known number 6; and the lowest term (or known part of the said Equation) is the number 55, which for distinction sake is usually called the Absolute number given.

The like may be observed in this Equation $aa + ca = b$, where we may suppose b and c to represent two known numbers, and a some number unknown; then the highest term is the Square aa ; the middle term is ca , to wit, the Product made by the multiplication of a the Root of the said Square aa by the known number c ; and the lowest term of the said Equation is the known absolute number b .

Again, in this Equation $5aa - aaaa = 4$, the highest term is the square number $5aa$; the middle term is $aaaa$, to wit, the Product made by the multiplication of aa the square Root of the said square number $aaaa$ into the known number 5; and the lowest term is the absolute number 4.

III. In every Equation which falls under any of the three before-mentioned forms, there are two different Powers or Degrees of the number sought, and those such, that the Index or Exponent of the higher Power is the double of the Index of the lower: As in this Equation $aa + 6a = 55$, the Index or number of dimensions in aa is 2, which is the double of 1 the Index of a (in the middle term $6a$); so also in this Equation $5aa - aaaa = 4$, the Index of the highest term $5aa$ is 2, which is the double of 1 the Index of aa in the middle term. Likewise in this Equation $9aaa - aaaaa = 8$, the Index of the highest term $9aaa$ is 3, which is the double of 1 the Index of aaa in the

the middle term. But in this Equation $aaa + 6a = 39$ the Index of the highest term aaa is not the double of the Index of a in the middle term, (for the Index of the former is 3, and of the latter 1;) and therefore the Equation last proposed cannot be ranked under any of the three Forms aforesaid, and consequently it is not resolvable by the following Rules of this Chapter, but belongs to the 10, and 11. Chapters of my second Book.

IV. Known numbers which are drawn into, or multiplied by some Degree or Power of the number sought are by *Pieta* and others called Coefficients, viz. fellow-factors, or copartners in multiplication with unknown Powers: So in this Equation $aa + 6a = 55$ the number 6 is called the Coefficient, to wit, the fellow-multipier with the unknown number a to make the Product $6a$. Likewise in this Equation $aa + ca = b$, we may suppose the letters b and c to represent known numbers, and the letter a some unknown number whose Coefficient is c .

But sometimes the Coefficient will happen to be express'd by many letters, as in this Equation $aa + \frac{sc}{r}a$ (or $\frac{sc}{r}a$) = $\frac{15sc}{4r}$, where a only is supposed to be unknown, and the known number $\frac{sc}{r}$ is the Coefficient, which signifies but one number,

to wit, the Quotient that ariseth, when the Product of the number s multiplied by the number c is divided by the number r , viz. if $s = 2$; $c = 4$; and $r = 1$, then $\frac{sc}{r}$ or 8 is the Coefficient, and consequently $\frac{sc}{r}a$ is the same with $8a$.

Likewise in this Equation $\frac{2r+1}{s}a$ (or $\frac{2r+1}{s}a$) = $\frac{2r}{s}$, the Coefficient is $\frac{2r+1}{s}$, which is to be esteemed but as one number, to wit, the Quotient

that ariseth by dividing the sum of $2r$ and 1 by s ; so that if we suppose $r = 3$ and $s = 2$, then the Equation last proposed may be express'd thus, $4a - aa = 3$.

Note. When no known number appears to be drawn into the middle term of the Equation, then 1 (or Unity) must in that case be always taken for the Coefficient; so in this Equation $aa + a = 30$, the middle term a implies $1a$, to wit, the Product of a multiplied by 1, and therefore 1 is the Coefficient.

Note also. When the highest unknown Power or Degree is multiplied by any number greater than 1, then every term or member of the Equation must be divided by that number, to the end the said highest unknown Power may be cleared from any Coefficient unless it be 1; as before hath been shewn in Sect. 5. Chap. 12.

These things being premised by way of Explication, I proceed to the Resolution of Equations which fall under any of the three forms before specified.

V. The Arithmetical Resolution of Equations which fall under the first of the three Forms before specified in Sect. I. of this Chapter.

QUEST. 1.

1. What is the number represented by a in this Equation? $aa + 6a = 55$
2. Which Equation, if c be assumed to signify 6, and b 55, $aa + ca = b$ may be express'd thus,

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, viz. There is an unknown number (represented by a) which is such, that if to its Square you add the Product made by the multiplication of that unknown number by 6, (or c .) the sum will be 55, (or b ;) what is that unknown number a ? *Ans.* 5; found out thus,
4. Let the Square of half the Coefficient 6 (or c) be added to each part of the Equation proposed, to the end its first part may be made a complete Square, (according to Sect. 4. Chap. 9.) whence this Equation ariseth,

$$aa + 6a + 9 = 64; \quad \text{or,} \quad aa + ca + \frac{c^2}{4} = b + \frac{c^2}{4}.$$

L z

5. Then

5. Then by extracting the Square Root of each part of the last Equation (according to *Self. 4.* and 5. of *Chap. 8.*) this Equation ariseth;

$$a + 3 = 8,$$

Or,

$$a + \frac{1}{2}c = \sqrt{b + \frac{1}{2}cc}:$$

6. Wherefore by transposition (or equal subtraction) of 3, or $\frac{1}{2}c$, the number a sought will be made known, *viz.*

$$a = 5 = \sqrt{b + \frac{1}{2}cc} - \frac{1}{2}c.$$

I say the number a sought is 5, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 5,$
Then consequently	$aa = 25,$
And	$6a = 30;$
Therefore	$aa + 6a = 55.$

Which was the Equation proposed.

Note. Every Equation which falls under this first Form may be expounded by either of two Roots, whereof one is Affirmative or greater than nothing, and the other Negative or less than nothing. As in the Equation proposed, to wit, $aa + 6a = 55$; forasmuch as according to the Rules of Algebraical Multiplication, — multiplied by — produceth +, and so in this sense the Square Root of 64 may be —8 as well as +8; therefore the square Root of the Equation $aa + 6a + 9 = 64$ in the fourth step may be this, to wit,

Whence, by transposition of + 3, a Negative Root } $a = -11.$
or value of a is discovered, to wit, }
I say the Root a in the Equation $aa + 6a = 55$ may be expounded by —11, (besides + 5,) as will be manifest by

The Proof.

If	$a = -11,$	Here the Rules of + and — in Algebraical Multiplication and Addition are to be respected.
Then	$aa = +121,$	
And	$6a = -66$	
Therefore, as before, $aa + 6a = +55.$		

Negative Roots are oftentimes of good use to find our Affirmative Roots, as hereafter will appear in *Chap. 11.* of the second Book.

QUEST. 2.

1. What is the number represented by a in this Equation? . . . } $aaaa + 8aa = 48,$
2. Which Equation, if d be put for 8, and f for 48, may be? } $aaaa + daa = f.$
express this,

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by a , which is such, that if to its Biquadrate or squared Square you add the Product made by the multiplication of the Square of that unknown number a by 8, (or d ;) the sum will be 48, (or f ;) what is the unknown number a ? *Ans.* 2. found out in the same manner as before in *Quest. 1.* *viz.*
4. Let the Square of half the Coefficient 8 (or d) be added to each part of the Equation proposed, to the end its former part may be made a compleat Square, according to *Self. 4. Chap. 9.* whence this Equation ariseth;

$$aaaa + 8aa + 16 = 64,$$

$$\text{Or, } aaaa + daa + \frac{1}{4}dd = f + \frac{1}{4}dd.$$

5. Then by extracting the Square Root of each part of the last Equation (according to *Self. 4.* and 5. of *Chap. 8.*) this Equation ariseth,

$$aa + 4 = 8,$$

$$\text{Or, } aa + \frac{1}{2}d = \sqrt{f + \frac{1}{4}dd}:$$

6. Whence by equal subtraction or transposition of 4 (or $\frac{1}{2}d$) there will arise

$$aa = 4$$

$$\text{Or, } aa = \sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d.$$

7. There-

7. Therefore by extracting the Square Root of each part of the last Equation, the number a sought, will be made known, *viz.*

$$a = 2 = \sqrt{(2)} : \sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d:$$

I say the number a sought is 2, which will solve the Question proposed; as will appear by

The Proof.

If	$a = 2;$
Then consequently	$aa = 4,$
And	$aaaa = 16,$
Also	$8aa = 32,$
Therefore	$aaaa + 8aa = 48.$

Which was the Equation propos'd to be resolved.

QUEST. 3.

1. What is the number represented by a in } : $aaaaa + 4aaa = 837.$
this Equation? }
2. Which Equation, if g be put for 4, and b } : $aaaaa + gaaa = b.$
for 837, may be express thus }

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by a , which is such, that if to its cubed Cube or sixth Power, you add the Product made by the multiplication of the Cube of that unknown number by 4 (or g) the sum will be 837, what is that unknown number a ? *Ans.* 2. found out in the same manner as before, *viz.*
4. By adding the Square of half the Coefficient 4 (or g) to each part of the Equation proposed, this Equation ariseth;

$$aaaaa + 4aaa + 4 = 841,$$

$$\text{Or, } aaaaa + gaaa + \frac{1}{4}g = b + \frac{1}{4}gg.$$

5. And by extracting the Square Root of each part of the last Equation this ariseth;

$$aaa + 2 = 29;$$

$$\text{Or, } aaa + \frac{1}{2}g = \sqrt{b + \frac{1}{4}gg}:$$

6. Whence by transposition of 2 (or $\frac{1}{2}g$) this Equation ariseth;

$$aaa = 27,$$

$$aaa = \sqrt{b + \frac{1}{4}gg} - \frac{1}{2}g.$$

7. Therefore by extracting the Cubick Root of each part of the last Equation the number a sought will be made known, *viz.*

$$a = 3 = \sqrt[3]{(3)} : \sqrt[3]{b + \frac{1}{4}gg} - \frac{1}{2}g:$$

I say the number a sought is 3, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 3;$
Then consequently	$aaa = 27,$
And	$aaaaa = 729,$
Also	$4aaa = 108,$
Therefore	$aaaaa + 4aaa = 837.$

Which was the Equation propos'd to be resolved.

V I. From the Resolution of the three last Questions the following Canon is deduced for the resolving of all Equations which fall under the first of the three forms before specified in *Self. 1.* of this Chapter.

CANON.

Add the Square of half the Coefficient, or (which is the same thing) a quarter of the Square of the whole Coefficient, to the given Absolute number.
Extract the Square Root of that sum.

From

From the said Square Root subtract half the Coefficient, and reserve the Remainder.

Lastly, when the unknown number which is multiplied by the Coefficient in the middle term of the Equation is express'd by a single letter only, as a , then the Remainder before reserved is the number sought; but if the said unknown number in the middle term be a Square, as aa , then the Square Root of the Remainder reserved is the number sought; if a Cube, as aaa , then the Cubick Root of the said Remainder shall be the number sought; if any higher Power, then the Root for the kind must be extracted out of the said Remainder, which Root shall be the number sought.

An Example of the Canon.

1. Let the preceding *Quest.* 1. be here repeated, $\left. \begin{array}{l} \text{viz. What is the number represented by } a \\ \text{in this Equation?} \end{array} \right\} \dots aa + 6a = 55$
2. Or, what is the value of a in this Equation, $\dots aa + ca = b$

RESOLUTION.

3. To the given absolute number 55 $\left. \begin{array}{l} b. \\ \frac{1}{2}cc. \end{array} \right\} 9$
4. Add the Square of half the Coefficient 6 , $\left. \begin{array}{l} \text{to wit, the Square of } 3, \text{ which is } \dots 64 \\ \text{The sum is } \dots \end{array} \right\} b + \frac{1}{2}cc.$
5. The Square Root of that sum is 8 $\sqrt{b + \frac{1}{2}cc.}$
6. From that Square Root subtract half the Coefficient 6 , to wit, 3 $\frac{1}{2}c.$
7. The Remainder is the number a sought, to wit, 5 $\sqrt{b + \frac{1}{2}cc.} - \frac{1}{2}c.$

Whence it is manifest that the Answer is the same as was before found to *Quest.* 1.

A second Example of the Canon.

1. Let the preceding *Quest.* 2. be here repeated, $\left. \begin{array}{l} \text{viz. What is the number represented} \\ \text{by } a \text{ in this Equation?} \end{array} \right\} \dots aaaa + 8aa = 48$
2. Or what is the value of a in this Equation, $\dots aaaa + daa = f$

RESOLUTION.

3. To the given absolute number 48 $\left. \begin{array}{l} f. \\ \frac{1}{2}dd \end{array} \right\} 16$
4. Add the Square of half the Coefficient 8 , $\left. \begin{array}{l} \text{to wit, the Square of } 4, \text{ which is } \dots 64 \\ \text{The sum is } \dots \end{array} \right\} f + \frac{1}{2}dd.$
5. The Square root of that sum is 8 $\sqrt{f + \frac{1}{2}dd.}$
6. From which square root subtract half the Coefficient 8 , to wit, 4 $\frac{1}{2}d.$
7. The Remainder is the value of aa , to wit, 4 $\sqrt{f + \frac{1}{2}dd} - \frac{1}{2}d$
8. Lastly, the Square root of the said Remainder gives the number a , 2 $\sqrt{(\sqrt{f + \frac{1}{2}dd} - \frac{1}{2}d)}$

Whence it is evident that the Answer is the same as was before found to *Quest.* 2.

A third Example of the Canon.

1. Let the preceding *Quest.* 3. be here repeated, $\left. \begin{array}{l} \text{viz. What is the number represented by } a \text{ in this Equation?} \end{array} \right\} \dots aaaaaa + 4aaa = 837.$
2. Or what is the value of a in this Equation, $\dots aaaaaa + gaaa = h.$

RESOLUTION.

3. To the absolute number 837 $\left. \begin{array}{l} h. \\ \frac{1}{2}gg. \end{array} \right\} 4$
4. Add the Square of half the Coefficient 4 , to wit, 4 $\frac{1}{2}gg.$
5. The sum is 841 $\left. \begin{array}{l} b + \frac{1}{2}gg. \\ \sqrt{b + \frac{1}{2}gg.} \end{array} \right\} 29$
6. The Square root whereof is 29 $\frac{1}{2}g.$
7. From that square root subtract half the Coefficient 4 , to wit, 2

8. The

8. The Remainder is the value of aaa , to wit, 27 $\sqrt{b + \frac{1}{2}gg} - \frac{1}{2}g$
9. Therefore the Cubick Root of that Remainder shall be the number a sought, 3 $\sqrt[3]{(27)} = \sqrt[3]{b + \frac{1}{2}gg} - \frac{1}{2}g$

Whereby it is manifest that the Answer is the same as was before found to *Quest.* 3.

Example 4.

If $\dots aa + a = b$ (or 35), what is a ?

Ans. $a = \sqrt{b + \frac{1}{4}} - \frac{1}{2} = 5 \frac{22}{25}, \text{ \&c.}$

For the Coefficient drawn into the middle term a being 1 , its half is $\frac{1}{2}$, the Square whereof is $\frac{1}{4}$, which added to the absolute number 35 makes $35 \frac{1}{4}$, whose Square Root is $5 \frac{22}{25}$, &c. from which subtracting $\frac{1}{2}$ (or $\frac{12}{25}$) to wit, half the Coefficient 1 , the Remainder $5 \frac{10}{25}$, &c. is the number a sought, which here happens to be irrational, that is, inexpressible by any true number, but by continuing the extraction of the said Square Root of the said $35 \frac{1}{4}$, you may approach infinitely near the exact number a .

Example 5.

If $\dots aa + \frac{1}{2}a = \frac{121}{2}$, what is a ?

Ans. $a = \sqrt{\frac{121}{2} + \frac{1}{4}} - \frac{1}{4} = \frac{11}{2}.$

The Learner must remember to reduce a Fraction to its least Terms, before he goes about to extract any Root out of it.

Example 6.

If $\dots \left. \begin{array}{l} r = 1, \\ s = 2, \\ e = 4, \end{array} \right\}$

And if $\dots aa + \frac{3c}{r}a = \frac{155cc}{4rr}$

What is $\dots a = ?$

Ans. $\dots a = \frac{3c}{2r} = 12 \frac{1}{2}$

Example 7.

If $\dots aaaa + \frac{11}{3}aa = \frac{12164}{27}$, what is a ?

Ans. $\dots a = \frac{11}{3}.$

VII. The Arithmetical Resolution of Equations which fall under the second of the three Forms before expressed in Sect. 1. of this Chapter.

QUEST. 1.

1. What is the number represented by a in $\dots aa - 10a = 24.$
2. Which Equation, by assuming b to represent 10 , and k to signify 24 , may be expressed thus, $\dots aa - ba = k.$

RESOLUTION.

3. Let the Square of half the Coefficient 10 (or b) be added to each part of the Equation proposed, to the end its first part may be made a complete Square, (according to *Sect.* 4. Chap. 9.) whence this Equation ariseth;

$$aa - 10a + 25 = 49,$$

$$\text{Or, } aa - ba + \frac{1}{4}bb = k + \frac{1}{4}bb.$$

4. Then by extracting the Square Root of each part of the last Equation (according to *Sect.* 4. and 5. of Chap. 8.) this Equation ariseth;

$$a - 5 = 7,$$

$$\text{Or, } a - \frac{1}{2}b = \sqrt{k + \frac{1}{4}bb.}$$

5. Wherefore by equal addition of 5 , or $\frac{1}{2}b$, the number a sought will be made known, viz.

$$a = 12 = \frac{1}{2}b + \sqrt{k + \frac{1}{4}bb.}$$

6. But forasmuch as the square Root of $aa - 10a + 25$ in the third step may be $5 + a$ as well as $a - 5$, (for either of those Roots being multiplied by itself will produce the

the same Square $aa - 10a + 25$, therefore let $5 - a$ be set instead of $a - 5$ in the fourth step, whence this Equation ariseth, viz.

$$\begin{aligned} 5 - a &= 7, \\ \frac{1}{2}b - a &= \sqrt{k + \frac{1}{2}bb}: \end{aligned}$$

7. Therefore by transposition, another value of a ariseth, to wit,

$$a = -2 = \frac{1}{2}b - \sqrt{k + \frac{1}{2}bb}:$$

Which latter value of a is less than nothing, and such it will always be, as may easily be proved from the last Equation. For $k + \frac{1}{2}bb$ is manifestly greater than $\frac{1}{2}bb$, and consequently the square Root of the former will be greater than the square Root of the latter, viz. $\sqrt{k + \frac{1}{2}bb}$ is greater than $\frac{1}{2}b$, therefore $\frac{1}{2}b - \sqrt{k + \frac{1}{2}bb}$ (that is a) will be less than nothing, for if a greater quantity be subtracted from a less, the Remainder will be a negative quantity, that is less than nothing, as before hath been shewn in Algebraical Subtraction. From the premises it is evident that the Equation propounded, to wit, $aa - 10a = 24$ (and likewise every Equation which falleth under the second form of Equations before-mentioned) is explicable by two Roots, whereof one is real or affirmative, whose value is before exprest in the fifth step; and the other negative or less than nothing, the value whereof is exprest in the seventh step.

I say the real or true number a sought in the Question propoed is 12, as will appear by

The Proof.

$$\begin{aligned} \text{If} & \dots \dots \dots a = 12, \\ \text{Then consequently} & \dots \dots \dots aa = 144, \\ \text{And} & \dots \dots \dots 10a = 120, \\ \text{Therefore} & \dots \dots \dots aa - 10a = 24. \end{aligned}$$

Which was the Equation propoed.

Moreover, according to the Rules of Algebraical Multiplication and Subtraction, the negative value of a , to wit -2 before found, will constitute the Equation first propoed:

$$\begin{aligned} \text{For if} & \dots \dots \dots a = -2, \\ \text{Then consequently} & \dots \dots \dots aa = +4, \\ \text{And} & \dots \dots \dots 10a = -20, \\ \text{Therefore} & \dots \dots \dots aa - 10a = +24; \text{ as before.} \end{aligned}$$

QUEST. 2.

1. What is the number represented by a in } this Equation? $aaaa - 6aa = 27$
2. Which Equation, if p be put for 6, and } a for 27, may be exprest thus, $aaaa - paa = d$

RESOLUTION.

3. Let the Square of half the Coefficient 6 (or p) be added to each part of the Equation propoed, to the end its first part may be made a compleat Square (according to Sect. 4. Chap. 9.) whence this Equation ariseth;

$$aaaa - 6aa + 9 = 36,$$

$$\text{Or, } aaaa - paa + \frac{1}{4}pp = d + \frac{1}{4}pp.$$

4. Then by extracting the square Root of each part of the last Equation (according to Sect. 4. and 5. of Chap. 8.) this Equation ariseth, viz.

$$aa - 3 = 6,$$

$$aa - \frac{1}{2}p = \sqrt{d + \frac{1}{4}pp}:$$

5. Whence, by equal addition of 3 (or $\frac{1}{2}p$.) there will arise

$$aa = 9,$$

$$\text{Or, } aa = \sqrt{d + \frac{1}{4}pp} + \frac{1}{2}p.$$

6. Wherefore by extracting the square Root of each part of the last Equation, the number a sought will be made known, viz.

$$a = 3 = \sqrt{(2) : \sqrt{d + \frac{1}{4}pp} + \frac{1}{2}p}:$$

I say the number a sought is 3, which will solve the Question propoed; as will appear by

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The Proof.

$$\begin{aligned} \text{If} & \dots \dots \dots a = 3; \\ \text{Then consequently} & \dots \dots \dots aa = 9, \\ \text{And} & \dots \dots \dots aaaa = 81, \\ \text{Also} & \dots \dots \dots 6aa = 54, \\ \text{Therefore} & \dots \dots \dots aaaa - 6aa = 27. \end{aligned}$$

Which was the Equation propoed to be resolved.

QUEST. 3.

1. What is the number represented by a in } this Equation? $aaaaaa - 2aaa = 48$
2. Which Equation, if m be put for 2, and } g for 48, may be exprest thus, $aaaaaa - maaa = g$

RESOLUTION.

3. Let the Square of half the Coefficient 2 (or m) be added to each part of the Equation propoed, to the end its former part may be made a compleat Square (according to Sect. 4. Chap. 9.) whence this Equation ariseth;

$$aaaaaa - 2aaa + 1 = 49;$$

$$\text{Or, } aaaaaa - maaa + \frac{1}{4}mm = g + \frac{1}{4}mm.$$

4. Then by extracting the square Root of each part of the last Equation (according to Sect. 4. and 5. of Chap. 8.) this Equation ariseth;

$$aaa - 1 = 7;$$

$$\text{Or, } aaa - \frac{1}{2}m = \sqrt{g + \frac{1}{4}mm}:$$

5. Whence by equal addition of 1 (or $\frac{1}{2}m$) there ariseth

$$aaa = 8,$$

$$\text{Or, } aaa = \sqrt{g + \frac{1}{4}mm} + \frac{1}{2}m.$$

6. Wherefore by extracting the Cubick Root of each part of the last Equation, the number a sought will be made known, viz.

$$a = 2 = \sqrt[3]{(3) : \sqrt{g + \frac{1}{4}mm} + \frac{1}{2}m}:$$

I say the number a sought is 2, which will solve the Question propoed; as will appear by

The Proof.

$$\begin{aligned} \text{If} & \dots \dots \dots a = 2; \\ \text{Then consequently} & \dots \dots \dots aaa = 8, \\ \text{And} & \dots \dots \dots aaaaaa = 64, \\ \text{Also} & \dots \dots \dots 2aaa = 16, \\ \text{Therefore} & \dots \dots \dots aaaaaa - 2aaa = 48. \end{aligned}$$

Which was the Equation propoed to be resolved.

VIII. From the Resolution of the three last Questions the following Canon is deduced, for the resolving of all Equations which fall under the second of the three Forms before specified in Sect. 1. of this Chap.

CANON.

Add the Square of half the Coefficient, or, (which is the same thing) a quarter of the Square of the whole Coefficient, to the given Absolute number;

Extract the Square Root of that sum.

To the said Square Root add half the Coefficient, and reserve this sum.

Lastly, when the unknown number which is drawn into the Coefficient in the middle term of the Equation is exprest by a single letter only, as a , then the Summ before reserved is the number sought; but if the said unknown number in the middle term be a Square, as aa , then the Square Root of the Summ reserved is the number sought; if a Cube, as aaa , then the Cubick Root of the said Summ shall be the number sought; if any higher Power, then the Root for the kind must be extracted out of the said Summ, which Root shall be the number sought.

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An Example of the said Canon.

1. Let the preceding *Quest.* 1. in *Sect.* 7. of this *Chapt.* be here repeated, viz. What is the number represented by a in this Equation? $aa - 10a = 24$
2. Or, what is the value of a in this Equation? $aa - 10a = 24$

RESOLUTION.

3. To the given absolute number . . . 24 k
 4. Add the Square of half the Coefficient 10, which is . . . 25 $\frac{1}{4}bb$
 5. The sum is . . . 49 $k + \frac{1}{4}bb$
 6. The Square Root of that sum is . . . 7 $\sqrt{k + \frac{1}{4}bb}$
 7. To which Square Root add half the Coefficient 10, to wit, . . . 5 $\frac{1}{2}b$
 8. The Sum is the number a sought, to wit, 12 $\sqrt{k + \frac{1}{4}bb} + \frac{1}{2}b$
- Whence it is manifest that the Answer is the same as was before found to *Quest.* 1. in *Sect.* 7.

A second Example of the Canon in *Sect.* 8.

1. Let the preceding *Quest.* 2. in *Sect.* 7. of this *Chapt.* be here repeated, viz. What is the number represented by a in this Equation? $aaaa - 6aa = 27$
2. Or, What is the value of a in this Equation? $aaaa - 6aa = 27$

RESOLUTION.

3. To the given absolute number . . . 27 d
 4. Add the Square of half the Coefficient 6, which is . . . 9 $\frac{1}{4}pp$
 5. The sum is . . . 36 $d + \frac{1}{4}pp$
 6. The Square Root of that sum is . . . 6 $\sqrt{d + \frac{1}{4}pp}$
 7. To which Square Root add half the Coefficient 6, to wit, . . . 3 $\frac{1}{2}p$
 8. The Sum is the value of aa , to wit, 9 $\sqrt{d + \frac{1}{4}pp} + \frac{1}{2}p$
 9. Therefore the square Root of the said Sum shall be the number sought, to wit, 3 $\sqrt{(2) : \sqrt{d + \frac{1}{4}pp} + \frac{1}{2}p}$
- Whence it is manifest that the Answer is the same as was before found to *Quest.* 2. in *Sect.* 7.

A third Example of the Canon in *Sect.* 8.

1. Let the preceding *Quest.* 3. in *Sect.* 7. of this *Chapt.* be here repeated, viz. What is the number represented by a in this Equation? $aaaaaa - 2aaa = 48$
2. Or, What is the value of a in this Equation? $aaaaaa - 2aaa = 48$

RESOLUTION.

3. To the given absolute number . . . 48 g
4. Add the Square of half the Coefficient 2, to wit, the Square of 1, which is . . . 1 $\frac{1}{4}mm$
5. The sum is . . . 49 $g + \frac{1}{4}mm$
6. The Square Root of that Sum is . . . 7 $\sqrt{g + \frac{1}{4}mm}$
7. To which square Root add half the Coefficient 2, to wit, . . . 1 $\frac{1}{2}m$
8. The sum is the value of aaa , to wit, 8 $\sqrt{g + \frac{1}{4}mm} + \frac{1}{2}m$
9. Therefore the cubick Root of the said sum shall be the number a sought, to wit, 2 $\sqrt{(3) : \sqrt{g + \frac{1}{4}mm} + \frac{1}{2}m}$

Whereby it is manifest that the Answer is the same as was before found to *Quest.* 3. in *Sect.* 7.

Example

Example 4.

- If . . . $aa - a = g$ (or 1122,) what is a = ?
 Answ. . . $a = \sqrt{g + \frac{1}{4}} + \frac{1}{2} = 34$.

Example 5.

- If . . . $aa - \frac{1}{3}a = 373 \frac{1}{3}$, what is a = ?
 Answ. . . $a = 20 \frac{2}{3}$.

Example 6.

- If . . . $\begin{cases} x = 1 \\ y = 2 \\ z = 4 \end{cases}$
 And if . . . $aa - \frac{50}{27}a = \frac{15500}{27}$
 What is . . . a = ?
 Answ. . . $a = \frac{50}{27} + 20 = 20 \frac{50}{27}$

IX. The Arithmetical Resolution of Equations; which fall under the last of the three Forms before express'd in *Sect.* I. of this Chapter.

QUEST. I.

1. What is the number represented by a in this Equation? $10a - aa = 24$
2. Which Equation, if c be assumed to signifie 10, and n put for 24, may be express'd thus; $ca - aa = n$

RESOLUTION.

3. Let the Equation propos'd, by transposition of its Terms, be reduced to an Equation of the second of the three Forms before express'd in *Sect.* 1. viz. First, by transposition of $-aa$, this Equation ariseth;

$$10a = 24 + aa$$

$$\text{Or, } ca = n + aa$$

4. Likewise by transposition of 24 (or n) this Equation ariseth;

$$10a - 24 = aa$$

$$\text{Or, } ca - n = aa$$

5. And from the last Equation by transposition of $10a$ (or ca) there will arise

$$-24 = aa - 10a$$

$$\text{Or, } -n = aa - ca$$

6. Which last Equation, by transposing each part of it to the contrary coast, may be express'd thus;

$$aa - 10a = -24$$

$$\text{Or, } aa - ca = -n$$

7. Now let the following process be made as before in the Resolution of Equations of the second Form (in *Sect.* 7.) viz. Let the Square of half the Coefficient 10 (or c) be added to each part of the last Equation, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aa - 10a + 25 = 25 - 24 = 1$$

$$\text{Or, } aa - ca + \frac{1}{4}cc = \frac{1}{4}cc - n$$

8. Then by extracting the Square root of each part of the last Equation, (according to *Sect.* 4. and 5. of *Chap.* 8.) this Equation ariseth, viz.

$$a - 5 = 1$$

$$\text{Or, } a - \frac{1}{2}c = \sqrt{\frac{1}{4}cc - n}$$

9. Whence by equal addition of 5 (or $\frac{1}{2}c$) one value of a will be made known, viz.

$$a = 5 + \sqrt{\frac{1}{4}cc - n}$$

10. But sofar as the Square root of $aa - 10a + 25$ in the seventh step may be 5 - a as well as $a - 5$, (for either of those Roots being multiplied into it self, will produce

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produce $aa - 10a + 25$, therefore let $5 - a$ be set instead of $a - 5$ in the eighth step, whence this Equation will arise, *viz.*

$$5 - a = 1, \\ \frac{1}{2}c - a = \sqrt{\frac{1}{2}cc - n}.$$

Or, When by due transposition another value of a is discovered, to wit,

$$a = 4 = \frac{1}{2}c - \sqrt{\frac{1}{2}cc - n}.$$

12. I say the number a sought may be either 6 or 4, for either of these numbers will constitute the Equation proposed, as will appear by.

The Proof.

$$\begin{array}{ll} \text{If} & a = 6, \\ \text{Then consequently} & aa = 36, \\ \text{And} & 10a = 60, \\ \text{Therefore} & 10a - aa = 24. \end{array}$$

Which was the Equation propos'd to be resolved.

Again,

$$\begin{array}{ll} \text{If} & a = 4, \\ \text{Then consequently} & aa = 16, \\ \text{And} & 10a = 40, \\ \text{Therefore} & 10a - aa = 24; \text{ as before.} \end{array}$$

13. But to the end that both the values of a before express'd in the ninth and eleventh Equations may be real or Affirmative numbers, (that is, each greater than nothing) the given numbers in the Equation propos'd, and likewise in every Equation of the third Form aforesaid must be subject to the following

DETERMINATION.

The Absolute number given must not exceed the Square of half the Coefficient.

The reason of this Determination is evident by the said ninth and eleventh Equations; for the latter part of each of them shews, that the given Absolute number is to be subtracted from the Square of half the Coefficient, and therefore it ought to be less, or equal to the said Square: Therefore when in any Equation of the third form, the given Absolute number exceeds the Square of half the Coefficient that Equation is impossible, and likewise the Question that produced it.

It is also evident by the said ninth and eleventh Equations, That when it happens that $n = \frac{1}{2}cc$, then $\frac{1}{2}c - n = 0$, and consequently each value of a is equal to $\frac{1}{2}c$, *viz.* When the Absolute number happens to be equal to the Square of half the Coefficient, then the two values of a will be equal to one another, each value in that case being equal to half the Coefficient: But when it happens that the Absolute number is less than the Square of half the Coefficient, then those two Roots or values of a will be unequal. But here is to be noted, that although in this latter case the Equation be always explicable by either of those two unequal Roots or numbers, yet the Question that produced the Equation will sometimes be answered only by one of those Roots or numbers, (as hereafter will appear in *Quest.* 10. *Chap.* 16. and by the latter way of resolving the 16. *Quest.* of the same *Chapt.*)

QUEST. 2.

- What is the number represented by a in this Equation? $5aa - aaaa = 4$.
- Which Equation, if r be put for 5, and s for 4, may be express'd thus? $raa - aaaa = s$.

RESOLUTION.

3. Let the Equation propos'd, by Transposition of its Terms (after the same manner as in the third, fourth, fifth, and sixth steps of the preceding *Quest.* 1. *Sett.* 9.) be reduced to an Equation of the second of the three Forms before express'd in *Sett.* 1. so this Equation will arise, *viz.*

$$\begin{array}{ll} aaaa - 5aa = -4, \\ \text{Or,} & aaaa - raa = -s. \end{array}$$

4. Then

4. Then by adding (as in the former Examples) the Square of half the Coefficient 5 (or r) to each part of the last Equation, there ariseth

$$aaaa - 5aa + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4},$$

$$\text{Or,} \quad aaaa - raa + \frac{r^2}{4} = \frac{r^2}{4} - s.$$

5. And by extracting the Square Root of each part of the last Equation this ariseth;

$$aa - \frac{1}{2}r = \frac{3}{2},$$

$$\text{Or,} \quad aa - \frac{1}{2}r = \sqrt{\frac{1}{4}rr - s};$$

6. Whence by equal addition of $\frac{1}{2}$ (or $\frac{1}{2}r$) this Equation ariseth, *viz.*

$$aa = \frac{1}{2} \text{ or } 4,$$

$$\text{Or,} \quad aa = \frac{1}{2}r + \sqrt{\frac{1}{4}rr - s};$$

7. Therefore by extracting the Square Root of each part of the last Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(2) : \frac{1}{2}r - \sqrt{\frac{1}{4}rr - s}};$$

8. But so far as much as the Square Root of $aaaa - 5aa + \frac{25}{4}$ in the fourth step may be $\frac{3}{2} - aa$, as well as $aa - \frac{1}{2}$, (for either of those Roots being multiply'd by it self will produce $aaaa - 5aa + \frac{25}{4}$), therefore let $\frac{3}{2} - aa$ be set instead of $aa - \frac{1}{2}$ in the fifth step, whence this Equation will arise;

$$\frac{3}{2} - aa = \frac{3}{2},$$

$$\text{Or,} \quad \frac{1}{2}r - aa = \sqrt{\frac{1}{4}rr - s};$$

9. Whence by due transposition this Equation ariseth;

$$aa = \frac{1}{2} \text{ or } 1,$$

$$\text{Or,} \quad aa = \frac{1}{2}r - \sqrt{\frac{1}{4}rr - s};$$

10. Wherefore by extracting the Square Root of each part of the last Equation, another value of a is discovered, to wit,

$$a = 1 = \sqrt{(2) : \frac{1}{2}r - \sqrt{\frac{1}{4}rr - s}};$$

I say the number a sought may be either 2 or 1, for either of these numbers will constitute the Equation propos'd, as will appear by

The Proof.

$$\begin{array}{ll} \text{If} & a = 2, \\ \text{Then consequently} & aa = 4, \\ \text{And} & aaaa = 16, \\ \text{Also} & 5aa = 20, \\ \text{Therefore} & 5aa - aaaa = 4. \end{array}$$

Which was the Equation propos'd to be resolved.

Again,

$$\begin{array}{ll} \text{If} & a = 1, \\ \text{Then} & aa = 1, \\ \text{And} & aaaa = 1, \\ \text{Also} & 5aa = 5, \\ \text{Therefore,} & 5aa - aaaa = 4; \text{ as before.} \end{array}$$

QUEST. 3.

- What is the number represented by a in this Equation? $9aad - aaaaa = 8$.
- Which Equation, if d be put for 9, and t for 8, may be express'd thus, $daaa - aaaaa = t$.

RESOLUTION.

3. Let the Equation propos'd, by transposition of its Terms (after the same manner as in the third, fourth, fifth, and sixth steps of the preceding *Quest.* 1. *Sett.* 9.) be reduced to an Equation of the second of the three forms before express'd in *Sett.* 1. so this Equation will arise, *viz.*

$$aaaaa - 9aad = -8,$$

$$\text{Or,} \quad aaaaa - daaa = -t.$$

4. Then by adding the Square of half the Coefficient 9 (or d) to each part of the last Equation, there ariseth

$$aaaaa - 9aad + \frac{81}{4} = \frac{81}{4} - 8 = \frac{49}{4},$$

$$\text{Or,} \quad aaaaa - daaa + \frac{d^2}{4} = \frac{d^2}{4} - t.$$

5. And

5. And by extracting the Square Root of each part of the last Equation this aritheth,
- $$aaa - \frac{1}{2}d = \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
- Or,
- $$aaa - \frac{1}{2}d = \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
6. Whence by equal addition of $\frac{1}{2}d$ (or $\frac{1}{2}d$) this Equation aritheth,
- $$aaa = \frac{1}{2}d \text{ or } 8,$$
- Or,
- $$aaa = \frac{1}{2}d - \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
7. Therefore by extracting the Cubick Root of each part of the Equation, one value of a will be made known, viz.
- $$a = 2 = \sqrt[3]{(3) : \frac{1}{2}d - \sqrt{\frac{1}{4}dd - \frac{1}{4}d}}$$
8. But sofar as the Square Root of $aaaaaa - 9aaa + \frac{1}{4}d$ in the fourth step may be $\frac{1}{2}d - aaa$ as well as $aaa - \frac{1}{2}d$, (for either of these Roots being multiplied by it self, will produce the same Square $aaaaaa - 9aaa + \frac{1}{4}d$), therefore let $\frac{1}{2}d - aaa$ be set instead of $aaa - \frac{1}{2}d$ in the fifth step, whence this Equation will be made, viz.
- $$\frac{1}{2}d - aaa = \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
- Or,
- $$\frac{1}{2}d - aaa = \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
9. Whence by due transposition this Equation aritheth, viz.
- $$aaa = \frac{1}{2}d - \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
- Or,
- $$aaa = \frac{1}{2}d - \sqrt{\frac{1}{4}dd - \frac{1}{4}d}$$
10. Wherefore by extracting the Cubick Root of each part of the last Equation, another value of a is made known, viz.
- $$a = 1 = \sqrt[3]{(3) : \frac{1}{2}d - \sqrt{\frac{1}{4}dd - \frac{1}{4}d}}$$

I say the number a sought is either 2 or 1, for either of these numbers will constitute the Equation proposed; as will appear by

The Proof.

If $a = 2$,
Then consequently $aaa = 8$,
And $aaaaaa = 64$,
Also $9aaa = 72$,
Therefore $9aaa - aaaaaa = 8$.

Which was the Equation proposed to be resolved.

Again,

If $a = 1$,
Then consequently $aaa = 1$,
And $aaaaaa = 1$,
Also $9aaa = 9$,
Therefore $9aaa - aaaaaa = 8$; as before.

X. From the Resolution of the three last Questions the following Canon is deduced for the resolving of all Equations which fall under the last of the three Forms before specified in Sect. 1. of this Chap.

CANON.

From the Square of half the Coefficient, or (which is the same thing) from a quarter of the Square of the whole Coefficient, subtract the Absolute number given.

Extract the Square Root of that Remainder.

Add the said Square Root to half the Coefficient, and also subtract it from half the Coefficient, reserving the Summ and Remainder.

Lastly, when the unknown number which is multiplied by the Coefficient in the middle term of the Equation is express'd by a single letter only, as a , then the Summ and Remainder before reserved are the two numbers sought, each of which will constitute the Equation proposed; but if the said unknown number in the middle term be a Square, as aa , then the Square Root severally extracted out of the Summ and Remainder reserved shall be the two numbers sought; if a Cube, as aaa , then the Cubick Root severally extracted out of the said Summ and Remainder shall be the two numbers sought; if any higher Power, then the Root for the kind must be extracted severally out of the said Summ and Remainder, which Roots shall be the two numbers sought.

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An Example of the said Canon.

1. Let the preceding Quest. 1. in Sect. 9. of this Chap. be here repeated, viz. What is the number represented by a in this Equation? $10d - aa = 24$
2. Or, What is the value of a in this Equation? $ca - aa = n$

RESOLUTION.

3. From the square of half the Coefficient 10, to wit, the square of 5, which is 25 $\frac{1}{2}cc$.
4. Subtract the given absolute number 24 n .
5. The remainder is 1 $\frac{1}{4}cc - n$.
6. The square root of that remainder is 1 $\sqrt{\frac{1}{4}cc - n}$.
7. To which square root add half the Coefficient 10, to wit, 5 $\frac{1}{2}c$.
8. The sum is the greater value of a sought, to wit, 6 $\frac{1}{2}c + \sqrt{\frac{1}{4}cc - n}$.
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of a , to wit, 4 $\frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$.

Either of which numbers 6 and 4 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the Answer to Quest. 1. in Sect. 9. of this Chap.

A second Example of the Canon in Sect. 10.

1. Let the preceding Quest. 2. in Sect. 9. of this Chap. be here repeated, viz. What is the number represented by a in this Equation? $5aa - aaaa = 4$
2. Or, What is the value of a in this Equation? $raa - aaaa = s$.

RESOLUTION.

3. From the square of half the Coefficient 5, to wit, the square of $\frac{5}{2}$, which is $2\frac{1}{4}$ $\frac{1}{4}rr$.
4. Subtract the given absolute number 4 s .
5. The remainder is $\frac{1}{4}$ $\frac{1}{4}rr - s$.
6. The square root of that remainder is $\frac{1}{2}$ $\sqrt{\frac{1}{4}rr - s}$.
7. To which square root add half the Coefficient 5, to wit, $2\frac{1}{2}$ $\frac{1}{2}r$.
8. The sum is the greater value of aa , to wit, 4 $\frac{1}{2}r + \sqrt{\frac{1}{4}rr - s}$.
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of aa , to wit, 1 $\frac{1}{2}r - \sqrt{\frac{1}{4}rr - s}$.
10. Therefore the square root of the sum in the 8th step is the greater value of a , to wit, 2 $\sqrt{(2) : \frac{1}{2}r + \sqrt{\frac{1}{4}rr - s}}$.
11. And the square root of the remainder in the ninth step is the lesser value of a , to wit, 1 $\sqrt{(2) : \frac{1}{2}r - \sqrt{\frac{1}{4}rr - s}}$.

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the Answer to Quest. 2. in Sect. 9. of this Chap.

A third Example of the Canon in Sect. 10.

1. Let the preceding Quest. 3. in Sect. 9. of this Chap. be here repeated, viz. What is the number represented by a in this Equation? $9aaa - aaaaaa = 8$
2. Or, What is the value of a in this Equation? $daaa - aaaaaa = t$

RESOLUTION.

3. From the square of half the Coefficient 9, to wit, the square of $\frac{9}{2}$, which is $20\frac{1}{4}$ $\frac{1}{4}dd$.
4. Subtract the given absolute number 8 t .

5. The

5. The remainder is $\frac{1}{2}d$
 6. The Square root of that remainder is $\sqrt{\frac{1}{2}d}$
 7. To which Square root add half the Coefficient 9, to wit, $\frac{1}{2}d$
 8. The sum is the greater value of aaa , to wit, $\frac{1}{2}d + \sqrt{\frac{1}{2}d}$
 9. But subtracting the said Square root from half the Coefficient, the remainder is the lesser value of aaa , to wit, $\frac{1}{2}d - \sqrt{\frac{1}{2}d}$
 10. Therefore the Cubick root of the sum in the eighth step is the greater value of a , to wit, $\sqrt[3]{\frac{1}{2}d + \sqrt{\frac{1}{2}d}}$
 11. And the Cubick root of the remainder in the ninth step is the lesser value of a , to wit, $\sqrt[3]{\frac{1}{2}d - \sqrt{\frac{1}{2}d}}$
 Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the Answer to Quest. 3. in Sect. 9. of this Chap.

Example 4.

1. If b, d, f, g represent such known numbers that bf is greater than dg ; and,

$$\frac{bf - dg}{bg + dg + bf + df} a - aa = \frac{bf - dg}{bg + dg + bf + df}$$

 2. If $\frac{bf - dg}{bg + dg + bf + df} a - aa = \frac{bf - dg}{bg + dg + bf + df}$

What is a equal to?

Ans. a is equal to 1, and also to $\frac{bf - dg}{bg + dg + bf + df}$

Which values of a are also found out by the Canon in the tenth Section of this Chap. but I shall leave the Operation as an exercise for the industrious Learner, and in the next place shew the use of the Rules before delivered in this fifteenth Chap. in the Resolution of various Arithmetical Questions.

CHAP. XVI.

Various Arithmetical Questions, producing Equations that fall under some of the three Forms in Sect. 1. of the foregoing Chap. 15. and are resolvable by their respective Canons in Sect. 6, 8, and 10. of the same Chap.

QUEST. 1.

There are two numbers whose difference is 16 (or c) and the Product of their multiplication is 36 (or b); what are the numbers?

RESOLUTION.

- | | Numeral. | Literal. |
|---|--|---------------|
| 1. For the lesser of the two numbers sought put | a | a |
| 2. Then by adding to the said lesser number the given difference 16 (or c) the greater number sought will be | $a + 16$ | $a + c$ |
| 3. Therefore from the two last steps the Product made by the mutual multiplication of the two numbers sought will be | $aa + 16a$ | $aa + ca$ |
| 4. Which Product must be equal to the given Product 36 (or b) whence this Equation ariseth, viz. | $aa + 16a = 36$ | $aa + ca = b$ |
| Or, | $aa + 16a = 36$ | $aa + ca = b$ |
| 5. Which Equation being resolved by the Canon in Sect. 6. of Chap. 15. the value of a , or the lesser number sought by this Question will be discovered, viz. | $a = 2 = \sqrt{b + \frac{1}{4}c} - \frac{1}{2}c$ | |

6. To

6. To which lesser number adding the given difference 16 (or c) the greater number sought will also be made known, viz.

$$2 + 16 = 18 = \sqrt{b + \frac{1}{4}c} + \frac{1}{2}c$$

Otherwise thus;

- | | | |
|--|--|---------------|
| 1. For the greater of the two numbers sought put | a | a |
| 2. Then by subtracting from the said greater number the given difference 16; (or c) the lesser number sought will be | $a - 16$ | $a - c$ |
| 3. Therefore from the two last steps, the Product made by the mutual multiplication of the two numbers sought will be | $aa - 16a$ | $aa - ca$ |
| 4. Which Product must be equal to the given Product 36, (or b) whence this Equation ariseth, viz. | $aa - 16a = 36$ | $aa - ca = b$ |
| 5. Which Equation being resolved by the Canon in Sect. 8. of Chap. 15. the value of a , to wit, the greater number sought will be discovered, viz. | $a = 18 = \sqrt{b + \frac{1}{4}c} + \frac{1}{2}c$ | |
| 6. And by subtracting from the said greater number the given difference 16 (or c) the lesser number sought will also be discovered, viz. | $18 - 16 = 2 = \sqrt{b + \frac{1}{4}c} - \frac{1}{2}c$ | |

From either of those ways of Resolution, the numbers sought are found 18 and 2, which will solve the Question proposed; for their difference is 16; and the Product of their multiplication is 36, as was prescribed.

Moreover, the two last steps of each Resolution by Literal Algebra give one and the same Canon to solve the Question proposed.

CANON.

To the given Product add the Square of half the given difference; and extract the Square Root of that sum; then to the said Square Root adding half the given difference; and from the said Square Root subtracting the said half difference, the Sum and Remainder shall be the two numbers sought.

Therefore the difference and the Rectangle (or Product of the multiplication) of any two numbers being severally given, the numbers themselves shall also be given by the said Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continuing in the difference of the extremes, that is, of the first and third is 16 (or c), and the mean is 6 (or m); what are the extreme Proportionals?

RESOLUTION.

- | | | |
|---|---|----------------|
| 1. For the lesser of the two extreme Proportionals sought put | a | a |
| 2. Then by adding to the said lesser extreme the given difference of the extremes, to wit, 16 (or c), the greater extreme will be | $a + 16$ | $a + c$ |
| 3. Therefore the Rectangle contained under the extreme Proportionals, to wit, the Product made by their mutual multiplication shall be | $aa + 16a$ | $aa + ca$ |
| 4. Which Rectangle (or Product) must (by Sect. 10. Chap. 13.) be equal to the Square of the given mean Proportional 6 (or m), hence this Equation | $aa + 16a = 36$ | $aa + ca = mm$ |
| 5. Which Equation being resolved by the Canon in Sect. 6. Chap. 15. the value of a , or the lesser of the two extreme Proportionals sought will be made known, viz. | $a = 2 = \sqrt{mm + \frac{1}{4}c} - \frac{1}{2}c$ | |

N

6. To

5. The remainder is $\frac{43}{4}$ $\frac{1}{4}dd - t$.
6. The square root of that remainder is $\frac{1}{2}$ $\sqrt{\frac{1}{4}dd - t}$.
7. To which square root add half the Coefficient 9, to wit, $\frac{3}{2}$ $\frac{1}{2}d$.
8. The sum is the greater value of aaa , to wit, 8 $\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t}$.
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of aaa , to wit, 1 $\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t}$.
10. Therefore the Cubick root of the sum in the eighth step is the greater value of a , to wit, 2 $\sqrt{(3)\frac{1}{2}d + \sqrt{\frac{1}{4}dd - t}}$.
11. And the Cubick root of the remainder in the ninth step is the lesser value of a , to wit, 1 $\sqrt{(3)\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t}}$.

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer to Quest. 3. in Sect. 9. of this Chap.*

Example 4.

1. If b, d, f, g represent such known numbers that bf is greater than dg ; and,
2. If $\frac{bg + 2bf + df}{bg + dg + bf + df}a - aa = \frac{bf - dg}{bg + dg + bf + df}$;

What is a equal to?

Answer. a is equal to 1, and also to $\frac{bf - dg}{bg + dg + bf + df}$.

Which values of a are also found out by the Canon in the tenth *Section* of this *Chap.* but I shall leave the Operation as an exercise for the industrious Learner, and in the next place shew the use of the Rules before delivered in this fifteenth *Chap.* in the Resolution of various Arithmetical Questions.

CHAP. XVI.

Various Arithmetical Questions, producing Equations that fall under some of the three Forms in Sect. 1. of the foregoing Chap. 15. and are resolvable by their respective Canons in Sect. 6, 8, and 10. of the same Chap.

QUEST. 1.

There are two numbers whose difference is 16 (or c), and the Product of their multiplication is 36 (or b); what are the numbers?

RESOLUTION.

- | | Numeral. | Literal. |
|---|---|-----------|
| 1. For the lesser of the two numbers sought put | a | a |
| 2. Then by adding to the said lesser number the given difference 16 (or c), the greater number sought will be | $a + 16$ | $a + c$ |
| 3. Therefore from the two last steps the Product made by the mutual multiplication of the two numbers sought will be | $aa + 16a$ | $aa + ca$ |
| 4. Which Product must be equal to the given Product 36 (or b) whence this Equation ariseth, viz. | $aa + 16a = 36$,
Or, $aa + ca = b$. | |
| 5. Which Equation being resolved by the Canon in Sect. 6. of Chap. 15. the value of a , or the lesser number sought by this Question will be discovered, viz. | $a = 2 = \sqrt{b + \frac{1}{4}cc} - \frac{1}{2}c$. | |

6. To

6. To which lesser number adding the given difference 16 (or c) the greater number sought will also be made known, viz.

$$2 + 16 = 18 = \sqrt{b + \frac{1}{4}cc} + \frac{1}{2}c.$$

Otherwise thus;

- | | | |
|--|---|-----------|
| 1. For the greater of the two numbers sought put | a | a |
| 2. Then by subtracting from the said greater number the given difference 16, (or c) the lesser number sought will be | $a - 16$ | $a - c$ |
| 3. Therefore from the two last steps, the Product made by the mutual multiplication of the two numbers sought will be | $aa - 16a$ | $aa - ca$ |
| 4. Which Product must be equal to the given Product 36, (or b); whence this Equation ariseth, viz. | $aa - 16a = 36$,
Or, $aa - ca = b$. | |
| 5. Which Equation being resolved by the Canon in Sect. 8. of Chap. 15. the value of a , to wit, the greater number sought will be discovered, viz. | $a = 18 = \sqrt{b + \frac{1}{4}cc} + \frac{1}{2}c$. | |
| 6. And by subtracting from the said greater number the given difference 16 (or c), the lesser number sought will also be discovered, viz. | $18 - 16 = 2 = \sqrt{b + \frac{1}{4}cc} - \frac{1}{2}c$. | |

From either of those ways of Resolution, the numbers sought are found 18 and 2, which will solve the Question proposed; for their difference is 16, and the Product of their multiplication is 36, as was prescribed.

Moreover, the two last steps of each Resolution by *Literal Algebra* give one and the same Canon to solve the Question proposed:

CANON.

To the given Product add the Square of half the given difference, and extract the Square Root of that sum; then to the said square Root adding half the given difference, and from the said square Root subtracting the said half difference, the Sum and Remainder shall be the two numbers sought.

Therefore the difference and the Rectangle (or Product of the multiplication) of any two numbers being severally given, the numbers themselves shall also be given by the said Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continued, the difference of the extremes, that is, of the first and third is 16 (or c), and the mean is 6 (or m); what are the extreme Proportionals?

RESOLUTION.

- | | | |
|---|--|-----------|
| 1. For the lesser of the two extreme Proportionals sought put | a | a |
| 2. Then by adding to the said lesser extreme the given difference of the extremes, to wit, 16 (or c), the greater extreme will be | $a + 16$ | $a + c$ |
| 3. Therefore the Rectangle contained under the extreme Proportionals, to wit, the Product made by their mutual multiplication shall be | $aa + 16a$ | $aa + ca$ |
| 4. Which Rectangle (or Product) must (by Sect. 10. Chap. 15.) be equal to the Square of the given mean Proportional 6 (or m), hence this Equation | $aa + 16a = 36$,
Or, $aa + ca = mm$. | |
| 5. Which Equation being resolved by the Canon in Sect. 6. Chap. 15. the value of a , or the lesser of the two extreme Proportionals sought will be made known, viz. | $a = 2 = \sqrt{mm + \frac{1}{4}cc} - \frac{1}{2}c$. | |

N

6. To

6. To which lesser extreme Proportional adding 16 (or c) the given difference of the extremes, the greater of the two extreme Proportionals will also be discovered, *viz.*

$$2 + 16 = 18 = \sqrt{mm} + \frac{1}{2}cc : - \frac{1}{2}c.$$

I say the two extreme Proportionals sought are 2 and 18, between which the given number 6 is a mean Proportional; for, as 2 is to 6, so is 6 to 18.

Moreover, the two last steps of the Resolution give the following Canon to find out the extreme Proportionals sought.

CANON.

To the Square of the given mean Proportional add the Square of half the given difference of the extremes, and extract the square Root of that sum; then to the said square Root adding half the said difference, and from the said square Root subtracting the same half difference, the Summ and Remainder shall be the extreme Proportionals sought.

Therefore if of three numbers in continual proportion the mean be given, as also the difference of the extremes, the extremes shall be given severally by the said Canon.

QUEST. 3.

There are two numbers whose sum is 20 (or c), and the Product of their multiplication is 36 (or n); what are the numbers?

RESOLUTION.

- For one of the numbers sought put a
- Then by subtracting that number from the given sum 20 (or c) the Remainder will be the other number sought, to wit, $20 - a$
- Therefore the Product of the multiplication of those two numbers will be $20a - aa$
- Which Product must be equal to the given Product 36 (or n) whence this Equation ariseth, *viz.*

$$20a - aa = 36$$

Or,

$$ca - aa = n.$$
- Which Equation being resolved by the Canon in *Sett.* 10. *Chap.* 15. the two values of a , which are the numbers sought by this Question will be discovered, *viz.*

$$a = \frac{1}{2} \left(18 = \frac{1}{2}c + \sqrt{\frac{1}{4}cc - n} : \right. \\ \left. 2 = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n} : \right)$$

I say the numbers sought are 18 and 2, for their sum is 20, and the Product of their multiplication is 36, as was prescribed.

Moreover, if the two values of a , which are express'd by letters in the last step of the Resolution, be express'd by words, they will give the following Canon to solve the Question proposed.

CANON.

From the Square of half the given Summ subtract the given Product, and extract the square Root of the Remainder, then to the said half Summ adding the said square Root, and from the said half Summ subtracting the same square Root, the Summ and Remainder shall be the two numbers sought.

Therefore the Summ and Rectangle (or Product of the multiplication) of any two numbers being severally given, the numbers themselves shall also be given severally by the said Canon.

QUEST. 4.

There are three numbers in continual proportion; the sum of the extremes is 20, (or c), and the mean proportional is 6, (or m); what are the extremes?

RESOLUTION.

- For one of the two extreme proportionals sought put a
- Then

- Then by subtracting that extreme from 20 (or c) the given sum, the Remainder will be the other extreme, to wit, $20 - a$
- Therefore the Rectangle contained under the extreme proportionals, (to wit, the Product of their multiplication) shall be $20a - aa$
- Which Rectangle (or Product) must (according to *Sett.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m), whence this Equation ariseth, *viz.*

$$20a - aa = 36$$

$$\text{Or, } ca - aa = mm.$$

- Which Equation being resolved by the Canon in *Sett.* 10. *Chap.* 15. the two values of a , which are the numbers sought by this Question will be discovered, *viz.*

$$a = \frac{1}{2} \left(18 = \frac{1}{2}c + \sqrt{\frac{1}{4}cc - mm} : \right. \\ \left. 2 = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - mm} : \right)$$

I say the two extreme Proportionals sought are 18 and 2, between which the given number 6 is a mean Proportional; for, as 18 is to 6, so is 6 to 2.

Moreover, if the two values of a which are express'd by letters in the last step of the Resolution be express'd by words, they will give the following Canon to find out the extreme Proportionals sought.

CANON.

From the Square of half the given sum of the extreme Proportionals subtract the Square of the given mean, and extract the square Root of the Remainder, then to the said half sum adding the said square Root, and from the said half sum subtracting the same square Root, the Summ and Remainder shall be the two extreme Proportionals sought.

Therefore if of three numbers in continual proportion the mean be given, as also the sum of the extremes, the extremes themselves shall be given severally by the said Canon.

QUEST. 5.

There are two numbers whose difference is 15, (or d), and if the Product of the multiplication of the said two numbers be divided by 2, (or c), the Quotient will give the Cube of the lesser number; what are the numbers?

RESOLUTION.

- For the lesser number sought put a
- To which adding the given difference 15 (or d) the sum shall be the greater number, to wit, $a + 15$
- Therefore the Product of the multiplication of the two numbers is $aa + 15a$
- Which Product being divided by 2 (or c) the Quotient will be $\frac{aa + 15a}{2}$
- From the first step the Cube of the lesser number is aaa
- Which Cube must (as the Question requires) be equal to the Quotient in the fourth step, whence this Equation;

$$aaa = \frac{aa + 15a}{2}$$

$$\text{Or, } aaa = \frac{aa + da}{c}$$

- Which Equation being duly reduced (according to *Sett.* 2, 4, 3, 5 of *Chap.* 12.) there will arise

$$aa - \frac{1}{2}a = \frac{1}{2}d$$

$$\text{Or, } aa - \frac{1}{c}a = \frac{d}{c}$$

- Therefore the last Equation being resolved by the Canon in *Sett.* 8. *Chap.* 15. the value of a , to wit, the lesser number sought will be discovered, *viz.*

$$a = 3 = \sqrt{\frac{d}{c} + \frac{1}{4cc} - \frac{1}{2c}}$$

9. To

9. To which lesser number adding the given difference 15 (or d) the sum shall be the greater number sought, to wit,

$$3 + 15 = 18 = \sqrt{\frac{d}{c} + \frac{1}{4cc} : - \frac{1}{2c} + d}.$$

10. I say the two numbers sought are 3 and 18, which will satisfy the conditions in the Question, for their difference is 15, and if the Product of their multiplication 54 be divided by 2, the Quotient is 27, which is the Cube of the lesser number 3, as was required.
11. But if the Equation in the eighth step be express'd by words, it will give the following Canon to find out the lesser number sought, to which adding the given difference, the greater number is also given.

CANON.

Divide the given difference by the given Divisor, also divide 1 (or Unity) by the quadruple of the Square of the given Divisor, add those two Quotients together, and extract the square Root of the sum; then to this square Root add the Quotient that ariseth by dividing 1 by the double of the given Divisor, so shall the sum be the lesser of the two numbers sought, which increased with their given difference will give the greater number.

QUEST. 6.

There are two numbers whose difference is 2 (or d), and the sum of their Squares is 130 (or c); what are the numbers?

RESOLUTION.

- | | | |
|--|-----------------------|------------------|
| 1. For the lesser number sought put | a | a |
| 2. Then to that lesser number adding the given difference 2 (or d) the sum shall be the greater number, to wit, | $a + 2$ | $a + d$ |
| 3. Therefore from the first step the Square of the lesser number is | aa | aa |
| 4. And from the second step the Square of the greater number is | $aa + 4a + 4$ | $aa + 2da + dd$ |
| 5. Therefore from the two last steps the sum of the Squares of the two numbers sought is | $2aa + 4a + 4$ | $2aa + 2da + dd$ |
| 6. Which sum must be equal to the given sum of the Squares 130 (or c), whence this Equation ariseth, viz. | $2aa + 4a + 4 = 130,$ | |

$$\text{Or, } 2aa + 4a + 4 = c.$$

7. Which Equation, after due Reduction according to the Rules of the twelfth Chap. will give this Equation, viz.

$$\text{Or, } aa + 2a = \frac{1}{2}c - \frac{1}{2}dd.$$

8. Therefore the Equation in the last step being resolved according to the Canon in Sect. 6. Chap. 15. the value of a , to wit, the lesser number sought by the Question will be made known, viz.

$$a = 7 = \sqrt{\frac{1}{2}c - \frac{1}{2}dd} - \frac{1}{2}d.$$

9. To which lesser number adding the given difference 2 (or d) the sum shall be the greater number sought, to wit,

$$7 + 2 = 9 = \sqrt{\frac{1}{2}c - \frac{1}{2}dd} + \frac{1}{2}d.$$

10. I say the two numbers sought are 9 and 7, for their difference is 2, and the sum of their Squares is 130, as was prescribed by the Question.

11. Moreover, from the eighth and ninth step ariseth this

CANON.

From half the given sum subtract the Square of half the given difference, and extract the square Root of the Remainder, then from this square Root subtract half the given difference, the Remainder shall be the lesser number sought, to which adding the given difference the sum shall be the greater number.

QUEST.

QUEST. 7.

There are two numbers whose sum is 14 (or b), and the sum of their Squares is 100 (or c), what are the numbers?

RESOLUTION.

- | | | |
|--|---|------------------|
| 1. For one of the numbers sought put | a | a |
| 2. Which subtracted from the given sum 14 (or b) leaves the other number | $14 - a$ | $b - a$ |
| 3. The Square of the first number is | aa | aa |
| 4. The Square of the other number is | $aa - 28a + 196$ | $aa - 2ba + bb$ |
| 5. The sum of the said Squares is | $2aa - 28a + 196$ | $2aa - 2ba + bb$ |
| 6. Which sum must be equal to 100 (or c) the given sum of the Squares, whence this Equation ariseth, viz. | $2aa - 28a + 196 = 100,$ | |
| | $\text{Or, } 2aa - 28a + bb = c.$ | |
| 7. Which Equation, after due Reduction, according to the Rules of the twelfth Chap. will give this following Equation, | $14a - aa = 48,$ | |
| | $\text{Or, } ba - aa = \frac{1}{2}bb - \frac{1}{2}c.$ | |

8. Which Equation being resolved by the Canon in Sect. 10. Chap. 15. the two values of a , which are the numbers sought by this Question, will be discovered, viz.

$$a = \begin{cases} 8 = \frac{1}{2}b + \sqrt{\frac{1}{2}c - \frac{1}{2}bb} \\ 6 = \frac{1}{2}b - \sqrt{\frac{1}{2}c - \frac{1}{2}bb} \end{cases}$$

9. I say the numbers sought are 8 and 6; for their sum is 14, and the sum of their Squares is 100, as was prescribed.

10. Moreover, if the two values of a which are express'd by letters in the eighth step be express'd by words there will arise this

CANON.

From half the given sum of the Squares subtract the Square of half the given sum of the two numbers, and extract the square Root of the Remainder; then adding the said square Root to the said half sum of the numbers, the sum of this addition shall be the greater number; but subtracting the said square Root from the said half sum of the numbers, the Remainder shall be the lesser number.

QUEST. 8.

There are three numbers in Geometrical proportion continued; and such, that if the difference between the sum of the extremes and the mean be multiplied by the sum of the extremes, the Product will be 1120 (or b); but if the said difference be multiplied by the sum of all the three Proportionals, the Product will be 1456 (or c); what are the Proportionals?

RESOLUTION.

- | | | |
|--|----------------------------|--------------------|
| 1. For the difference of the sum of the extremes and mean put | a | a |
| 2. Then, according to the Question, the sum of the extremes is | $\frac{1120}{a}$ | $\frac{b}{a}$ |
| 3. From which sum if the difference in the first step be subtracted, the Remainder will be the mean proportional, to wit, | $\frac{1120}{a} - a$ | $\frac{b}{a} - a$ |
| 4. Therefore from the two last steps the sum of all three proportionals is | $\frac{2240}{a} - a$ | $\frac{2b}{a} - a$ |
| 5. But (according to the Question) if the sum of all the three proportionals be multiplied by the difference of the sum of the extremes and the mean, the Product must be equal to 1456 (or c); therefore from the first and fourth steps this following Equation ariseth, viz. | $2240 - aa = 1456,$ | |
| | $\text{Or, } 2b - aa = c.$ | |

6. Which

6. Which Equation being reduced according to the Rules of the twelfth *Chapt.* the value of a will be discovered, *viz.*

$$a = 28 = \sqrt{2b - c}.$$

7. Therefore from the sixth and second steps, the sum of the extremes is also known, *viz.*

$$40 = \frac{b}{\sqrt{2b - c}} = \text{the sum of the extremes.}$$

8. And from the sixth and third steps, the mean proportional is also given, *viz.*

$$12 = \frac{c - b}{\sqrt{2b - c}} = \text{the mean.}$$

9. Lastly, the sum of the extremes of three continual proportionals being given 40, as also the mean 12, the extremes shall also be given severally by the Canon of the fourth *Question* of this *Chapt.* to wit, 4 and 36; therefore the three continual proportionals sought are 4, 12 and 36, which will satisfy the conditions in the *Question* proposed, as will appear by

The Proof.

I. 4, 12, 36 are $\div \div$ for, $4 \times 36 = 12 \times 12$.

II. $4 + 36 - 12$ into $36 \div 4 = 1120$.

III. $4 + 36 - 12$ into $4 + 12 + 36 = 1456$.

QUEST. 9.

There are two numbers whose sum is 10 (or b) and the sum of their Cubes is 520 (or c), what are the numbers?

Q

RESOLUTION.

1. For one of the numbers sought put . . . a
2. Then by subtracting that number from the given sum 10 (or b) the other number remains, to wit, . . . $10 - a$
3. The Cube of the former is . . . aaa

4. And from the second step the Cube of the latter number is
- $$1000 - 300a + 30aa - aaa,$$
- Or,
- $$bbb - 3bba + 3baa - aaa.$$

4. Therefore the sum of the two Cubes in the third and fourth steps is

5. Which sum must be equal to 520 (or c) the given sum of the Cubes, whence this Equation ariseth, *viz.*
- $$1000 - 300a + 30aa = 520;$$
- Or,
- $$bbb - 3bba + 3baa = c.$$

6. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will give this Equation,

$$\text{Or, } \frac{bbb - c}{3b} = ba - aa.$$

7. Therefore the last Equation being resolved by the Canon in *Sett. 10. Chap. 15.* the two values of a , which are the numbers sought by this *Question*, will be discovered, *viz.*

$$a = \begin{cases} \frac{1}{2}b + \sqrt{\frac{c}{3b} - \frac{bb}{12}} = 8. \\ \frac{1}{2}b - \sqrt{\frac{c}{3b} - \frac{bb}{12}} = 2. \end{cases}$$

8. I say the two numbers sought are 8 and 2, for their sum is 10, and the sum of their Cubes is 520, as was prescribed.

9. Moreover, if the two values of a which are exprest by letters in the seventh step be exprest by words, they will give this

CANON.

From the Quotient that ariseth by dividing the given sum of the two Cubes, by the triple of the given sum of their sides, subtract $\frac{1}{12}$ of the Square of the last mentioned sum,

sum, and extract the square Root of the Remainder; then adding the said square Root to half the said sum of the sides of the two Cubes, and also subtracting the said square Root from the said half sum, the Summ and Remainder shall be the sides or numbers sought.

QUEST. 10.

There are two numbers whose sum is 10 (or b) and the proportion which their difference beareth to the sum of their Squares is as 2 to 29, (or as r to s), what are the numbers?

RESOLUTION.

1. For the greater number sought put . . . a
2. Which subtracted from the given sum 10 (or b) leaves the lesser number . . . $10 - a$
3. Therefore the difference of the two numbers is . . . $2a - 10$
4. And from the first step the Square of the greater number is . . . aa
5. And from the second step the Square of the lesser number is

$$100 - 20a + aa,$$

$$\text{Or, } bb - 2ba + aa.$$

6. And from the two last steps the sum of the Squares of the two numbers sought is

$$100 - 20a + 2aa,$$

$$\text{Or, } bb - 2ba + 2aa.$$

7. Then according to the *Question*, the difference in the third step must be to the sum of the Squares in the sixth step as 2 to 29, (or as r to s), *viz.*

$$2 : 29 :: 2a - 10 : 100 - 20a + 2aa,$$

$$\text{Or, } r : s :: 2a - b : bb - 2ba + 2aa.$$

8. Which Analogy may be converted into this following Equation, (according to the Theorem in *Chap. 1. Sett. 13.*) *viz.*

$$200 - 40a + 4aa = 58a - 290;$$

$$\text{Or, } rbb - 2rba + 2raa = 2sa - sb.$$

9. Which Equation, after due Reduction according to the Rules in the 12. *Chapt.* will produce this Equation,

$$\text{Or, } \frac{rbb - sb}{2r} = \frac{s + rb}{r} a - aa.$$

10. Therefore by resolving the Equation in the last step according to *Sett. 10. Chap. 15.* the two values of a , or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \frac{1}{2} \left(\frac{s}{2r} + \frac{b}{2} + \sqrt{\frac{ss}{4rr} - \frac{bb}{4}} \right) \\ \frac{1}{2} \left(\frac{s}{2r} + \frac{b}{2} - \sqrt{\frac{ss}{4rr} - \frac{bb}{4}} \right) \end{cases}$$

11. The lesser of which two Roots or numbers, to wit 7, is the greater number sought by this *Question*; and consequently, the said 7 being subtracted from the given sum 10, the Remainder 3 is the lesser number sought.

I say 7 and 3 will solve the *Question*, for their sum is 10, and their difference 4 is to the sum of their Squares 58, as 2 to 29, which was prescribed.

12. *Note.* Although the value of a in the Equation in the tenth step may be either $\frac{1}{2}$ or 7, (for that Equation may be expounded by $\frac{1}{2}$ as well as 7, yet 7 only, to wit, the lesser value of a , shall be the greater number sought by this *Question*.

For that the greater value of a , to wit, $\frac{1}{2} \left(\frac{s}{2r} + \frac{b}{2} + \sqrt{\frac{ss}{4rr} - \frac{bb}{4}} \right)$, can never be equal to either of the two numbers sought, I prove thus; First, it is manifest by each of the values of a exprest by letters in the tenth step, That if $\frac{s}{2r} = \frac{b}{2}$, then

consequently $\frac{ss}{4rr} = \frac{bb}{4}$, and the two values of a are equal one to the other, each being

being equal to $\frac{s}{2r} + \frac{b}{2}$, that is, b ; and therefore in this first case, neither of the two values of a can possibly be equal to either of the two numbers sought; for that which is equal to the sum of two numbers must needs be greater than either of them.

Secondly, If $\frac{s}{2r} < \frac{b}{2}$, which is a necessary Determination to make the Question possible, then the greater value of a , that is, $\frac{s}{2r} + \frac{b}{2} + \sqrt{\frac{ss}{4rr} - \frac{bb}{4}}$ is manifestly greater than b the given sum of the two numbers sought, and therefore it cannot be equal to either of them. Wherefore the said greater value of a cannot in any case be equal to either of the two numbers sought. Which was to be proved.

But the said lesser value of a is the greater of the two numbers sought, and consequently they are given severally by this following

CANON.

13. From the Quotient that aritheth by dividing the Square of the latter term of the given Reason by the quadruple of the Square of the first term, subtract a quarter of the Square of the given sum of the two numbers sought, and extract the Square Root of the Remainder; then subtract that Square Root from the sum of the Quotient that aritheth by dividing the latter term of the given Reason by the double of the first, and the half of the given sum of the two numbers, so the Remainder shall be the greater number sought; which subtracted from the said given sum leaves the lesser number.
14. From the premises this following Question may easily be solved, viz. The sum of two numbers being given, suppose $\frac{s}{2}$ (or b), and their difference being equal to the sum of their Squares, to find the numbers.

First, suppose $r = s = 1$, (because the Terms of the Proportion in this Question are equal to one another,) then the two values of a before express in the tenth step will be converted into these, viz.

$$a = \frac{6}{5} = \frac{1+b}{2} + \sqrt{\frac{1-bb}{4}}$$

$$a = \frac{3}{5} = \frac{1+b}{2} - \sqrt{\frac{1-bb}{4}}$$

The lesser of which values of a , to wit, $\frac{3}{5}$, is the greater of the two numbers sought, and therefore the said $\frac{3}{5}$ being subtracted from $\frac{1}{2}$ the given sum, leaves $\frac{2}{5}$ for the lesser number. I say $\frac{3}{5}$ and $\frac{2}{5}$ will solve the Question, for their difference $\frac{1}{5}$ is equal to the sum of their Squares.

QUEST. 11.

There are two numbers, the Product of whose Multiplication is 48 (or p), and the difference of their Squares is 28 (or d), what are the numbers?

RESOLUTION.

1. For the greater number put a
2. Then dividing 48 (or p) by a , the Quotient $\frac{48}{a}$ is the lesser number, to wit, $\frac{p}{a}$
3. From the first step the Square of the greater number is aa
4. And from the second step the Square of the lesser number is $\frac{pp}{aa}$
5. Therefore the difference of the said Squares is $aaaa - 2304$
6. Which difference must be equal to the given difference of the Squares, whence this Equation aritheth, viz.

$$\frac{aaaa - 2304}{aa} = 28,$$

$$\text{Or, } \frac{aaaa - pp}{aa} = d.$$

7. Which

7. Which Equation, after due Reduction according to the Rules of the twelfth Chap. will produce this;

$$aaaa - 28aa = 2304,$$

$$\text{Or, } aaaa - daa = pp.$$

8. Therefore by resolving the last Equation according to the Canon in Sect. 8. Chap. 15, the value of a , to wit, the greater number sought will be discovered, viz.

$$a = 8 = \sqrt{(2)} : \sqrt{pp} + \frac{1}{2}d :$$

Whence the greater number is found 8, by which if the given Product 48 be divided, the Quotient 6 is the lesser number sought.

I say, the numbers 8 and 6 will solve the Question; for the Product of their multiplication is 48, and the difference of their Squares 64 and 36 is 28, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find the greater of the two numbers sought, by the help whereof and the given Product the lesser number shall be also given.

CANON.

9. To the Square of the given Product add the Square of half the given difference of the Squares, and extract the Square Root of that sum; then to the said Square Root add the said half difference, and extract the Square Root of this sum, so shall the last Square Root be the greater of the two numbers sought; lastly, by the said greater number divide the given Product of the multiplication of both numbers, and the Quotient shall be the lesser number.

QUEST. 12.

There are two numbers the Product of whose multiplication is 48 (or p), and the sum of their Squares is 100 (or c), what are the numbers?

RESOLUTION.

1. For one of the numbers sought put a
2. Then dividing 48 (or p) by a , the Quotient $\frac{48}{a}$ will give the other number, to wit, $\frac{p}{a}$
3. From the first step, the Square of one of the numbers is aa
4. And from the second step the Square of the other number is $\frac{pp}{aa}$
5. Therefore the sum of the said Squares is $aaaa + \frac{pp}{aa}$
6. Which sum must be equal to the given sum of the Squares, whence this Equation aritheth, viz.

$$\frac{aaaa + 2304}{aa} = 100,$$

$$\text{Or, } \frac{aaaa + pp}{aa} = c.$$

7. From which Equation, after due Reduction by the Rules in Chap. 12, this will arise,
- $$2304 = 100aa - aaaa,$$
- $$pp = caa - aaaa.$$
8. Which last Equation being resolved by the Canon in Sect. 10. Chap. 15, the two values of a , which are the numbers sought, will be discovered, viz.

$$a = \sqrt{(2)} : \sqrt{\frac{1}{2}c} + \sqrt{\frac{1}{2}cc - pp} :$$

$$a = \sqrt{(2)} : \sqrt{\frac{1}{2}c} - \sqrt{\frac{1}{2}cc - pp} :$$

9. I say, 8 and 6 are the numbers required; for the Product of their multiplication is 48, and the sum of their Squares 64 and 36 is 100, as was prescribed. From the last step also aritheth this

CANON.

From the Square of half the given sum of the Squares of the two numbers sought, subtract the Square of the given Product of their multiplication, and extract the Square Root of the Remainder; then to half the said sum add the said Square Root, and from the

the said half sum subtract the said square Root; lastly, extract the square Root of the sum of that Addition, and also of the Remainder of the latter Subtraction, so shall these two square roots be the numbers sought by the Question propos'd.

QUEST. 13.

There are two numbers whose sum is 14 (or b), and if the sum of their Squares be multiplied by the sum of their Cubes, the Product is 72800 (or c); what are the numbers?

RESOLUTION.

- For one of the numbers sought put $a + 7$
- Then, that their sum may be 14 (or b), the other number must be $a + 7$
- The Square of the first number is $aa + 14a + 49$
- The Square of the latter number is $aa - 14a + 49$
- Therefore the sum of their Squares is $2aa + 98$
- Again, the Cube of the first number will be $aaa + 21aa + 147a + 343$,
Or, $aaa + 21aa + 147a + 343$.
- And the Cube of the latter number will be $aaa - 21aa + 147a + 343$,
Or, $aaa - 21aa + 147a + 343$.
- Therefore the sum of the Cubes in the two last steps is $42aa + 686$,
Or, $42aa + 686$.
- Which sum of the Cubes in the last step being multiplied by the sum of the Squares in the fifth step, produceth $84aaaa + 5488aa + 67228$,
Or, $84aaaa + 5488aa + 67228$.
- Which Product in the last step must be equal to 72800 (or c) the Product given in the Question, whence this Equation arith, viz.
 $84aaaa + 5488aa + 67228 = 72800$,
Or, $84aaaa + 5488aa + 67228 = c$.
- And from that Equation, after due Reduction according to the Rules of the twelfth Chapter, this will arise, $aaa + \frac{128}{3}aa = \frac{128}{3}$,
Or, $aaa + \frac{128}{3}aa = \frac{c}{6}$.
- Which Equation being solved by the Canon in Sect. 6. of Chap. 15. the value of a will be discovered, viz.
 $a = 1 = \sqrt{(2)} : \sqrt{\frac{c}{6b} + \frac{1}{14}bbb} - \frac{1}{2}bb$.

- Therefore from the twelfth, first and second steps the two numbers sought are made known:

$$7 + 1 = 8 = \frac{1}{2}b + \sqrt{(2)} : \sqrt{\frac{c}{6b} + \frac{1}{14}bbb} - \frac{1}{2}bb$$

$$7 - 1 = 6 = \frac{1}{2}b - \sqrt{(2)} : \sqrt{\frac{c}{6b} + \frac{1}{14}bbb} - \frac{1}{2}bb$$

I say the numbers sought are 8 and 6; for their sum is 14, and if 100 the sum of their Squares be multiplied by 728, the sum of their Cubes, the Product will be 72800, as was prescribed.

Moreover, the thirteenth step gives a Canon to find out the numbers sought.

CANON.

Divide the given Product by six times the given Sum; then to the Quotient add $\frac{1}{14}$ of the Biquadrate of the given sum, and extract the square Root of the sum of that addition; then from the said square Root subtract $\frac{1}{2}$ of the Square of the given sum, and extract the square Root of the Remainder; lastly, add this square Root to half the given sum and subtract it from the said half sum, so shall the Sum and Remainder be the two numbers sought.

QUEST.

QUEST. 14.

There are two numbers the Product of whose multiplication is 20 (or b), and the sum of their Cubes is 189 (or c); what are the numbers?

RESOLUTION.

- For one of the numbers sought put a
- Then, by dividing the given Product 20 (or b) by a , the other number will be $\frac{b}{a}$
- Therefore from the first step, the Cube of the first number is aaa
- And from the second step the Cube of the other number is $\frac{bbb}{aaa}$
- Therefore the sum of the said Cubes is $aaaaa + 8000$
- Which sum must be equal to 189 (or c) the sum given in the Question, whence this Equation arith, viz.
 $aaaaa + 8000 = 189$,
Or, $aaaaa + bbb = 189$.
- Which Equation being reduced according to Sect. 2, 3, and 5. of Chap. 12. there will arise
 $8000 = 189aaa - aaaaa$,
Or, $bbb = caa - aaaaa$.
- And by resolving the Equation in the last step by the Canon in Sect. 10. Chap. 15. the two values of a , which are the numbers sought by this Question, will be made known, viz.
 $a = \frac{5}{2} = \sqrt{(3)} : \frac{1}{2}c - \sqrt{\frac{1}{2}cc - bbb}$,
 $a = \frac{4}{2} = \sqrt{(3)} : \frac{1}{2}c - \sqrt{\frac{1}{2}cc - bbb}$.
- I say, the numbers sought are 5 and 4; for the Product of their multiplication is 20, and the sum of their Cubes 125 and 64 is 189, as was prescribed.

Moreover, from the two values of a express'd by letters in the eighth step, the following Canon arith to find out the numbers sought.

CANON.

From the Square of half the given sum subtract the Cube of the given Product, and extract the square Root of the Remainder; then add the said square Root to half the given sum, and also subtract it from the said half sum; lastly, extract the cubick Root of the sum of that addition, and likewise extract the cubick Root of the latter Remainder, so shall these Cubick Roots be the numbers sought.

QUEST. 15.

There are two numbers the Product of whose multiplication is 20 (or b), and the difference of their Cubes is 61 (or d); what are the numbers?

RESOLUTION.

- For the greater of the two numbers sought put a
- Then, by dividing the given Product 20 (or b) by a , the lesser number will be $\frac{b}{a}$
- Therefore from the first step the Cube of the greater number is aaa
- And from the second step the Cube of the lesser number is $\frac{bbb}{aaa}$
- Therefore from the two last steps, the difference of the Cubes of the two numbers sought is $aaaaa - 8000$
- Which difference must be equal to 61 (or d) the difference given in the Question, whence this Equation arises, viz.
 $aaaaa - 8000 = 61$,
Or, $aaaaa - bbb = d$.

O 2

7. Which

7. Which Equation, after due Reduction, (according to *Sett.* 2, 3, and 5, of *Chap.* 12.) will give this that follows, *viz.* $aaaaa - 61aaa = 8000$,
 Or, $aaaaa - daaa = bbb$.
8. Therefore by resolving the Equation in the last step by the Canon in *Sett.* 8, *Chap.* 15, the value of a , to wit, the greater number sought will be made known, *viz.*
 $a = 5 = \sqrt{(3) \cdot \frac{1}{2}d + \sqrt{\frac{1}{4}dd + bbb}}$.
9. Whence the greater number sought is found 5, by which if the given Product 20 be divided, the Quotient will give 4 for the lesser number required.

I say, the numbers 5 and 4 will solve the Question proposed, for the Product of their multiplication is 20, and the difference of their Cubes 125 and 64 is 61, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find out the greater of the two numbers sought, by the help whereof and the given Product the lesser number is also given.

CANON.

To the Square of half the given difference add the Cube of the given Product, and extract the Square Root of the sum of that addition; then add the said Square Root to half the given difference and extract the cubick Root of this sum, so shall the said cubick Root be the greater of the two numbers sought, by which greater number if the given Product be divided the Quotient shall be the lesser number sought.

QUEST. 16.

A Merchant having bought certain Clothes, sells them at $17\frac{1}{2}l.$ (or b) the Cloth; and then found that by every 100 $l.$ (or c) that he had laid out, he gained as many pounds as he paid for one Cloth; what was the first cost of a Cloth?

RESOLUTION.

1. For the first cost of one Cloth put a .
2. Which first cost being subtracted from the money for which the Merchant sold one Cloth, there will remain the gain of one Cloth, to wit, $17\frac{1}{2} - a$ $b - a$.
3. Then find what was gained in laying out 100 $l.$ (or c), *viz.* say by the Rule of Three;

$$\text{If } a \cdot 17\frac{1}{2} - a :: 100 \cdot \frac{17\frac{1}{2} - 100a}{a}$$

$$\text{Or, } a \cdot b - a :: c \cdot \frac{cb - ca}{a}$$

$$\text{Whence the gain of 100 } l. \text{ is found } \frac{17\frac{1}{2} - 100a}{a}, \text{ or } \frac{cb - ca}{a}.$$

4. But according to the Question the gain of 100 $l.$ (or c) must be equal to the first cost of one Cloth, therefore from the first and third steps this Equation ariseth, *viz.*
 $a = \frac{17\frac{1}{2} - 100a}{a}$, Or, $a = \frac{cb - ca}{a}$.
5. Which Equation, after due Reduction (according to *Sett.* 2, and 3, of *Chap.* 12.) will give this that follows, *viz.* $aa + 100a = 17\frac{1}{2}a$,
 Or, $aa + ca = cb$.
6. Therefore by resolving the Equation in the last step by the Canon in *Sett.* 6, *Chap.* 15, the value of a , to wit, the first cost of a Cloth will be discovered, *viz.*
 $a = 15 = \sqrt{cb + \frac{1}{2}cc} - \frac{1}{2}c$

I say the first cost of a Cloth was 15 $l.$ as will appear by the Proof: For if a Cloth be bought for 15 $l.$ and sold for $17\frac{1}{2}l.$ the gain is $2\frac{1}{2}l.$ Then if 15 $l.$ gain $2\frac{1}{2}l.$ it will follow that 100 $l.$ will gain 15 $l.$ which is equal to the first cost of a Cloth, as was prescribed.

Answer

Another way of resolving the preceding Quest. 16.

1. Let the same things be given as before, then for the gain of one Cloth put a .
2. Which gain, being subtracted from the money for which one Cloth was sold, will leave the first cost of a Cloth, to wit, $17\frac{1}{2} - a$ $b - a$.
3. Then find what was gained in laying out 100 $l.$ (or c), and say by the Rule of Three;

$$\text{If } 17\frac{1}{2} - a \cdot a :: 100 \cdot \frac{100a}{17\frac{1}{2} - a}$$

$$\text{Or, } b - a \cdot a :: c \cdot \frac{ca}{b - a}$$

$$\text{Whence the gain of 100 } l. \text{ is found } \frac{100a}{17\frac{1}{2} - a}, \text{ or } \frac{ca}{b - a}.$$

4. But, according to the Question, the gain of 100 $l.$ (or c) must be equal to the first cost of one Cloth, therefore from the second and third steps this Equation ariseth, *viz.*
 $\frac{100a}{17\frac{1}{2} - a} = 17\frac{1}{2} - a$, Or, $\frac{ca}{b - a} = b - a$.

5. Which Equation, after due Reduction according to *Sett.* 2, and 3, of *Chap.* 12, will give this that follows, *viz.* $aa + ca = cb$,
 Or, $aa + 100a = 17\frac{1}{2}a$.

6. Therefore by resolving the Equation in the last step by the Canon in *Sett.* 6, *Chap.* 15, the two values of a , or the two Roots of that Equation will be made known, *viz.*
 $a = \frac{1}{2}c + \frac{1}{2}b \pm \sqrt{\frac{1}{4}cc + cb}$

The lesser of which two Roots or numbers, to wit $\frac{1}{2}c + b - \sqrt{\frac{1}{4}cc + cb}$, is the gain of a Cloth, which subtracted from $17\frac{1}{2}l.$ leaves 15 $l.$ for the first cost of a Cloth, as before.

Note. Although the value of a in the Equation in the fifth step may be either $\frac{1}{2}c + b$ or $\frac{1}{2}c + b - \sqrt{\frac{1}{4}cc + cb}$, yet $\frac{1}{2}c$ only, to wit, the lesser value of a shall be the gain of a Cloth, for $\frac{1}{2}c$ is greater than $17\frac{1}{2}$, and consequently the gain of one Cloth would exceed the money for which one Cloth was sold. Which absurdity appears also by the greater value of a as is express'd by Letters in the sixth step, for $\frac{1}{2}c + b + \sqrt{\frac{1}{4}cc + cb}$ is manifestly greater than b .

QUEST. 17.

Each of two Captains, whereof one had a lesser number of Souldiers in his Company by 40 (or b) than the other, distributed equally among the Souldiers of his own Company 1200 (or c) Crowns, whereby it happened that the Souldiers of the lesser Company had 5 (or d) Crowns a piece more than the Souldiers of the greater Company; the Question is to find the number of Souldiers in each Company, and how many Crowns each Souldier received.

RESOLUTION.

1. For the number of Souldiers in the lesser Company put a .
2. To which adding 40 (or b) the sum will give the number of Souldiers in the greater Company, to wit, $a + 40$ $a + b$.
3. Then if 1200 (or c) Crowns be equally divided among the Souldiers of the lesser Company, the Quotient or share of every Souldier will be $\frac{1200}{a}$ $\frac{c}{a}$.
4. Likewise, if 1200 (or c) Crowns be equally divided among the Souldiers of the greater Company, the Quotient or share of every Souldier will be $\frac{1200}{a + 40}$ $\frac{c}{a + b}$.
5. To which latter Quotient adding 5 (or d) Crowns, the sum is $\frac{5a + 1200}{a + 40}$ $\frac{da + db + c}{a + b}$.

6. But

6. But according to the Question the summ in the last step must be equal to the Quotient in the third step, whence this Equation ariseth, *viz.*

$$\frac{5a+1400}{a+40} = \frac{1200}{a}, \quad \text{Or,} \quad \frac{da+db+c}{a+b} = \frac{c}{a}.$$

7. From which Equation after due Reduction according to *Self. 2, 3, and 5. of Chap. 12.* this will arise, *viz.*

$$aa+40a = 9600;$$

$$\text{Or,} \quad aa+ba = \frac{bc}{d}.$$

8. Therefore the Equation in the last step being resolved by the Canon in *Self. 6. Chap. 15.* the value of *a*, to wit, the number of Souldiers in the lesser Company will be discovered, *viz.*

$$a = 80 = \sqrt{\frac{bc}{d} + \frac{bb}{4}} = -\frac{1}{2}b.$$

From the eighth, first, and second steps it is evident that the lesser Company consisted of 80, and the greater 120 Souldiers, which numbers will satisfy the Conditions in the Question. For the difference of the two Companies is 40 Souldiers, also $\frac{1200}{80} = 15$, and $\frac{1400}{80} = 17.5$; whence it is manifest that the Souldiers of the lesser Company received 15 Crowns a piece, the Souldiers of the greater Company 10 Crowns a piece, and consequently the Souldiers of the lesser Company had 5 Crowns a piece more than the Souldiers of the greater Company, as was prescribed.

QUEST. 18.

Two Merchants sell linnen Cloth in this manner, *viz.* each sells 60 (or *b*) Ells, and the first Merchant selling 2 (or *c*) Ells less for one pound than the second, receives for his 60 Ells 5 (or *d*) pounds more than the second Merchant for his 60 Ells. The Question is to find how many Ells each Merchant sold for 1 pound?

RESOLUTION.

1. For the number of Ells which the first Merchant sold for 1 *l.* put
2. To which number of Ells adding 2 (or *c*), the summ will be the number of Ells which the latter Merchant sold for 1 *l.* to wit,
3. Then find how much money the first Merchant received for his 60 Ells, *viz.* say by the Rule of Three,

$$\text{If } a : 1 :: 60 : \frac{60}{a};$$

$$\text{Or, } a : 1 :: b : \frac{b}{a}.$$

whence the first Merchants total money is found

4. Find likewise how much money the latter Merchant received for his 60 Ells, *viz.* say,

$$\text{If } a+2 : 1 :: 60 : \frac{60}{a+2};$$

$$\text{Or, } a+c : 1 :: b : \frac{b}{a+c}.$$

whence the latter Merchants total money is found

5. To which latter summ of money adding 2 (or *c*) pounds, the summ will be

6. But according to the Question the summ of money in the last step must be equal to the summ in the third step, whence this Equation ariseth, *viz.*

$$\frac{5a+70}{a+2} = \frac{60}{a}, \quad \text{Or,} \quad \frac{da+dc+b}{a+c} = \frac{b}{a}.$$

7. Which

7. Which Equation, after due Reduction according to *Self. 2, 3, and 5. of Chap. 12.* will give this that follows, *viz.*

$$aa+2a = \frac{24}{a},$$

$$\text{Or,} \quad aa+ca = \frac{bc}{d}.$$

8. Which Equation in the last step being resolved by the Canon in *Self. 6. Chap. 15.* the value of *a*, to wit the number of Ells which the first Merchant sold will be made known,

$$\text{viz.} \quad a = 4 = \sqrt{\frac{bc}{d} + \frac{cc}{4}} = -\frac{1}{2}c.$$

I say the first Merchant sold 4 Ells for 1 pound, and the second 6 Ells for 1 pound, as will appear by the Proof. For if 4 Ells give 1 pound, then 60 Ells will give 15 pounds. Again, if 6 Ells give one pound, then 60 Ells will give 10 pounds. Whence it is manifest that the first Merchant sold his 60 Ells for 5 pounds more than the second sold his 60 Ells, and sold two Ells less for 1 pound than the second Merchant sold for one pound.

QUEST. 19.

Two Societies, whereof one exceeds the other by 4 (or *b*) men, divide two equal summs of Crowns; the men of the lesser Society have 8 (or *c*) Crowns a piece more than those of the greater: and the number of Crowns which each Society receives exceeds, the number of men of both Societies by 172 (or *d*). The Question is, to find the number of Men in each Society, and the number of Crowns which each Society had?

RESOLUTION.

1. For the number of men of the lesser Society put
2. To which number adding 4 (or *b*), the summ will be the number of men of the greater Society, to wit,
3. Then, according to the Question, if 172 (or *d*) be added to the summ of the men of both Societies, it will give the number of Crowns shared by each Society, to wit,
4. Which number of Crowns being divided by (*a*) the number of men of the lesser Society, the Quotient or share of every man in that Society will be
5. Likewise if the same number of Crowns before express in the third step be divided by $a+4$, (or $a+b$, the number of men of the greater Society,) the Quotient will give the share of every man in this Society, to wit,
6. To which Quotient in the last step adding 8 (or *c*) the summ will be
7. But, according to the Question, the summ in the last step must be equal to the Quotient in the fourth step, whence this Equation ariseth, *viz.*
8. From which Equation, after due Reduction according to *Self. 2, 3, and 5. of Chap. 12.* this Equation will arise, *viz.*
9. Therefore by resolving the last Equation according to the Canon in *Self. 6. Chap. 15.* the value of *a*, to wit, the number of men in the lesser Society will be discovered, *viz.*
10. Lastly, from the ninth, first, second, and third steps, it is manifest that the number of men in the lesser Society was 8, that of the greater 12, and the number of Crowns divided by each Society 192; which numbers will satisfy the conditions in the Question,

Question, as will appear by the Proof: For $\frac{122}{3} = 24$, and $\frac{122}{3} = 16$; whence it is evident that the men of the lesser Society had 8 Crowns a piece more than those of the greater; also 192, the number of Crowns which each Society divided, exceeded 20 the number of men in both Societies by 172, and 1: the number of men in the greater Society exceeded 8 the number of men in the lesser by 4; as was prescribed.

QUEST. 20.

A Grafter having bought certain Oxen for 270 (or b) pounds, finds, that if he had paid that sum for 5 (or c) Oxen fewer, every Ox would have cost him $\frac{1}{2}l$ (or d) more than he paid for an Ox: What was the number of Oxen bought?

RESOLUTION.

- For the number of Oxen bought put a
- Then find out the cost of an Ox, and say,
If $a \cdot 270 :: 1 \cdot \frac{270}{a}$;
Or, $a \cdot b :: 1 \cdot \frac{b}{a}$;
whence the price of an Ox is $\frac{b}{a}$
- Subtract 5 (or c) from the number of Oxen bought, and then find what the rest would cost a piece, saying,
If $a - 5 \cdot 270 :: 1 \cdot \frac{270}{a - 5}$;
Or, $a - c \cdot b :: 1 \cdot \frac{b}{a - c}$;
Whence the price of an Ox is found $\frac{b}{a - c}$
- Then according to the Question, the last mentioned price of an Ox must exceed that in the second step by $\frac{1}{2}l$ (or d); therefore if the former price be subtracted from the latter, the Remainder must be equal to $\frac{1}{2}l$ (or d); whence this Equation arith, viz.
 $\frac{270}{a - 5} - \frac{270}{a} = \frac{1}{2}$ Or, $\frac{b}{a - c} - \frac{b}{a} = d$.
- Which Equation, after due Reduction according to the Rules in Chap. 12. will give this that follows,
Or, $aa - ca = \frac{bc}{d}$.
- Therefore the Equation in the last step being resolved by the Canon in Sect. 12. Chap. 15. the value of a , to wit the number of Oxen bought will be discovered, viz.
 $a = 45 = \sqrt{\frac{bc}{d} + \frac{ca}{4}} + \frac{1}{2}c$.

I say the number of Oxen bought was 45, and every Ox cost 6 pounds, as will appear by the Proof: For first, $\frac{122}{3} = 6$; then from 45 Oxen subtracting 5, the remaining 40 Oxen valued at 270 l . will yield $6\frac{1}{2}l$ a piece, which exceeds the former price $6l$ by $\frac{1}{2}l$. as was prescribed.

QUEST. 21.

A Merchant buyes linnen Clothes of two sorts, viz. 90 (or b) Ells of one sort, together with 40 (or c) Ells of a worser sort for 42 (or d) pounds; and he finds that in laying out 1 pound upon each sort he hath $\frac{1}{2}$ (or m) of an Ell more of the worser sort than the other: What was the price of an Ell of each sort?

RESOLUTION.

- For the number of Ells of the better sort of Cloth which the Merchant bought for 1 l . put a
- Then according to the Quest. the number of Ells of the worser sort bought for 1 l . will be $a + m$

3. Then

- Find the cost of all the Ells of the worser sort, and say,

$$\text{If } a - \frac{1}{2} \cdot 1 :: 40 \cdot \frac{40}{a - \frac{1}{2}};$$

$$\text{Or, } a - m \cdot 1 :: c \cdot \frac{c}{a - m}.$$

whence the said full Cost is found

- Find likewise the cost of all the Ells of the better sort, and say,

$$\text{If } a \cdot 1 :: 90 \cdot \frac{90}{a};$$

$$\text{Or, } a \cdot 1 :: b \cdot \frac{b}{a}.$$

whence the said full Cost is

- Then the two sums of money found out in the third and fourth steps being added together will give the full cost of both sorts of Cloth, to wit,

$$\frac{130a + 30}{aa + \frac{1}{2}a}$$

$$\frac{ca + ba + bm}{aa + ma}$$

- Which total Cost express'd in the last step, must (according to the Question) be equal to 42 (or d); whence this Equation arith, viz.

$$42 = \frac{130a + 30}{aa + \frac{1}{2}a}$$

$$\text{Or, } d = \frac{ca + ba + bm}{aa + ma}$$

- Which Equation, after due reduction (according to the Rules in Chap. 12.) will give this that follows, viz.

$$\text{Or, } aa - \frac{1}{2}a = \frac{7}{2};$$

$$\text{Or, } aa - \frac{c + b - dm}{d}a = \frac{mb}{d}.$$

In which last Equation, if instead of the known Coefficient $\frac{c + b - dm}{d}$ we take f , that Equation may be express'd thus;

$$aa - fa = \frac{mb}{d}.$$

- Therefore by resolving the last Equation according to the Canon in Sect. 8. Chap. 15. the value of a , to wit, the number of Ells of the better sort of Cloth which were bought for 1 l . will be discovered, viz.

$$a = 3 = \sqrt{\frac{mb}{d} + \frac{fa}{4}} + \frac{1}{2}f.$$

Thus it is found that 3 Ells of the better sort of Cloth did cost 1 l . and consequently 1 Ell cost $\frac{1}{3}l$. and 90 Ells 30 l . which subtracted from 42 l . (the full cost of both sorts,) leaves 12 l . for the full cost of 40 Ells of the worser sort; and consequently 1 Ell cost $\frac{3}{10}l$. and at this rate 1 l . will buy $3\frac{1}{3}$ Ells; which is more by $\frac{1}{2}$ of an Ell than was bought of the better sort of Cloth for 1 l . Therefore all the conditions in the Question are satisfied.

QUEST. 22.

A Merchant having Spices, to wit, 80 lb weight (or b) of Mace; and 100 lb weight (or c) of Cloves, sells both quantities for 65 (or d) pounds in money; whereby it happened that he sold a quantity of Mace for 10 l . (or m), and the like quantity of Cloves with 60 lb weight (or a) more of Cloves for 20 l . (or p .) The Question is, to find how many lb weight of Mace he sold for 10 l .

RESOLUTION.

- Let the number of lb weight of Mace that the Merchant sold for 10 l . be represented by a
- To which number adding 60, the sum will give the number of lb weight of Cloves that he sold for 20 l . to wit,

$$a + 60$$

3. Then

3. Then find how much money 80 lb weight of Mace was sold for, and say,

$$\text{If } a : 10 :: 80 : \frac{800}{a};$$

$$\text{Or, } a : m :: b : \frac{mb}{a};$$

whence the money for which the said 80 lb of Mace was sold is

4. Find likewise how much money 100 lb weight of Cloves was sold for, and say,

$$\text{If } a : 60 :: 100 : \frac{2000}{a};$$

$$\text{Or, } a : n :: c : \frac{rc}{a};$$

whence the money for which the said 100 lb of Cloves was sold is

5. The sum of both the said sums of money found out in the third and fourth steps is

6. Which sum in the last step must (according to the Question) be equal to 65 l. (or d,) hence this Equation arith, viz.

$$65 = \frac{2800a + 48000}{aa + 60a}; \quad \text{Or, } d = \frac{mba + mbn + rca}{aa + na}$$

7. Which Equation, after due Reduction (according to Sect. 12. Chap. 2, 3, 5.) will give this following Equation, viz.

$$\text{Or, } aa + \frac{225a}{13} = \frac{2800}{13};$$

In which last Equation if we take f instead of the known Coefficient $\frac{dn - mb - rc}{d}$,

and g instead of the known number $\frac{mbn}{d}$, that Equation may be express thus,

$$aa + fa = g;$$

8. Therefore by resolving the last Equation according to the Canon in Sect. 6. Chap. 15, the value of a , to wit, the number of lb weight of Mace that was sold for 10 l. will be made known, viz.

$$a = 20 = \sqrt{g + \frac{1}{4}ff} - \frac{1}{2}f.$$

Thus it is found that 20 lb weight of Mace was sold for 10 l. and consequently 80 lb weight for 40 l.

Moreover, adding 60 to 20 (before found,) the sum 80 is the number of lb weight of Cloves that was sold for 20 l. and consequently 100 lb of Cloves was sold for 25 l. which added to 40 l. (the price of 80 lb of Mace,) makes 65 l. the prescribed sum of money for both quantities of Spices sold.

QUEST. 23.

Two Merchants entered into Partnership, the first brought in a certain sum of pounds which continued in Company 12 (or b) months, and the second put in 30 l. (or c) for 17 (or d) months; they gained together 18½ l. (or m ;) whereof the first Merchant had 26 l. (or n) for his principal and gain. It is required to find how many pounds the first Merchant brought into the common Stock?

RESOLUTION.

1. For the first Merchants Stock put
2. Which Stock being multiplied by the time it continued in Company, produceth
3. The second Merchants Stock being multiplied by the time it remained in Company, produceth

$$\begin{array}{r|l} a & \\ 12a & \\ \hline 30 & \\ cd & \end{array}$$

4. Then

4. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the gain of the first Merchant, and say,

$$\text{If } 12a + 510 : 18\frac{1}{2} :: 12a : \frac{225a}{12a + 510};$$

$$\text{Or, } ba + cd : m :: ba : \frac{mba}{ba + cd};$$

Whence the gain of the first Merchant is found $\frac{225a}{12a + 510}$ or $\frac{mba}{ba + cd}$.

5. Which gain added to the first Merchants Stock a , gives for the sum of his Stock and gain;

$$\frac{12aa + 735a}{12a + 510}; \quad \text{Or, } \frac{baa + cda + mba}{ba + cd};$$

6. Which sum must be equal to the 26 l. (or n) given in the Question; whence this Equation arith, viz.

$$\frac{12aa + 735a}{12a + 510} = 26; \quad \text{Or, } \frac{baa + cda + mba}{ba + cd} = n.$$

7. Then by reducing that Equation according to the Rules in Chap. 12, there will arise;

$$aa + 35\frac{1}{2}a = 1105;$$

$$\text{Or, } aa + \frac{cd + mb - nb}{b}a = \frac{ncd}{b}.$$

8. Which last Equation being resolved by the Canon in Sect. 6. of the 15. Chap. the value of a , to wit, the first Merchants Stock will be found 20 pounds, viz. If instead of the known Coefficient $\frac{cd + mb - nb}{b}$ we take f , and g instead of the given number $\frac{ncd}{b}$;

Then by the said Canon,

$$a = 20 = \sqrt{g + \frac{1}{4}ff} - \frac{1}{2}f.$$

Whence the first Merchants Stock is found 20 l. The Proof may be made by the Rule of Fellowship with Time, in manner following.

$$\begin{array}{r} 20 \times 12 = 240 \\ 30 \times 17 = 510 \\ \hline 750 \end{array} \quad ; \quad 18\frac{1}{2} :: \begin{array}{r} 240 \\ 510 \end{array} \quad ; \quad \begin{array}{r} 6 \\ 12\frac{1}{2} \end{array}$$

QUEST. 24.

Two Merchants entered into Partnership, the first put in a certain number of Pounds for 3 (or b) months; the second put in 50 l. (or c) more than the first for 5 (or d) months: they gained together 140 l. (or m ;) whereof the first Merchant had such part, that if 60 l. (or n) be added to it, the sum must be equal to the Stock wherewith he entered Partnership: What was the Stock and gain of each Merchant?

RESOLUTION.

1. For the Stock of the first Merchant put
2. To which adding 50 l. (or c ;) the sum will give the second Merchant's Stock, to wit,
3. Then multiplying the first Merchant's Stock by the time it remained in Company, the Product is
4. Likewise by multiplying the second Merchant's Stock by the time it continued in Company, the Product is
5. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the first Merchant's Gain, and say,

$$\text{If } 8a + 250 : 140 :: 3a : \frac{420a}{8a + 250};$$

$$\text{Or, } ba + da + dc : m :: ba : \frac{mba}{ba + da + dc};$$

Whence the gain of the first Merchant is found $\frac{420a}{8a + 250}$ Or, $\frac{mba}{ba + da + dc}$.

6. To which gain add 60 (or n), to the sum will be

$$\frac{900a + 15000}{8a + 250}; \quad \text{Or,} \quad \frac{mba + nba + nda + ndc}{ba + da + dc}.$$
7. But, according to the Question, the sum in the last step must be equal to (a) the first Merchant's Stock, whence this Equation aritheth;

$$\frac{900a + 15000}{8a + 250} = a = \frac{mba + nba + nda + ndc}{ba + da + dc}.$$
8. Which Equation, after due Reduction according to the Rules in Chap. 12. will produce this following Equation, viz.

$$aa - 81\frac{1}{2}a = 1875,$$
Or,

$$aa - \frac{mb + nb + nd - dc}{b + d}a = \frac{ndc}{b + d}.$$
9. In which Equation the value of a , to wit, the first Merchant's Stock, will be discovered by the Canon in *Self. 8. Chap. 15. viz. $a = 100$ l.* And consequently from the premises the second Merchant's Stock was 150 l. the gain of the first 40 l. and the gain of the second 100 l. All which will be evident by the following Proof wrought by the Rule of Fellowship with Time.

$$\begin{array}{r} 100 \times 3 = 300 \\ 150 \times 5 = 750 \\ \hline 1050 \cdot 140 :: \left\{ \begin{array}{l} 300 : 40 \\ 750 : 100 \end{array} \right. \end{array}$$

QUEST. 25.

A Citizen having bought a House for a certain sum of pounds, sells it for 64 l. (or d) and finds that his loss in 100 pounds (or c) was equal to a fourth part (or m) of the money that he paid for the House. What number of pounds did the Citizen pay for the House?

RESOLUTION.

1. For the number of pounds which the Citizen paid for the house put a
2. Then will the whole loss by sale of the house be $a - 64$
3. Find how much was lost by 100 l. (or c) and say,
 If $a \cdot a - 64 :: 100 \cdot \frac{100a - 6400}{a};$
 Or,
 $a \cdot a - d :: c \cdot \frac{ca - cd}{a}.$

Whence the loss per Cent. is found $\frac{100a - 6400}{a};$ Or, $\frac{ca - cd}{a}.$

4. But according to the Question the loss per Cent. was equal to $\frac{1}{4}$ part of the money which the Citizen paid for the House, therefore from the first and third steps this Equation aritheth, viz.

$$\frac{100a - 6400}{a} = \frac{a}{4}; \quad \text{Or,} \quad \frac{ca - cd}{a} = \frac{ma}{m}.$$
5. Which Equation, after due Reduction according to the Rules in Chap. 12. will give

$$400a - aa = 25600; \quad \text{Or,} \quad \frac{c}{m}a - aa = \frac{cm}{m}.$$
6. Therefore by resolving the said Equation according to the Canon in *Self. 10. Chap. 15.* both the values of a will be discovered, either of which will solve the Question, which values or numbers are these following, viz.

$$a = \left\{ \begin{array}{l} 320 = \frac{c}{2m} + \sqrt{\frac{cc - 4cdm}{4mm}} \\ 80 = \frac{c}{2m} - \sqrt{\frac{cc - 4cdm}{4mm}} \end{array} \right.$$

I say either of the numbers 320 and 80 will satisfy the Conditions in the Question, as will be evident by the Proof: For if a House cost 320 l. and be sold for 64 l. the loss is 256 l. and 100 l. at that rate of loss will lose 80, which is $\frac{1}{4}$ part of the first Cost 320 l. Again,

Again, if a House cost 80 l. and be sold for 64 l. the loss is 16 l. and 100 l. at this rate of loss will lose 20 l. which is likewise $\frac{1}{5}$ part of the first Cost 80 l.

QUEST. 26.

Two Merchants entered into Partnership, the sum of their Stocks was 165 (or b) pounds: the first Merchant's Stock continued in Company 12 (or c) months, and the Stock of the second 8 (or d) months: they gained a certain sum of pounds, which together with their Stocks they divided between themselves in such manner, that the first Merchant received 67 (or f) pounds for his Stock and gain, and the second 126 (or g) pounds for his Stock and gain. It is desired to find out each Merchant's Stock and Gain.

RESOLUTION.

1. For the first Merchant's Stock put a
2. Then, by subtracting that Stock (a) from 165 (or b) there remains the second Merchant's Stock; to wit,

$$165 - a \quad b - a$$
3. And if you subtract (a) the first Merchant's Stock from 67 (or f) the sum of his Stock and Gain, there will remain his Gain only; to wit,

$$67 - a \quad f - a$$
4. Likewise, if you subtract the second Merchant's Stock (in the second step) from 126 (or g) the sum of his Stock and Gain, there will remain his Gain only; to wit,

$$126 - a - b \quad a + g - b$$
5. Now according to the nature of the Rule of Fellowship with Time, the Gain of the first Merchant $67 - a$ must be in such proportion to $a - 39$ the Gain of the second, as the Product of the first Merchant's Stock a multiplied by its time 12 months, is to the Product of the second Merchant's Stock $165 - a$ multiplied by its time 8 months; hence this Analogy, viz.

$$67 - a \cdot a - 39 :: 12a \cdot 1320 - 8a,$$

- That is, $f - a \cdot a - g - b :: ca \cdot db - da.$
6. Which Analogy, by comparing the Product made by the multiplication of the Means one into the other, to the Product of the Extremes, produceth this Equation, viz.

$$12aa - 468a = 8aa - 1856a + 88440,$$

That is, $caa + cga - cba = daa - dba - dfa + dbf.$

7. From which Equation after due Reduction thus aritheth, viz.

$$aa + 347a = 22110,$$

That is, $aa + \frac{db + df + cg - cb}{c - d}a = \frac{dbf}{c - d}.$

8. Wherefore by resolving the last Equation according to the Canon in *Self. 6. Chap. 15.* the value of a , that is, the number of pounds expressing the first Merchant's Stock will be found 55; which subtracted from 165 l. the sum of both their Stocks, leaves 110 l. for the second Merchant's Stock; then each of their Stocks being subtracted from their respective Stock and Gain, viz. 55 l. from 67 l. and 110 l. from 126 l. there remains 12 l. for the Gain of the first Merchant, and 16 l. for the gain of the second, whence the total Gain was 28 l. Which numbers will solve the Question, as may easily be proved by the Rule of Fellowship with Time; thus;

$$\begin{array}{r} 55 \times 12 = 660 \\ 110 \times 8 = 880 \\ \hline 1540 \cdot 28 :: \left\{ \begin{array}{l} 660 : 12 \\ 880 : 16 \end{array} \right. \end{array}$$

QUEST. 27.

A certain Foot-man A departeth from London towards Lincoln, and at the same time another Foot-man B departeth from Lincoln towards London, each keeping the same Road. When they met, A said to B ; I find that I have travelled 20 (or b) miles more than you, and have gone as many miles in $6\frac{1}{2}$ (or d) days, as you have gone miles

miles in all hitherto: 'Tis true faith *B*, I am not so good a Foot-man as you, but I find that at the end of 15 (or *f*) days hence, I shall be at *London*, if I travel as many miles in every one of those 15 days, as I have done in every day hitherto. The Question is, to find how many miles those two Cities are distant one from another, and how many miles each Foot-man had travelled when they met one another.

RESOLUTION.

1. For the desired distance between the two Cities put a
2. Then forasmuch as the number of miles each Foot-man had travelled when they met, being added together make the sum (a) , and the difference between those two numbers was 20 (or c), for *A* had travelled 20 miles more than *B*: Therefore (by the Theorem at the end of *Quest. 1. Chap. 14.*) the number of miles which *A* had travelled was $\frac{1}{2}a + 10$
3. And (by the same Theorem) the number of miles which *B* had travelled was $\frac{1}{2}a - 10$
4. Then say, If in $\frac{1}{2}a - 10$ miles *A* had travelled $\frac{1}{2}a - 10$ miles, how many miles did he travel in one day? so by the Rule of Three, you will find $\frac{\frac{1}{2}a - 10}{6\frac{1}{2}}$
5. Say again, If in 15 days *B* must travel $\frac{1}{2}a - 10$ miles, (that is, all the miles which *A* had travelled,) how many miles must *B* travel in one day? so you will find $\frac{\frac{1}{2}a - 10}{15}$
6. Say again, If $\frac{\frac{1}{2}a - 10}{15}$ miles were travelled by *B* in one day, in how many days did he travel $\frac{1}{2}a - 10$ miles? so you will find $\frac{7\frac{1}{2}a - 150}{\frac{1}{2}a - 10}$
7. Say again, If $\frac{\frac{1}{2}a - 10}{6\frac{1}{2}}$ miles were travelled by *A* in one day, in how many days did he travel $\frac{1}{2}a + 10$ miles? so you will find $\frac{3\frac{1}{2}a + 66\frac{1}{2}}{\frac{1}{2}a - 10}$
8. But the numbers of days found out in the two last steps must be equal to one another, for when *A* and *B* met, each had travelled the same number of days, because they began their Journey at one and the same time: Hence this Equation ariseth, *viz.*

$$\frac{3\frac{1}{2}a + 66\frac{1}{2}}{\frac{1}{2}a - 10} = \frac{7\frac{1}{2}a - 150}{\frac{1}{2}a - 10}$$

That is,

$$\frac{\frac{1}{2}da + \frac{1}{2}dc}{\frac{1}{2}a - 10} = \frac{\frac{1}{2}fa - \frac{1}{2}fc}{\frac{1}{2}a - 10}$$

9. In which Equation, if you double both the Numerators and Denominators, and then reduce the Equation resolving, to a common Denominator, and cast away the common Denominator, the new Numerators being compared to one another will give this following Equation, *viz.*

$$\frac{2a}{daa} + \frac{2a}{daa} + \frac{2a}{daa} = 15aa - 600a + 6000;$$

That is,

$$\frac{2da}{daa} + \frac{2dc}{daa} + \frac{2cc}{daa} = \frac{15a}{daa} - \frac{600}{daa} + \frac{6000}{daa};$$

10. Which last Equation duly reduced gives this that follows, *viz.*

$$104a - aa = 400,$$

That is,

$$\frac{2c - \frac{1}{2}fc}{f - d} a - aa = cc.$$

11. Wherefore by resolving the Equation in the last step according to the Canon in *Sol. 10. Chap. 15.* the two values of a will be found these, *viz.*

$$a = 100 = \frac{dc + fc + \sqrt{4dfcc}}{f - d}$$

$$a = 4 = \frac{dc + fc - \sqrt{4dfcc}}{f - d}$$

12. But

12. But although by either of those values of a , to wit, 100 and 4; the Equation in the tenth step may be expounded, yet the greater value only is the desired number of miles expressing the distance between the two Cities; for 'tis evident by the Question, that 20 is but part of the number of miles between the two Cities, and therefore 4 the lesser value of a is much less than the said distance: Wherefore 100 the greater value of a is the desired number of miles between the two Cities. And consequently the second, third, fourth and fifth steps being resolved into numbers, will shew, that when the two Foot-men *A* and *B* met one another, *A* had travelled 60 miles, and *B* 40 miles: Also, *A* travelled 6 miles, and *B* 4 miles every day; as will easily appear by the Proof.
13. But the numbers in this Question must not be given at random, for the Denominator of the Fraction $\frac{2dc + 2fc}{f - d}$ in the Equation in the tenth step shews that the number d must be less than the number f , otherwise the Question is impossible; as may easily be infer'd from the literal Equation in the ninth step: for if in that Equation d be supposed greater than f , then consequently dc is greater than fc , and after due transposition this Equation will arise, *viz.* $dc - fc = faa - daa - 2dca - 2fca$; where if d be greater than f , then the first part of the Equation will be a real quantity, that is, greater than nothing, and the latter part less than nothing; but to affirm that a quantity greater than nothing is equal to a quantity less than nothing is absurd, the like absurdity will follow if we suppose $d = f$.
14. Having shew'd that d must necessarily be less than f , I shall prove that the lesser value of a , as it is express'd by letters in the eleventh step, can never be equal to the whole distance between the two Cities. For if we should suppose the lesser value to be equal to the said distance, it must necessarily be greater than c , which the Question shews to be but part of the said distance: But from that supposition, it will follow by undeniable consequence, that d is greater than f , which is contrary to what hath been before proved: Now to prove the said consequence,

15. Suppose the lesser value of a to exceed c , *viz.* $\frac{dc + fc - \sqrt{4dfcc}}{f - d} > c$
 16. Then by multiplying each part by $f - d$, it follows that $dc + fc - \sqrt{4dfcc} < fc - dc$
 17. And by adding $\sqrt{4dfcc}$ to each part, $dc + fc < fc - dc + \sqrt{4dfcc}$
 18. And by adding dc to each part, $2dc + fc < fc + \sqrt{4dfcc}$
 19. And by subtracting fc from each part, $2dc < \sqrt{4dfcc}$
 20. And by squaring each part, $4d^2c^2 < 4dfcc$
 21. And by dividing each part by $4dc$, $d < f$
 22. Thus from a supposition that the lesser value of a in the eleventh step is greater than c , it follows by just consequence that d is greater than f , which is impossible, for it hath been proved that d must be less than f . And because the Series of Inferences deduced from the said Supposition ends in an impossibility, therefore that which was supposed cannot be true, *viz.* The lesser value of a is not greater than c , and consequently it cannot be equal to the distance between the two Cities. Which was to be proved.
 23. Again, by supposing d to be less than f , as it ought to be, to the end the Question may be possible, we may prove the lesser value of a to be lesser than c , by returning backwards from the 21 step to the 15, in this manner, *viz.*
 24. Suppose $d < f$
 25. Then by multiplying each part by $4dc$, $4d^2c^2 < 4dfcc$
 26. And by extracting the square Root out of each part, $2dc < \sqrt{4dfcc}$
 27. And by adding fc to each part, $2dc + fc < fc + \sqrt{4dfcc}$
 28. And by subtracting dc from each part, $dc + fc < fc - dc + \sqrt{4dfcc}$
 29. And by subtracting $\sqrt{4dfcc}$ from each part, $dc + fc - \sqrt{4dfcc} < fc - dc$
 30. Wherefore by dividing each part by $f - d$, it is manifest that the lesser value of a is less than c , *viz.* $\frac{dc + fc - \sqrt{4dfcc}}{f - d} < c$
- Which was to be proved. Wherefore the lesser value of a cannot possibly be equal to the distance between the two Cities, for the said distance must necessarily be greater than part of it self,

31. But

31. But it may be objected, That although f be greater than d , yet how doth it appear that $dc + fc$ is greater than $\sqrt{4dfcc}$, to the end that this may be subtracted from that; as the lesser value of a requires, to make it self a possible Root of the Equation in the tenth step? In answer to this Objection I shall in the next place prove that $dc + fc$ is greater than $\sqrt{4dfcc}$.
32. Forasmuch as these quantities are Proportionals, (for the Product of the extremes is equal to the Product of the means,) $dd : df :: df : ff$
33. Therefore (per 25 Prop. 5. Elem. Euclid.) $dd + ff > 2df$
34. And by multiplying all in the last step by cc , $ddcc + ffcc > 2dfcc$
35. And by adding $2dfcc$ to each part, $ddcc + ffcc + 2dfcc > 4dfcc$
36. Wherefore by extracting the Square Root out of each part in the last step, $dc + fc > \sqrt{4dfcc}$.
- Which was to be proved.

C H A P. XVII.

Concerning ARITHMETICAL PROGRESSION.

I. **A** *Arithmetical Progression* is, when many numbers (or other quantities of one and the same kind) proceed by a common difference or excess, as in these, 2, 4, 6, 8, 10, 12, 14, &c. here 2 is the common difference betwix 2 and 4, 4 and 6, 6 and 8, 8 and 10, &c. So 1, 2, 3, 4, 5, 6, &c. are in *Arithmetical Progression*, 1 being the common difference: Likewise 3, 7, 11, 15, 19, &c. or 19, 15, 11, 7, and 3, where 4 is the common difference.

II. *Arithmetical Progression* is either continued, as in the Examples above express, where every two terms that stand next to one another, have one common difference, or else discontinued or interrupted, as in these numbers, 3, 5 : 9, 11, where 5 exceeds 3 by 2, and so doth 11 exceed 9, but 9 doth not exceed 5 by 2, for the excess of 9 above 5 is 4. In like manner 18, 14 : 21, 17, are in *Arithmetical Progression* discontinued.

III. For the better manifestation of the following Propositions concerning *Arithmetical Progression*, let there be a rank of numbers in a continued *Arithmetical Progression*, as, 3, 7, 11, 15, 19, 23, 27, &c. which numbers may be represented by a, b, c, d, e, f, g , &c. Also, let 105 the sum of all the terms of the Progression be represented by Z ; the common excess or difference 4 by X ; and the number of terms 7 by T : all which are here orderly express'd underneath.

Quantities in Arithmetical Progression continued:	3	=	a	=	a
	7	=	b	=	$a + X$
	11	=	c	=	$a + 2X$
	15	=	d	=	$a + 3X$
	19	=	e	=	$a + 4X$
	23	=	f	=	$a + 5X$
	27	=	g	=	$a + 6X$

The Summ of all the Terms is $105 = Z = Z$

The common difference is $4 = X = X$

The number of Terms is $7 = T = T$

IV. Whence it is manifest, that if a be put for the first and least term of an *Arithmetical Progression* continued, and X for the common difference, then (according to the Definition in Sect. 1.) the second term shall be $a + X$, the third $a + 2X$, the fourth $a + 3X$, the fifth $a + 4X$, &c. Moreover, according to the Suppositions in Sect. 3. $a = a$, $b = a + X$, $c = a + 2X$, $d = a + 3X$, $e = a + 4X$, &c.

V. Therefore it follows, that the last and greatest term of every *Arithmetical Progression* continued is compos'd of the first (to wit, the least) term, and of the Product of the common difference multiplied by a number less by 1 (or Unity) than the number

number of terms, as g ; or $a + 6X$ is compos'd of the first term a and the Product of X multiplied by 6, which is less by 1 than 7 the number of terms.

VI. Therefore the first and last terms, as also the number of terms being severally given, the common difference shall be also given; for if the first (to wit, the smallest) term be subtracted from the last, and the Remainder be divided by a number less by 1 (or Unity) than the number of terms, the Quotient is the common difference, viz. $\frac{g-a}{T-1} = X$.

VII. It is also manifest from Sect. 3. That if the first (to wit, the least) term be equal to the common difference, then the last term is equal to the Product of the common difference (or first term) multiplied by the number of terms, viz. If $a = X$, then $g = X + 6X = 7X$.

VIII. Therefore in an *Arithmetical Progression* continued whose first or least term is equal to the common difference, if the last term and the number of terms be severally given, the first term (or the common difference) shall also be given: For if the last term be divided by the number of terms, the Quotient is the first term or common difference; as, if $a = X$, then $g = X + 6X = 7X$; therefore $\frac{7X}{7} = X = a$.

IX. It is also manifest from Sect. 7. That when the common difference divideth any term just without any Remainder then the common difference is the same with the least term in that Progression, and the Quotient is the number of terms; but if any number remain after the Division is finished, then that Remainder is the least term, and the Quotient increased with 1 (or Unity) gives the number of terms (per Sect. 4. & 5.) Therefore if any term greater than the least be given, as also the common difference, the least term, as also the number of terms in that Progression shall also be given; as if 27 be some term greater than the least, and 3 the common difference, by dividing 27 by 3, the Quotient 9 is the number of terms, and the least term is equal to the common difference 3; as in this Progression, 3, 6, 9, 12, 15, 18, 21, 24, 27.

But if 27 be given as before, and 4 be prescribed for the common difference, then 27 divided by 4 gives 6 in the Quotient, and there remains 3 for the least term, and 7 (to wit $6 + 1$) is the number of terms; as in this Progression, 3, 7, 11, 15, 19, 23, 27.

X. If three numbers, suppose a, b, c , be in a continued *Arithmetical Progression*, viz. If the excess of c above b be equal to the excess of b above a , the sum of the Extremes, that is, of the first and last Terms shall be equal to the double of the Mean or middle Term; viz. $a + c = 2b$. For,

1. By supposition, $c - b = b - a$.

2. Therefore by adding b to each part, it gives $c = 2b - a$.

3. And by adding a to each part of the last Equation $a + c = 2b$.

Which was to be proved.

XI. If four numbers, suppose a, b, c, d , be in *Arithmetical Progression* whether continued or interrupted, viz. If the excess of b above a be equal to the excess of d above c , the sum of the Extremes shall be equal to the sum of the Means, viz. $a + d = b + c$. For,

1. By supposition, $d - c = b - a$.

2. Therefore by equal addition of a , $a + d - c = b$.

3. Therefore by equal addition of c , $a + d = b + c$.

Which was to be proved.

XII. If there be as many numbers as you please in a continued *Arithmetical Progression*, the sum of the Extremes is equal to the sum of any two Means equally distant from the Extremes, and also to the double of the Mean when the number of Terms is odd.

Let a, b, c, d, e, f , be in *Arithmetical Progression* continued; and increasing from a . I say the sum of the Extremes a and f is equal to the sum of any two Terms equally distant from the Extremes, that is, to the sum of b and e , and to the sum of c and d . For,

1. By supposition, in regard of the continued Progression, $f - e = b - a$.

2. Therefore by equal addition of e and a to each part, $a + f = b + e$.

3. Again, by supposition, $e - d = c - b$.

4. Therefore by equal addition of d and b to each part, $b + e = c + d$.

Q

4. There

4. Therefore by equal addition of d and b , to each part $c + d = b + e$,
 5. Therefore from the second and fourth steps (per $\frac{1}{2}$ Axiom. 1. Elem. Euclid.) $a + f = c + d = b + e$.

Which was to be proved.

And if more numbers were propos'd, the Demonstration would not be otherwise; therefore the first part of the Theorem is manifest.

But if the number of terms be odd as in this continued Progression, a, b, c, d, e, f, g , then the sum of the extremes a and g is equal to the double of the middle term d , viz. $a + g = 2d$; which I prove thus:

1. By supposition, in regard of the continued Progression, $d - c = e - d$,
 2. And consequently by equal addition of c and d , $2d = c + e$,
 3. But by what hath been proved concerning the first part of the Theorem in this twelfth Sect. $a + g = c + e$,
 4. Therefore from the two last steps, (per Axiom. 1. Elem. 1. Euclid.) $a + g = 2d$.

Which was to be demonstrated. Therefore the Theorem is every way manifest.

XIII. In every Arithmetical Progression continued, the sum of the extremes multiplied by the number of terms produceth the double of the sum of all the terms.

The number of terms is either even or odd; First, let there be an even number of terms, viz. suppose these six numbers a, b, c, d, e, f to be in Arithmetical Progression continued;

$$\text{I say, } 6a + 6f = 2a + 2b + 2c + 2d + 2e + 2f.$$

DEMONSTRATION.

1. It is evident that $2a + 2f = 2a + 2f$,
 2. And by Sect. 12. $2a + 2f = 2b + 2e$,
 3. Likewise, by the same Sect. $2a + 2f = 2c + 2d$,
 4. Therefore by adding the three last Equation together, $6a + 6f = 2a + 2b + 2c + 2d + 2e + 2f$.

Which was to be demonstrated. And so of others when the number of terms is even.

Secondly, let there be an Arithmetical Progression consisting of an odd number of terms, suppose these five, a, b, c, d, e .

$$\text{I say, } 5a + 5e = 2a + 2b + 2c + 2d + 2e.$$

DEMONSTRATION.

1. It is manifest that $2a + 2e = 2a + 2e$,
 2. And by Sect. 12. $2a + 2e = 2b + 2d$,
 3. Likewise by Sect. 12. $a + e = 2c$,
 4. Therefore by adding the three last Equations together, $5a + 5e = 2a + 2b + 2c + 2d + 2e$.

And so of others when the number of terms is odd.

XIV. Therefore from the last Sect. the first and last terms, as also the number of terms in an Arithmetical Progression continued being given, the sum of all the terms shall be also given: For if the sum of the first and last terms be multiplied by the number of terms the Product is the double sum of all the terms, and consequently the half of that Product is the sum of all the terms. For example, If a, b, c, d, e, f, g be in Arithmetical Progression continued, and T be put for the number of terms, also Z for their sum (as before,) Then $Ta + Tg = 2Z$, and consequently $Ta + Tg = 2Z$.

XV. Mr. William Oughtred in Probl. 4. Chap. 19. of his incomparable *Clevis Mathematicus*. hath very elegantly handled 20 Propositions about Arithmetical Progression continued, which (for the more ample Illustration of the preceding Rules in this Book,) I shall explain in this Section, using his own Symbols, which are these, viz.

$\left. \begin{array}{l} a \\ b \\ T \\ X \\ Z \end{array} \right\}$ Stands for $\left\{ \begin{array}{l} \text{The least (or first) term.} \\ \text{The greatest (or last) term.} \\ \text{The number of terms.} \\ \text{The common difference of the terms.} \\ \text{The sum of all the terms.} \end{array} \right.$

Any

Any three of these five things being given, the other two shall be also given, by the respective Canons of the following 20 Propositions, which Mr. Oughtred states thus:

Given,	Sought,	By Propof.
a, w, T	Z and X	1 and 2
a, w, X	T and Z	3 and 4
a, w, Z	T and X	5 and 6
a, T, X	w and Z	7 and 8
a, T, Z	w and X	9 and 10
a, X, Z	w and T	11 and 12
w, T, X	a and Z	13 and 14
w, T, Z	a and X	15 and 16
w, X, Z	a and T	17 and 18
T, X, Z	a and w	19 and 20

PROP. I.

1. $\therefore \sum a, w, T$ are given severally;
 Z is sought.

RESOLUTION.

2. By Sect. 14. of this Chap. $Ta + Tg = 2Z$.
 Which Equation, if express'd by words, gives this

CANON.

Multiply the sum of the first and last terms by the number of terms, the Product shall be the double of the sum of all the terms, and consequently the half of that Product is the required sum of all the terms.

Which Canon may be exemplified by the following (or any other) rank of numbers in Arithmetical Progression continued, viz.

3, 7, 11, 15, 19, 23, 27.

PROP. II.

1. $\therefore \sum a, w, T$ are given severally;
 X is sought.

RESOLUTION.

2. By Sect. 6. of this seventeenth Chap. $\frac{a - g}{T - 1} = X$.

Which Equation gives this following

CANON.

Divide the excess of the greatest (or last) term above the least, by the number of terms lessened by 1 (or Unity,) and the Quotient is the common difference required.

Which Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued, viz.

3, 7, 11, 15, 19, 23, 27.

From the Equation in the second step of Prop. 1. and the Equation in the second step of Prop. 2. the Canons of all the following 18 Propositions are deduced.

PROP. III.

1. $\therefore \sum a, w, X$ are given severally;
 T is sought.

RESOLUTION.

2. The letters put for the things given and sought, without any other letter, are contained in the Equation in the second step of Prop. 2. therefore the work here is only to set T alone in that Equation, which may be done thus, viz.

Q_2

3. By

3. By the Canon of Prop. 2. $\frac{a-a}{T-1} = X$,
 4. Therefore by multiplying each part of that Equation }
 by $T-1$, this aritheth, viz. $a-a = TX-X$;
 5. And by addition of X to each part of the last Equation, }
 this aritheth; $a-a+X = TX$,
 6. Therefore each part of the last Equation being divided }
 by X , the number T will be made known, viz. $\frac{a-a}{X} + 1 = T$.

The last Equation gives this following

CANON.

From the last (to wit, the greatest) term subtract the first, and divide the Remainder by the common difference; then to the Quotient add 1 (or Unity,) so shall the sum be the required number of terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IV.

1. $\sum a, a, X$ are given severally;
 Z is required.

RESOLUTION.

2. By the Canon of Prop. 1. $Ta+Ta = 2Z$;
 3. And by the Canon of Prop. 3. $\frac{a-a}{X} + 1 = T$,
 4. Now if instead of T in the first part of the Equation in the second step, you multiply into $a-a$ that which in the last Equation is found equal to T , the former Equation will be converted into this, viz.

$$\frac{a-a}{X} + a + a = 2Z.$$

Which in words is this following

CANON.

From the Square of the greatest (or last) term subtract the Square of the least (or first) then dividing the Remainder by the common difference, and to the Quotient adding the sum of the first and last terms, the half of the sum of this addition shall be the required Summ of all the terms.

The Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. V.

1. $\sum a, a, Z$, are given severally;
 T is required.

RESOLUTION.

2. By the Canon of Prop. 1. $Ta+Ta = 2Z$;
 3. Therefore by dividing each part of that Equation by }
 $a+a$, this aritheth, viz. $T = \frac{2Z}{a+a}$,
 Which Equation gives this following

CANON.

Divide the double of the sum of all the terms by the sum of the first and last terms, the Quotient is the number of terms sought; as may be proved by this following (or any other) Rank of numbers in Arithmetical Progression:

3, 7, 11, 15, 19, 23, 27.

PROP.

PROP. VI.

1. $\sum a, a, Z$ are given severally;
 X is required.

RESOLUTION.

2. By the Canon of Prop. 4. $\frac{a-a}{X} + a + a = 2Z$;
 3. Which Equation multiplied by X produceth, $a-a+ax+ax = 2ZX$;
 4. And by subtracting $aX+aX$ from each part of }
 the last Equation, this aritheth, viz. $a-a = 2ZX-aX-aX$;
 5. Therefore by dividing each part of the last E- }
 quation by the Coefficients that are drawn into X , }
 you will find, $\frac{a-a}{2Z-a-a} = X$.
 Which last Equation gives this

CANON.

From the Square of the last term subtract the Square of the first (to wit, the least) term; divide the Remainder by the excess whereby the double sum of all the terms exceeds the sum of the first and last terms, so shall the Quotient be the common difference required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression:

3, 7, 11, 15, 19, 23, 27.

PROP. VII.

1. $\sum a, T, X$ are given severally;
 a is sought.

RESOLUTION.

2. By the Canon of Prop. 2. $\frac{a-a}{T-1} = X$,
 3. Therefore by multiplying each part of the said }
 Equation by $T-1$, this will be produced, } $a-a = TX-X$;
 4. And by adding a to each part of the last Equation }
 this aritheth, viz. $a = TX+a-X$.
 Which last Equation gives this

CANON.

To the Product made by the multiplication of the number of terms into the common difference, add the first (to wit, the least) term, and from the sum subtract the said difference, so shall the Remainder be the last term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. VIII.

1. $\sum a, T, X$ are given severally;
 Z is sought.

RESOLUTION.

2. By the Canon of Prop. 1. $Ta+Ta = 2Z$;
 3. And by the Canon of Prop. 7. $TX+a-X = a$;
 4. Now to find an Equation that may consist only of the things given and sought in this Prop. 8. multiply each part of the Equation in the third step by T , and there will be produced

$$TTX+Ta-TX = Ta.$$

5. Then

5. Then if instead of T in the second step, you take that which in the fourth step is found equal to T , the Equation in the second step will be reduced to this, to wit,

$$TIX + 2Ta - TX = 2Z,$$

That is,
Which last Equation gives this

CANON.

6. To the Product of the multiplication of the number of terms by the common difference, add the double of the first (to wit, the least) term, and from the sum of that Addition subtract the common difference; then multiply the Remainder by the number of terms; so shall the Product be the double sum of all the terms, and consequently the half of that Product is the required sum of all the terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IX.

1. $\{ \alpha, T, Z$ are given severally;
 α is sought.

RESOLUTION.

2. By the Canon of Prop. 1. $\{ Ta + Ta = 2Z;$
3. Therefore by equal subtraction of Ta , . . . $\{ T\alpha = 2Z - Ta;$
4. Therefore by dividing each part of the last $\{ \alpha = \frac{2Z - Ta}{T}.$
Equation by T , this aritheth;

Which last Equation gives this

CANON.

From the double of the sum of all the terms subtract the Product of the multiplication of the number of terms by the first (to wit, the least) term, and divide the Remainder by the number of terms; so shall the Quotient be the last term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. X.

1. $\{ \alpha, T, Z$ are given severally;
 X is sought.

RESOLUTION.

2. By the Canon of Prop. 8. $\{ TTX + 2Ta - TX = 2Z;$
3. Therefore by equal subtraction of $2Ta$ from $\{ TTX - TX = 2Z - 2Ta;$
each part, this will arise; to wit,
4. And by dividing each part of the last Equation $\{ X = \frac{2Z - 2Ta}{TT - T}.$
by $TT - T$, the common difference X will be
made known, viz.

Which last Equation gives this

CANON.

From the double sum of all the terms subtract the double Product made by the multiplication of the number of terms by the least term, and divide the Remainder by the excess of the Square of the number of terms above the number of terms, so shall the Quotient be the common difference sought.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XI.

1. $\{ \alpha, X, Z$ are given severally;
 α is sought.

RESOLUTION.

2. By the Canon of Prop. 4. $\{ \frac{\alpha\alpha - \alpha\alpha}{X} + \alpha \cdot \alpha = 2Z;$

3. There-

3. Therefore by multiplying that Equation by X , $\{ \alpha\alpha - \alpha\alpha + X\alpha + X\alpha = 2ZX;$
this will be produced; to wit,
4. And by transposition of $-\alpha\alpha$, this aritheth; $\{ \alpha\alpha + X\alpha + X\alpha = 2ZX + \alpha\alpha;$
5. And from the last Equation by transposition $\{ \alpha\alpha + X\alpha = 2ZX + \alpha\alpha - X\alpha;$
of $X\alpha$ this aritheth;
6. Which last Equation falling under the first of the three Forms in Sect. 1. Chap. 15.
of this Book, the value of α shall be given by the Canon in Sect. 6. of the same
Chapt. viz.

$$\alpha = \sqrt{\frac{1}{4}XX + 2ZX + \alpha\alpha - X\alpha} - \frac{1}{2}X.$$

Which Equation gives this

CANON.

From the sum of these three numbers, to wit, the Square of half the common difference; the double Product of the multiplication of the sum of all the terms by the common difference; and the Square of the first (to wit, the least) term; subtract the Product of the first term multiplied by the common difference, and extract the Square Root of the Remainder; then from the said Square Root subtract half the common difference, so shall this last Remainder be the last and greatest term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XII.

1. $\{ \alpha, X, Z$ are given severally;
 T is sought.

RESOLUTION.

2. The Canon of Prop. 8. gives this Equation, $\{ XTT + 2\alpha T - XT = 2Z;$
3. Where in regard X is drawn into TT (which
is the highest degree of the quantity sought,) let
every term of the Equation be divided by X , $\{ TT + \frac{2\alpha T - XT}{X} = \frac{2Z}{X};$
whence this Equation will arise,
4. Now it must be discovered from the things given whether 2α exceeds X ; or is less, or
equal to X . First then suppose $2\alpha < X$, and then the last Equation may be express'd thus;

$$TT + \frac{2\alpha - X}{X}T = \frac{2Z}{X}.$$

5. Which Equation falling under the first of the three Forms in Sect. 1. Chap. 15. the
value of T shall be given by the Canon in Sect. 6. of the same Chapt. viz.

$$T = \sqrt{\frac{\alpha\alpha - \alpha X + \frac{1}{4}XX + 2ZX}{XX}} - \frac{2\alpha - X}{2X}.$$

6. Secondly, if $2\alpha = X$, then the Equation in the third step shall be express'd thus;

$$TT - \frac{X - 2\alpha}{X}T = \frac{2Z}{X}.$$

7. Which Equation falling under the second of the three Forms in Sect. 1. Chap. 15. the
value of T shall be given by the Canon in Sect. 8. of the same Chapt. viz.

$$T = \sqrt{\frac{\frac{1}{4}XX - \alpha X + \alpha\alpha + 2ZX}{XX}} + \frac{X - 2\alpha}{2X}.$$

8. Lastly, if $2\alpha > X$, then the Equation in the third step will be express'd thus;

$$TT \pm \frac{2Z}{X}; \quad \text{Whence,} \quad T = \sqrt{\frac{2Z}{X}}.$$

The three Equations in the 5, 7, and 8 steps give a threefold Canon to solve this 12 Prop. viz.

Canon I. When the double of the least term exceeds the common difference.

9. To the Square of the excess of the least term above half the common difference add the double Product of the multiplication of the sum of all the terms by the common difference, divide the sum of that Addition by the Square of the common difference and extract the square Root of the Quotient; then from the double of the least term subtract the common difference and divide the Remainder by the double of the common difference; lastly, subtracting this Quotient from the square Root before found, the Remainder shall be the number of terms sought.

This

This Canon may be exemplified by the following or the like Series of numbers in Arithmetical Progression continued, where the double of the least term exceeds the common difference of the terms:

3, 5, 7, 9, 11, 13, 15, &c.

Canon II. When the double of the least term is less than the common difference of the terms.

10. To the Square of the excess of half the common difference above the least term, add the double Product of the multiplication of the sum of all the terms by the common difference; divide the sum of that Addition by the Square of the common difference, and extract the square Root of the Quotient; then from the common difference subtract the double of the least term, and divide the Remainder by the double of the common difference; lastly, adding this Quotient to the square Root before found, the sum shall be the number of terms sought.

This Canon may be exemplified by the following or the like Rank of numbers in Arithmetical Progression continued, where the double of the least term is less than the common difference:

2, 7, 12, 17, 22, 27, 32, 37.

Canon III. When the double of the least term is equal to the common difference of the terms.

11. Divide the double of the sum of all the terms by the common difference, so shall the square Root of the Quotient be the number of terms sought.

This Canon may be exemplified by the following Rank of numbers in Arithmetical Progression continued, where the double of the least term is equal to the common difference of the terms:

3, 9, 15, 21, 27, 33, 39.

PROP. XIII.

1. $\sum a, T, X$ are given severally;
a is sought.

RESOLUTION.

2. By the Canon of Prop. 7. $TX - X + a = a$,
3. Therefore by transposition of $TX - X$, this Equation will arise, which makes known the value of a ; $a = a + X - TX$.
Which Equation gives this

CANON.

To the last (that is, the greatest) term add the common difference, and from the sum subtract the Product of the number of terms multiplied by the common difference; so shall the Remainder be the first (or least) term sought.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XIV.

1. $\sum a, T, X$ are given severally;
Z is sought.

RESOLUTION.

2. By the Canon of Prop. 1. $Ta + Ta = 2Z$;
3. And by the Canon of Prop. 13. $a + X - TX = a$,
4. Which latter Equation if it be multiplied by T, will produce $Ta + TX - TTX = Ta$,
5. Then if instead of Ta in the Equation in the second step, you take that which in the fourth step is found equal to Ta , the Equation in the second step will be converted into this, $2Ta + TX - TTX = 2Z$,
6. That is, $2a + X - TX$ into $T = 2Z$.
Which Equation gives this

CANON.

To the double of the last (to wit, the greatest) term, add the common difference; from the sum subtract the Product of the number of terms multiplied by the common difference:

difference: then multiply the Remainder by the number of terms; the Product shall be the double of the sum of all the terms; and consequently the half of that Product is the required sum of all the terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27, 31.

PROP. XV.

1. $\sum a, T, Z$ are given severally;
a is sought.

RESOLUTION.

2. By the Canon of Prop. 9. $2Z - Ta = a$;
3. Therefore multiplying each part of that Equation by T, this will arise, $2Z - Ta = Ta$,
4. And by transposition of $-Ta$ in the last Equation, this will arise, $2Z = Ta + Ta$,
5. Likewise by transposition of Ta , this Equation ariseth, $2Z - Ta = Ta$,
6. Therefore each part of the last Equation being divided by T, the value of a will be made known, viz. $2Z - a = a$.
Which Equation gives this

CANON.

Divide the double sum of all the terms by the number of terms, and from the Quotient subtract the last (to wit, the greatest) term; so shall the Remainder be the first and least term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVI.

1. $\sum a, T, Z$ are given severally;
X is sought.

RESOLUTION.

2. By the Canon of Prop. 14. $2a + X - TX$ into $T = 2Z$;
3. That is, $2Ta + TX - TTX = 2Z$;
4. Therefore by due transposition this Equation will arise, $2Ta - 2Z = TTX - TX$,
5. Therefore by dividing all in the last Equation by T, $2Ta - 2Z = X$,
TT - T, the value of X will be made known, viz. $2Ta - T = X$.
Which Equation gives this

CANON.

From the double Product of the multiplication of the number of terms by the greatest term, subtract the double of the sum of all the terms; divide the Remainder by the excess of the Square of the number of terms above the number of terms, so shall the Quotient be the common difference sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVII.

1. $\sum a, X, Z$ are given severally;
a is sought.

RESOLUTION.

2. By the Canon of Prop. 6. $2a - a = X$;
3. Therefore each part of that Equation being multiplied by $2Z - a - a$, there will arise, $2a - a = 2Z - a - a$,
4. Whence by equal addition of $Xa + Xa$ you will find, $2a - a + Xa + Xa = 2ZX$,
R Now

Now before known quantities can be separated from unknown in the last Equation, we must discover from the things given in the Proposition, Whether $aa + Xa$ be equal, greater, or less than $2ZX$? First therefore,

5. Suppose $aa + Xa = 2ZX$,
6. And then by setting $aa + Xa$ in the place of $2ZX$ in the Equation in the fourth step, there will arise,
7. Whence by subtracting $aa + Xa$ from each part, and by transposition of $-aa$, this Equation ariseth;
8. Which last Equation being divided by a , gives $X = a$.

From the premises ariseth this

CANON I.

9. When the sum of the Square of the last (to wit, the greatest) term and the Product of the multiplication of the said last term by the common difference of the terms is equal to the double of the Product made by the multiplication of the sum and common difference of the terms, then the said difference is equal to the first or least term sought. This Canon may be exemplified by the following Series of numbers in Arithmetical Progression continued:

2, 4, 6, 8, 10, 12, 14.

10. Secondly, suppose $aa + Xa < 2ZX$.
11. Then from the Equation in the fourth step, after due Reduction, there will arise,
12. In which last Equation all things are known but a , and the said Equation falls under the second of the three Forms in Sect. 1. Chap. 15. Therefore the value of a , to wit, the first (or least) term sought shall be given by the Canon in Sect. 8. of the same Chap. viz.

$$a = \frac{1}{2}X + \sqrt{\frac{1}{4}X^2 + Xa + \frac{1}{4}XX - 2ZX}.$$

From the tenth and twelfth steps ariseth

CANON II.

13. If the sum of the Square of the last (to wit, the greatest) term, and the Product of the multiplication of the said last term by the common difference of the terms, exceeds the double of the Product made by the multiplication of the sum and common difference of the terms; then to the sum first mentioned add the Square of half the common difference; from this sum subtract the double Product above mentioned, and extract the Square Root of the Remainder: lastly, add the said Square Root to half the common difference, so shall the Sum be the first (or least) term sought. This Canon may be exemplified by the following Progression:

3, 5, 7, 9, 11, 13.

14. Thirdly, suppose $aa + Xa > 2ZX$,
15. But in this third case, to the end a possible Equation may arise, this Determination is necessary, viz.
16. Then from the Equation in the fourth step by transposition of $aa + Xa$, this will arise,
17. In which last Equation all things are known but a , and the Equation falls under the last of the three Forms in Sect. 1. Chap. 15. Therefore the two values of a in that Equation shall be given by the Canon in Sect. 10. of the same Chap. viz.

$$a = \frac{1}{2}X + \sqrt{\frac{1}{4}X^2 + Xa + \frac{1}{4}XX - 2ZX}.$$

$$\text{Or, } a = \frac{1}{2}X - \sqrt{\frac{1}{4}X^2 + Xa + \frac{1}{4}XX - 2ZX}.$$

18. Whence it is manifest, that if in this third Case it happens that $aa + Xa + \frac{1}{4}XX = 2ZX$, then $a = \frac{1}{2}X$; that is to say, the first (or least) term sought shall be equal to half the given difference of the terms. But if in the said third Case it happens that

that $aa + Xa + \frac{1}{4}XX = 2ZX$, then there will be two unequal Roots or values of a , to wit, those above exprest, by either of which the Equation in the sixteenth step may be expounded; yet (as may easily be apprehended) only one of those values of a can be such a first (or least) term as will agree with the things given in the Proposition: But which of those two values of a is the least term sought, you may discover by the Proof formed thus, viz. First, by the help of one of those unequal values of a found out as above, together with the given last (to wit, the greatest) term and the given common difference of the terms, you may find out (by the Canon of the third Prop.) the number of terms, (which must always be a whole number,) and then by the same value of a , together with the said last term and the number of terms found by the Canon of Prop. 1. find out the sum of all the terms; then if this sum be equal to the sum given in the Propos. propos'd, that value of a , by which the Proof was made, is the least term sought: But if that Proof will not succeed, then the other value of a shall be the least term sought, as will be evident by the Proof made as before.

From the five last steps there will arise

CANON III.

19. When the sum of the Square of the last (to wit, the greatest) term, and the Product of the multiplication of the said last term by the common difference, is less than the double of the Product made by the multiplication of the sum and common difference of the terms; but the Aggregate of the sum first mentioned and the Square of half the common difference is not less than the said double Product; then from the said Aggregate subtract the said double Product and extract the Square Root of the Remainder: that done, add the said Square Root to half the common difference of the terms; and also subtract the said Square Root from the said half difference, so the Sum or else the Remainder, (viz. such of them, which by the Proof made according to the direction in the preceding eighteenth step will be found to agree with the things given in the Proposition,) shall be the first (or least) term sought.

This Canon may be exemplified by the two following Ranks of numbers in Arithmetical Progression continued:

I. 2, 5, 8, 11, 14, 17.
II. 2, 7, 12, 17, 22, 27.

PROP. XVIII.

1. . . . $\sum a, X, Z$ are given severally, T is required.

RESOLUTION.

2. By the Canon of Prop. 14. . . . $2aT + XT - XTT = 2Z$.
3. Therefore dividing every member of the said Equation by X , (because it is drawn into TT the highest degree of the number sought,) this following Equation will arise, viz.

$$\frac{2aT + XT}{X} - TT = \frac{2Z}{X}.$$

$$\text{That is, } \frac{2a + X}{X} T - TT = \frac{2Z}{X}.$$

4. In which all things are known but T , and the said Equation falls under the last of the three Forms in Sect. 1. Chap. 15. Therefore the two values of T will be made known by the Canon in Sect. 10. of the same Chap. viz.

$$T = \frac{a + \frac{1}{2}X}{X} + \sqrt{\frac{a^2 + aX + \frac{1}{4}XX - 2ZX}{X^2}}.$$

$$\text{Or, } T = \frac{a + \frac{1}{2}X}{X} - \sqrt{\frac{a^2 + aX + \frac{1}{4}XX - 2ZX}{X^2}}.$$

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S. But

5. But although the Equation in the third step may be expounded by either of the two Roots or values of T above express'd in the fourth step, yet only one of them can be the number of terms sought; but which of the said numbers, or values of T will solve the Proposition you may discover thus: First, If one of the two numbers or values of T before found out be a Fraction or a mixt number, that value cannot be the number of terms sought; for the number of terms in an Arithmetical Progression is always a whole number. Secondly, If both the values of T happen to be whole numbers, then the true number of terms sought may be discovered by this Proof: *viz.* First, by the help of one of those values of T in whole numbers, together with the given last (or greatest) term; and the given common difference, find out (by the Canon of *Prop. 13.*) the first (to wit, the least) term; and then by the same number T , together with the first and last terms, find out (by the Canon of *Prop. 1.*) the sum of all the terms; lastly, If the sum so found out be equal to the sum given in the Proposition propos'd, then that number or value of T by which the Proof was made shall be the true number of terms sought. But if the Proof will not succeed to find out a number equal to the sum first given, then the other value of T is the number of terms sought, which will be evident by the Proof made therewith in the same manner as before.

From the premisses there arises this

CANON.

6. From the Square of the sum of the last (to wit, the greatest) term, and half the common difference, subtract the double of the Product of the multiplication of the sum of all the terms by the common difference; divide the Remainder by the Square of the said difference, and extract the square Root of the Quotient. That done, add the said square Root to the Quotient which ariseth by dividing the sum of the last term and half the common difference by the difference it self, and also subtract the said square Root from the said Quotient, so the Summ, or else the Remainder (*viz.* such of them which according to the preceding fifth step will be found to agree with the things given in the *Propos.*) shall be the number of terms sought.

This Canon may be exemplified by the three following Progressions; in the first of which the greater of the two values of T (in the fourth step) is the number of terms sought; but in each of the two latter Progressions the lesser value of T is the number of terms sought.

I.	2, 7, 12, 17, 22, 27, 32.
II.	2, 5, 8, 11, 14, 17, 20.
III.	12, 20, 28, 36, 44, 52, 60.

PROP. XIX.

1. . . . $\left\{ \begin{array}{l} T, X, Z \text{ are given severally;} \\ a \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop. 10.* . . . $\frac{2Z - aTa}{TT - T} = X,$
3. Therefore multiplying each part of that Equation by $TT - T$, this will be produced, to wit, $2Z - 2Ta = TTX - TX,$
4. In which last Equation all things are known but a , whose value after due Reduction of that Equation will be found out, *viz.* $a = \frac{Z}{T} + \frac{1}{2}TX - \frac{1}{2}TX,$

Which in words gives this

CANON.

5. Divide the given sum of all the terms by the given number of terms; to the Quotient add half the given difference of the terms, and from the sum of that addition subtract half the Product of the multiplication of the said number of terms by the common difference; so shall the Remainder be the first (to wit, the least) term required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued:

2, 7, 12, 17, 22, 27, 32.

PROP.

PROP. XX.

1. . . . $\left\{ \begin{array}{l} T, X, Z \text{ are given severally;} \\ a \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop. 16.* . . . $\frac{2Ta - aZ}{TT - T} = X,$
3. Therefore multiplying each part of that Equation by $TT - T$, this will be produced, to wit, $2Ta - aZ = TTX - TX,$
4. In which last Equation all things are known but a , whose value, after due Reduction of that Equation, will be discovered, *viz.* $a = \frac{Z}{T} + \frac{1}{2}TX - \frac{1}{2}TX.$

Which in words gives this

CANON.

5. Divide the given sum of all the terms by the given number of terms; to the Quotient add half the Product of the multiplication of the number of terms by the common difference given, and from the sum of that Addition subtract half the said difference; the Remainder shall be the last (to wit, the greatest) term required.

This Canon may be exemplified by the following, or any other Rank of numbers in Arithmetical Progression continued:

2, 5, 8, 11, 14, 17, 20.

Questions to exercise some of the Canons of the preceding Propositions.

Quest. 1. Suppose 40 Stones be so placed in a straight line, that the first is distant from a Basket one yard, the second two, the third three, and the rest in the same excess; Now if some Foot-man undertakes to go from the Basket to fetch into it every Stone one after another, how many Yards must he go to perform that work? *Ans.* 1640 Yards.

Forasmuch as the Foot-man must go 2 Yards (to wit, one forwards, and the same backwards,) to fetch the first Stone into the Basket; 4 Yards for the second; 6 for the third, &c. here is an Arithmetical Progression continued whose first (or least) term is 2, the common difference of the terms is also 2, and the number of terms is 40; therefore the sum of all the terms, to wit, the number of Yards sought will be found 1640; by the Canon of the preceding eighth *Prop.*

Quest. 2. Two Foot-men, A and B , depart at the same time from London towards York, and travel in this manner, *viz.* A travelleth 8 (or 20) Miles every day; B travelleth 1 Mile the first day, 2 Miles the second day, 3 Miles the third day, and so forward, travelling every day one Mile more than in the day next preceding: the Question is, to find in how many days B will overtake A ? *Ans.* At the end of 15 days, found out by this following

RESOLUTION.

1. For the number of days that B had travelled when he overtook A , put x
2. Then to find how many Miles B had travelled when he overtook A , there is an Arithmetical Progression continued wherein the first and least term is 1, (to wit, 1 Mile which B travelled the first day,) also the common difference is 1, (for the Question saith that B travelled every day 1 Mile more than in the day next preceding,) and the number of terms is x , (which we assumed for the number of days that B had travelled when he overtook A ;) therefore the sum of all the terms (or number of Miles that B had travelled) will by the Canon of the preceding *Prop. 8.* be found to be,
3. And because A travelled 8 (or 20) Miles daily, and had travelled the same number of days as B when B overtook A , therefore 8 (or 20) multiplied by x produceth the number of Miles that A had then travelled, to wit,

4. But

4. But when *B* overtook *A*, each had travelled the same number of Miles; therefore the numbers found out in the two last steps must be equal the one to the other, *viz.* $\frac{1}{2}aa + \frac{1}{2}a = ca$
5. Which Equation after due Reduction gives $a = 2c - 1$
- Which in words is this

CANON.

From the double of the number of Miles that *A* travelled daily, subtract 1 (or Unity,) so shall the Remainder be the number of days fought.

Whence the number of days required will be found 15; for the double of 8 is 16, from which subtracting 1, the Remainder 15 is the number of days fought; *viz.* *B* will overtake *A* at the end of 15 days, as will be evident by

The Proof.

If 15 be the number of terms, and 1 the first (or least) term, as also the common difference of the terms of an Arithmetical Progression continued; the sum of all the terms will (per Canon of Prop. 8.) be found 120, being the number of Miles which *B* had travelled in 15 days, (according to the Progression of 1 Mile the first day, 2 Miles the second, 3 Miles the third, &c.) Also, *A* travelling 8 Miles every day, would in 15 days have travelled 120 Miles. Therefore the conditions in the Question are satisfied.

Quest. 3. A Merchant discharged a Debt of 1370 *l.* by several Payments made in this manner, *viz.* the first payment was $1\frac{1}{2}$ *l.* the second payment exceeded the first by $\frac{1}{2}$ *l.* the third exceeded the second by the same excess, and the rest of the payments in like manner. The Question is, to find how many Payments the Merchant made in discharging the said Debt? *Ans.* 120, found out thus:

There is given in the Question 12, to wit, the first and least term of an Arithmetical Progression continued; also $\frac{1}{2}$ the difference of the terms, and 1370 the sum of all the terms, to find the number of terms, which (by Canon 1 of the foregoing Prop. 12 of this Chap.) will be found 120.

Quest. 4. If a Debt of 1370 *l.* was discharged by several Payments made in such manner, that the second payment exceeded the first by $\frac{1}{2}$ *l.* the third the second, the fourth the third, &c. in the same excess, *viz.* every following payment exceeded the next preceding by $\frac{1}{2}$ *l.* and that the last payment was $21\frac{1}{2}$ *l.* What was the first (to wit, the least) Payment; and how many several Payments did the Debtor make? *Ans.* The first and least Payment was $1\frac{1}{2}$ *l.* (found out by Canon 2. of Prop. 17.) and the number of Payments was 120, found out by the Canon of Prop. 18.

Quest. 5. A Foot-man travelled 124 Miles in 8 days at this rate, *viz.* The second day's journey exceeded the first by 3 Miles, the third the second by 3 Miles, and so forward in that excess; How many Miles was his first day's journey, and how many his last? *Ans.* 5, and 26 Miles; found out by the Canons of Prop. 9 and 20.

Quest. 6. A Draper bought 20 Cloths for 20 Crowns a piece, and sold the first Cloth for a certain number of Crowns, the second for two Crowns more than the first, the third for two Crowns more than the second, and so by increasing the price of every following Cloth by two Crowns more than the next preceding Cloth, he sold the last Cloth for 41 Crowns. It is desired to find the number of Crowns for which he sold the first Cloth, and what he gained or lost by all the Cloths.

This Question implies an Arithmetical Progression, whose number of Terms is 20, the common difference of the Terms is 2; and the last Term is 41; therefore by the Canon of Prop. 13. of this Chap. the first and least Term will be found 3, and then by the Canon of Prop. 1. (or by the Canon of Prop. 14.) the sum of all the Terms will be found 440. Whence 'tis manifest that the Draper gained 40 Crowns by the 20 Cloths, for he bought them for 400 Crowns, and sold them for 440.

Quest. 7. One distributed 456 Pence among a certain number of poor persons in this manner, *viz.* To the first he gave 6 Pence, to the last 51 Pence; the number of Pence given to the second exceeded that given to the first, the third the second, and so forward to the last by an equal excess. The Question is, to find how many poor persons there were; and how many Pence every one between the first and last received? To

To solve this Question, an Arithmetical Progression must be conceived, whose first Term is 6; the last Term is 51; and the sum of all the Terms 456; then by the Canon of Prop. 5. the number of Terms will be found 16; and by the Canon of Prop. 6. the common difference of the Terms will be found 3; wherefore there were 16 poor persons; and if this Arithmetical Progression, to wit, 6, 9, 12, &c. be continued to the sixteenth Term inclusive, it will shew the number of Pence which every one of the poor persons received; and all those 16 Terms or numbers being added together, make the given sum 456.

Quest. 8. A Stationer sold 7 (or 8) Reams of Paper, the particular prices whereof were certain numbers of Shillings in Arithmetical Progression; the price of the second Ream, that is, of that next above the cheapest, was 8 (or 6) Shillings; and the price of the last or dearest Ream was 23 (or 6) Shillings: what was the price of each Ream?

RESOLUTION.

- For the price of the cheapest or first Ream put a
- Then because the price of the second Ream was 8 (or 6), therefore by subtracting a from 8, (or 6), there remains the common difference of the Terms of the Progression, *viz.* $8 - a$
- Then by the help of the least Term, the common difference of the Terms, and the number of Terms, seek (by the Canon of Prop. 7. of this Chap.) the last and greatest Term, which will be found $48 - 5a$
- Which greatest Term last found out must be equal to 23 (or 6), hence this Equation ariseth, *viz.* $48 - 5a = 23$; Or, $2a - 2a + 2b + b = c$

5. From which Equation after due Reduction this ariseth, *viz.*

$$a = 5 = \frac{2b - b - c}{2 - 2}$$

Which in words is this

CANON.

From the Product of the price of the second Ream of Paper (to wit, of that next above the cheapest;) multiplied by the number of Reams, subtract the sum of the prices of the second and last Reams, then divide the Remainder by the excess of the number of Reams above 2: so shall the Quotient be the price of the first (or cheapest) Ream.

Whence, by the help of the numbers given in the Question, these following numbers in Arithmetical Progression will be discovered, which solve the Question; *viz.* 5, 8, 11, 14, 17, 20, 23.

Quest. 9. One being asked what were the several ages of his five (or 4) Children, answered, that the age of the eldest exceeded that of the second by 2 (or 4) years; and by the same excess the second exceeded the third, the third the fourth, the fourth the fifth or youngest Child's age; and if the age of the eldest Child were multiplied by the age of the youngest it would produce 128 (or 6) years. It's desired to find out the age of every one of the five Children.

The numbers sought by the Question are in Arithmetical Progression.

RESOLUTION.

- For the age of the youngest Child (being the least Term of the Arithmetical Progression in the Question,) put a
- Then by the help of a , x and 2 , *viz.* the age of the youngest Child, the common difference of their ages, and the number of Children, seek (by the Canon of Prop. 7. of this Chap.) the age of the eldest, that is, the greatest Term of the Progression, so you will find $a + 1x - x$
- Therefore the Product of the multiplication of the first and last Terms of the Progression is $aa + 8a$

4. Which

4. Which Product must be equal to 128. (or 4,) the Product given in the Question; hence this Equation, viz. $aa + 8a = 128$; Or, $aa + 11a - xa = c$.
5. Wherefore, by resolving the last Equation according to the Canon in *Self. 6. Chap. 15* the value of a , that is, the age of the youngest Child will be discovered, viz.

$$a = 8 = \frac{\sqrt{11xx - 21xx + 11x - 4c} - 11x - x}{2}$$

Which in words is this

CANON:

From the Product of the number of Children multiplied into the common difference of their ages subtract the said difference, then to the Square of the Remainder add four times the Product of the age of the eldest Child multiplied into the age of the youngest, and extract the square Root of the sum of that Addition: then from the said square Root subtract the Product of the common difference of their Ages multiplied into the excess of the number of Children above Unity; so the half of the Remainder shall be the age of the youngest Child.

Whence these five numbers are discovered, viz. 8, 10, 12, 14, 16, which shew the number of years expressing the age of every one of the five Children: for the Product of the first and last numbers is 128; and the common difference is 2, as was required.

Quest. 10. If the sum of 6 (or 7) numbers or Terms in Arithmetical Progression be 48 (or 2,) and the Product of the common difference multiplied into the least Term be equal to the number of Terms, what are the numbers of that Progression?

RESOLUTION.

1. For the common difference of the Terms put a
2. Then according to the condition in the Question, if the number of Terms be divided by the common difference, the Quotient is the least Term; to wit, $\frac{6}{a}$
3. Now by the help of the common difference, the least Term, and the number of Terms, seek (by the eighth Prop. of this *Chap.*) the double sum of all the Terms, so you will find $30a + \frac{72}{a}$
4. Which double sum must be equal to twice 48, the sum given in the Question; hence this Equation ariseth, viz. $30a + \frac{72}{a} = 96$

$$\text{That is, } . . . 11a + \frac{21}{a} - 1a = 22$$

5. Which Equation duly reduced gives

$$\text{That is, } . . . \frac{22}{11 - 1} a - aa = \frac{21}{1 - 1}$$

6. Wherefore by resolving the last Equation according to the Canon in *Self. 10. Chap. 15* the two values of a will be found these, viz.

$$a = 2 = \frac{2 + \sqrt{22 + 2111 - 2111}}{11 - 1}$$

$$a = \frac{2}{5} = \frac{2 - \sqrt{22 + 2111 - 2111}}{11 - 1}$$

7. Each of which values of a , to wit, 2 and $\frac{2}{5}$ may be taken for the common difference sought. Then because 6 is prescribed in the Question for the Product of the least Term multiplied into the common difference, let 6 be divided by the said 2 and $\frac{2}{5}$ severally, and the Quotients 3 and 5 shall be the two least Terms of two Arithmetical Progressions, each of which will solve the Question: And therefore

The six numbers sought may be either these, $3, 5, 7, 9, 11, 13$; Or these, $5, 6\frac{2}{5}, 7\frac{4}{5}, 8\frac{2}{5}, 9\frac{4}{5}, 11$.

In each of which Progressions, the number of Terms is 6; the sum of all the Terms is 48; and the common difference multiplied by the least Term produceth the number of Terms. Which was prescribed in the Question.

The end of the First BOOK.



THE ELEMENTS OF THE ALGEBRAICAL ART.

BOOK II.

CHAP. I.

Concerning the Genesis or production of Powers, from Roots
Binomial, Trinomial, &c.

I. Shall take it for granted, that the Reader of this Second Book of Algebraical Elements is well exercis'd in the First; and therefore without making any repetition of what hath been there explain'd at large, I shall proceed to the handling of new matter in this my serious Art. First then, Forasmuch as the extraction of Roots is undoubtedly the hardest lesson in Vulgar Arithmetick, and the reason of the Rules delivered in most Treatises of Arithmetick for extracting the Square and Cubick Roots is known but to few practical Arithmeticians, I shall explain what our learned Divine, and famous Mathematician; Mr. William Oughtred, hath succinctly delivered upon this subject in the twelfth, thirteenth and fourteenth Chapters of his incomparable *Clavis Mathematicæ*; to which end, in this and the following second Chapters, I shall first shew the Genesis or production of Powers, from Roots binomial, trinomial, &c. and then, in the third and fourth Chapters, their Analysis; or the extraction of the Root or Side out of any given Power, whether it be express'd by number or letters.

II. If a Line or number be divided into any two parts, suppose a the greater; and e the lesser, these connected by the sign $+$ or $-$ do constitute a binomial Root, as $a + e$, or $a - e$; the latter of which some call a residual Root, because it imports a Remainder, viz. the difference of the two Names or parts of the Root. In like manner these Compound quantities, $a + b + c$; $a - b - c$; and the like, may be called trinomial Roots, because each of them consists of three Names or parts; and $a + b + c + d$ a quadrinomial Root, that is, a Root consisting of four parts; and so of others.

III. From a Root binomial, trinomial, &c. Algebraical Powers may be produced in like manner as from a simple Root, viz. by a continued multiplication of the Root into itself: As, for example, The binomial Root $a + e$ being multiplied by itself that is, $a + e$ by $a + e$, produceth $aa + 2ae + ee$ the Square of $a + e$. Again, If the Square $aa + 2ae + ee$ be multiplied by its Root $a + e$, the Product will be $aaa + 3aae + 3aee + eee$, which is the Cube of the Root $a + e$; and if the said Cube be multiplied by its Root $a + e$, it will produce the fourth Power, and so you may proceed to find a fifth; sixth, or what Power you please from the binomial Root $a + e$. But for the greater evidence view the following Operation;

S

Binomial

Binomial Root, .	$a + e$
Square, . . .	$aa + 2ae + ee$
Cube, . . .	$aaa + 3aae + 3aee + eee$
Biquadrate, . .	$aaaa + 4aaae + 6aaee + 4aeee + eeee$

After the same manner, if the Residual Root $a - e$ be multiplied by it self, the Product will be $aa - 2ae + ee$ the Square of $a - e$. Again, if the Square $aa - 2ae + ee$ be multiplied by its Root $a - e$, the Product will be $aaa - 3aae + 3aee - eee$, which is the Cube of the Root $a - e$. And so you may proceed to find a fourth, fifth, or what Power you please from the residual Root $a - e$; view the following Work.

Residual Root, .	$a - e$
Square, . . .	$aa - 2ae + ee$
Cube, . . .	$aaa - 3aae + 3aee - eee$
Biquadrate, . .	$aaaa - 4aaae + 6aaee - 4aeee + eeee$

By those two Examples it is manifest, that the Powers from the Residual Root $a - e$ differ only in the signs $+$ and $-$ from like Powers formed from the Binomial Root $a + e$; for in every Power of a residual Root, the signs prefix before the parts or members of the Power are alternately $+$ and $-$, viz. the greatest or first member is affirmative, the second negative, the third affirmative, the fourth negative, and so forwards; as you may see in the Cube of $a - e$, where aaa the greatest extreme member is affirmative, the next number in order being $-3aae$, is negative, the third member $+3aee$ is affirmative, and the last (to wit, the least) member $-eee$ is negative. But in every Power produced from a binomial Root whose parts are connected by $+$, as $a + e$, all the members of the Power are affirmative.

IV. If according to the construction in the last preceding Section, a Scale or Rank of Powers be formed from a binomial Root, as from $a + e$; the members of each Power to the tenth inclusive, will be such as you see in the following Table, where the last Powers are compendiously exprest according to *Cartesius* his way.

A Table

A Table of Powers, produced from the Binomial Root $a + e$.

The Root,	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$a + e$	aa	aaa	$aaaa$	$aaaaa$	$aaaaaa$	$aaaaaaa$	$aaaaaaaa$	a^9	a^{10}
$2ae$	$2ae$	$3aae$	$6aae$	$10aae$	$15aae$	$21aae$	$28aae$	$36aae$	$45aae$
ee	ee	$3aee$	$6aee$	$10aee$	$15aee$	$21aee$	$28aee$	$36aee$	$45aee$
		eee	$4eee$	$10eee$	$20eee$	$35eee$	$56eee$	$84eee$	$120eee$
			$eeee$	$10eeee$	$35eeee$	$84eeee$	$210eeee$	$450eeee$	$1000eeee$

V. By the foregoing Table it is evident, That the Square of $a + e$ consists of $aa + 2ae + ee$; which shews, that if a number be divided into any two parts, the Square of that number shall be equal to the Squares of the parts, and to twice the Product made by the multiplication of the parts one into the other. As, If 12 be divided into 10 and 2, which may be signified by a and e ; Then

The Square of 10 is	100	aa
The Product of 10 multiplied by 2 is 20, which doubled makes	40	$2ae$
The Square of 2 is	4	ee

Which three numbers, to wit, 100, 40 and 4 added together, make the Square of 12, viz. $144 = aa + 2ae + ee$.

In like manner, the said Table shews that the Cube or third Power of the binomial Root $a + e$ consists of the Cubes of the names or parts of the Root a and e , together with the triple of the solid Product made by the multiplication of the Square of the greater part a into the lesser part e , and the triple of the solid Product made by the multiplication of the greater part a into the Square of the lesser part e . This may be illustrated by numbers thus; Suppose 12 to be divided into 10 and 2, which may (as before) be represented by a and e , then the Cube of 12, or of $a + e$, will be equal to the sum of these four solid numbers, viz.

S z

The

The Cube of 10 is	1000	aaa
The Square of 10 is 100, which multiplied by 2 produceth 200, this tripled makes	600	3aac
Again, 10 multiplied by 4 the Square of 2, produceth 40, the triple whereof is	120	3acc
The Cube of 2 is	8	ccc

Which four numbers, viz. 1000, 600, 120 and 8, added together make the Cube of 12, (or $12 \times 12 \times 12$), that is, $1728 = aaa + 3aac + 3acc + ccc$.

After the same manner, the rest of the Powers in the Table might be exprest by words. Whence 'tis evident that this Literal method discovers many properties in Powers, which in Numerical calculations do lie in obscurity.

V1. Moreover, by a bare inspection into the said Table it may be perceived, that the number prefix to every one of the mean members of every Power produced from the binomial Root $a + e$, is composed of the two numbers prefix to the next superiour and inferior members of the next preceding Power: As for example, if you conceive the line upon which $3aac$ is set to be continued forth at length, it will pass between aa , that is, $1aa$, and $2ae$ in the foregoing second Power (or Square,) now I say that the number 3 prefix to aac is the sum of 1 and 2 the numbers prefix to aa and ae . Likewise the number 6 prefix to acc one of the members of the fourth power, is composed of 3 and 3 the numbers prefix to aa and acc in the third Power. Again, the number 15 prefix to $aaaae$ in the sixth Power, is the sum of 5 and 10 the numbers prefix to $aaaae$ and $aaaae$ in the fifth Power. Hence a Table may be made to shew what numbers are to be prefix to the mean members of every Power.

A	2	For the Square.							
3	3	For the Cube.							
4	6	4	For the fourth Power.						
5	10	10	5	For the fifth Power.					
6	15	20	15	6	For the sixth Power.				
7	21	35	35	21	7	For the seventh Power.			
8	28	56	70	56	28	8	For the eighth Power.		
9	36	84	126	126	84	36	9	For the ninth Power.	
10	45	120	210	252	210	120	45	10	For the tenth Power.

B

C

In this Table, the numbers from A to B, and likewise from A to C, do proceed from 2 in an Arithmetical Progression having 1 (to wit, Unity) for a common difference; and every one of the mean numbers standing between the same Term of each Progression, is composed of the two numbers which stand next above each mean number, respectively: As, 6 which stands between 4 and 4, is the sum of 3 and 3 which stand above and on each side of 6; likewise 10, which is set between 5 and 5, is the sum of 6 and 4 which stand above 10; and so of the rest. So that this Table may be easily continued farther at pleasure.

VII. Any Power of a Binomial or Residual Root exprest by letters, may without a continued multiplication of the Root into it self be easily formed by the following method, which is deduced from the premises, viz. Suppose the fifth Power of the binomial

binomial Root $a + e$ be desired; First, I write all the simple Powers of a , descending orderly from the fifth Power downwards to the Root a , as $aaaaa$, $aaaa$, aaa , aa , and a , as here you see in the first Columel: then to all those Powers, except the uppermost $aaaaa$, I joyn such simple Powers of e , that the sum of the Indices of both Powers may make 5, viz. To $aaaa$ I joyn e , to aaa , ee ; to aa , eee ; and to a , $eeee$; then I write $eeee$ underneath, so there are six distinct Members or Terms, every one of which consists of five dimensions, as you see in the second Columel; that done, by the Table in the foregoing Sect. 6. I find that the numbers 5, 10, 10 and 5 are to be prefix before the mean members of the fifth Power, and accordingly I set 5 before $aaaae$, 10 before $aaaae$, likewise 10 before $aaaae$, and 5 before $aaaae$; lastly, by prefixing $-$, or supposing it to be prefix before every one of the said five members, the fifth Power of the binomial Root $a + e$ is completed, as you see in the third Columel; and in every respect agrees with the fifth Power in the Table in the foregoing Sect. 4. But if the signs $+$ and $-$ be alternately prefix before the members of the said fifth Power, according to what hath been said at the latter end of Sect. 3: it will be the fifth Power of the Residual Root $a - e$.

VIII. Lastly, from a Root consisting of three, four, or any number of parts, the Square, Cube, or any higher Power of the Root may be produced by a continued multiplication of the Root into it self: As, the Trinomial Root $a + b + c$ being multiplied by it self, its Square will be found $aa + 2ab + 2ac + bb + 2bc + cc$; and this Square multiplied again by its Root $a + b + c$ produceth the Cube of the same Root, that is, $aaa + 3aab + 3aac + 3abb + 6abc + 3acc + bbb + 3bbc + 3bcc + ccc$. After the same manner Powers may be produced from a Root consisting of four, or any number of parts. And if the constitution of Powers exprest by letters be seriously considered, it will be some help to discover whether an Algebraick quantity consisting of more than three Members or Terms be a perfect Power or not, and also give some light to discover its Root.

CHAP. II.

Concerning the composition of Powers in numbers, from a Binomial Root.

Sect. I. Of the composition of a Square, from a number given for the Side or Root.

1. Suppose the Square of the Root 28 be desired; First write down the Root 28 in such manner that there may be space enough to set one figure between 2 and 8, and let a line be drawn under them; as also two downright lines, the one next after 2, and the other after 8, to the end the numbers which are to be found out may be orderly placed for Addition: then let the Root 28 be conceived to be divided into these two parts 20 and 8, and let a be put for the greater part, and e for the lesser. Now forasmuch as the Square of $a + e$ is $aa + 2ae + ee$, therefore the Square of 28, or of $20 + 8$ may be composed thus, viz. The Square of 20 is 400 (or aa); the double of 20 is 40 (or $2a$), which multiplied by 8 (or e) produceth 320 (that is $2ae$); and the Square of 8 is 64 (or ee); lastly, the said three numbers 400, 320 and 64 being set under one another

another, in such order, that units may stand under units, tens under tens, &c. and added together the sum makes 784 the Square of the Root 28, as may easily be proved by multiplying 28 into it self.

2. When the given number or Root whose Square is desired consists of three or more places, as 47803; First, the Square of the two foremost figures towards the left hand, that is, of 47, must be found out in like manner as before in the first Example, so there will be produced 2209 for the Square of 47, as you see in the following Example 2. Secondly, write 47 in a void place and annex a cypher to it, so it makes 470, this number must now be esteemed *a*, and 8 the next following character of the Root must be taken for *e*; and then according to these values of *a* and *e*, the numbers signified by *aa*, *2ae*, and *ee* being added together make 228484 for the Square of 478, (as you see here underneath.) Where observe, that to find the Square of 470, (that is, of *a*;) you need only annex two cyphers to 2209, which was before found for the Square of 47. Thirdly, annex a cypher to 478 (in a void place,) and it makes 4780 for a new value of *a*, and the next following character of the Root, to wit, 0, is the new value of *e*; then according to these values of *a* and *e*, the value of *aa* + *2ae* + *ee* is 22848400, to wit, *aa* only; for *e* = 0, and consequently *2ae* + *ee* = 0: so the said 22848400 is found for the Square of 4780. Lastly, by annexing a cypher to 4780 it makes 47800 for a new value of *a*, and 3 the last figure of the Root is the new value of *e*: Then according to these values of *a* and *e*, the sum of the numbers signified by *aa*, *2ae*, and *ee*, makes 2285126809, which is the Square of the said given Root 47803, as may easily be proved by multiplying the said Root by it self. Compare the following Example with the precedent directions.

Example 2. of Sect. I.

	4	7	8	0	3	Root proposed.
<i>a</i> = 40	16	00				<i>aa</i>
<i>e</i> = 7	5	50				<i>2ae</i>
		49				<i>ee</i>
<i>a</i> = 470	22	09	00			<i>aa</i>
<i>e</i> = 8		75	20			<i>2ae</i>
			64			<i>ee</i>
<i>a</i> = 4780	22	84	84	00		<i>aa</i>
<i>e</i> = 0				00		<i>2ae</i>
				00		<i>ee</i>
<i>a</i> = 47800	22	84	84	00	00	<i>aa</i>
<i>e</i> = 3				28	58	<i>2ae</i>
					09	<i>ee</i>
	22	85	12	68	09	Square required.

Sect. II. Of the composition of a Cube from a number given for the Side or Root.

1. Let the Cube of the Root 28 be desired: First, I write the Root 28 in such manner that there may be space enough to set two figures between 2 and 8; then having drawn a line under 28, and down-right lines as before in the Square, I conceive the Root 28 to be divided into 20 and 8, that is, *a* and *e*. Now forasmuch as the Cube of *a* + *e* is composed of these four members, viz. *aaa*, *3aae*, *3aee* and *eee*, (as appears by the Table in Sect. 4. Chap. 1.)

therefore the Cube of 20 + 8 (that is, of 28) may be composed thus; viz. First, the Cube of 20 is 8000, that is, *aaa*; secondly, the triple of the Square of 20 being multi-

	2	8	Root proposed.
<i>a</i> = 20	8	000	<i>aaa</i>
<i>e</i> = 8	9	600	<i>3aae</i>
	3	840	<i>3aee</i>
		512	<i>eee</i>
	21	952	Cube desired.

multiplied by 8 produceth 9600, (that is, *3aae*;) thirdly, the triple of 20 being multiplied by the Square of 8 produceth 3840, (that is, *3aee*;) fourthly, the Cube of 8 is 512, (that is, *eee*;) lastly, the said four numbers 8000, 9600, 3840, 512, being set under one another in such order that units may stand under units, tens under tens, &c. and added together make 21952 the Cube of the given Root 28.

2. When the given number or Root whose Cube is desired consists of three or more places, as 28503; First, the Cube of the two foremost figures, that is, of 28, must be found out in like manner as before in Example 1. so there will be produced 21952. Secondly, write 28 in a void place, and annexing a cypher to it, it makes 280, this number must now be esteemed *a*, and 5 the next following character of the Root must be taken for *e*; then according to these values of *a* and *e*, the numbers signified by *aaa*, *3aae*, *3aee* and *eee* being added together make 23149125 for the Cube of 285, (as you see in Example 2.) where observe, that to find the Cube of 280, that is, of *a*;) you need only annex three cyphers to 21952 which was before found for the Cube of 28. Thirdly, annex a cypher to 285 after it is set in a spare place, and it makes 2850 for a new value of *a*, and the next following Character of the Root, to wit, 0, is the new value of *e*: Then according to these values of *a* and *e*, the value of *aaa* + *3aae* + *3aee* + *eee* is 23149125000, that is, *aaa* only; for *e* = 0, and consequently *3aae* + *3aee* + *eee* = 0, so the said 23149125000 is found for the Cube of 2850. Lastly, by annexing a cypher to 2850 it makes 28500 for a new value of *a*, and 3 the last figure of the Root is the new value of *e*; then according to these values of *a* and *e*, the sum of the numbers signified by *aaa*, *3aae*, *3aee* and *eee* makes 23156436019527, which is the Cube of the given Root 28503, as may easily be proved by multiplying the said Root into it self cubically. Compare the following Example with the precedent directions.

Example 2. of Sect. II.

	2	8	5	0	3	Root proposed.	
$a = 20$	8	000				aaa	
$e = 8$	9	600				3aae	
	3	840				3aee	
		512				eee	
$a = 280$	21	952	000			aaa	
$e = 5$	11	760	000			3aae	
		21	000			3aee	
			125			eee	
$a = 2850$	23	149	125	000		aaa	
$e = 0$				000		3aae	
				000		3aee	
					000	eee	
$a = 28500$	23	149	125	000	000	aaa	
$e = 3$				7	310	3aae	
					769	3aee	
						27	eee
	23	156	436	019	527	Cube desired.	

Seçt. III. Of the composition of a Biquadrate, or the fourth Power, from a number given for the Root.

1. Let the Root 28 be proposed, and its Biquadrate or fourth Power desired. First, I write the Root 28 in such manner that there may be space enough to set three figures between 2 and 8; then having drawn a line under 28, and downright lines as in former Examples, I conceive the Root 28 to be divided into 20 and 8, that is, a and e ; now so far as the Biquadrate, or fourth Power produced from the Binomial Root $a + e$ is $aaaa + 4aaae + 6aaee + 4aece + eeee$, (as appears by the Table in *Seçt. 4. Chap. 1.*) therefore the fourth Power of 20 + 8, (that is, of 28)

	2	8	Root proposed.
$a = 20$	16	0000	aaaa
$e = 8$	25	6000	4aaae
	15	3600	6aaee
	4	0960	4aece
		4096	eeee
	61	4656	Biquadrate desired.

may be composed thus, *viz.* First, the fourth Power of 20 is 160000, (that is, $aaaa$;) secondly, four times the Cube of 20 being multiplied by 8 produceth 256000, that is, $4aaae$;) thirdly, six times the Square of 20 being multiplied by the Square of 8 produceth 153600, (that is, $6aaee$;) fourthly, four times 20 multiplied by the Cube of 8 produceth 40960, (that is, $4aece$;) fifthly, the fourth Power of 8 is 4096, (that is, $eeee$;) lastly, the sum of all the said five numbers, to wit, 160000, 256000, 153600, 40960, and 4096 makes 614656, which is the fourth Power of 28 the Root proposed; as will easily appear by the multiplication of 28 four times into it self.

2. When the given number or Root whose fourth Power is desired consists of three places, as 285; First, the fourth Power of the two foremost figures 28 must be found out in like manner as in Example 1. of this *Seçt.* so there will be produced 614656 for the fourth Power of 28. Secondly, let 28 be set in a void place, and annex a cypher to it, so it makes 280 which must now be esteemed a ; and 5 the next following character of the Root must be taken for e ; and then according to these values of a and e , the numbers signified by $aaaa$, $4aaae$, $6aaee$, $4aece$ and $eeee$ being added together make 6597500625, which is the fourth Power of the given Root 285, and the work will stand as you see in the following Example 2. After the same manner the work is to be continued when the given Root consists of more than three places, as is manifest by the following Example 3.

Example 2. of Seçt. III.

	2	8	5	Root proposed.
$a = 20$	16	0000		aaaa
$e = 8$	25	6000		4aaae
	15	3600		6aaee
	4	0960		4aece
		4096		eeee
$a = 280$	61	4656	0000	aaaa
$e = 5$	4	3904	0000	4aaae
	11	7600	0000	6aaee
	14	0000	0000	4aece
			625	eeee
	65	9750	0625	Biquadrate required.

Example

Example 3. of Seçt. III.

	2	8	0	5	Root proposed.
$a = 20$	16	0000			aaaa
$e = 8$	25	6000			4aaae
	15	3600			6aaee
	4	0960			4aece
		4096			eeee
$a = 280$	61	4656	0000		aaaa
$e = 0$			0000		4aaae
			0000		6aaee
			0000		4aece
			0000		eeee
$a = 2800$	61	4656	0000	0000	aaaa
$e = 5$	4	3904	0000	3000	4aaae
	11	7600	0000	0000	6aaee
	14	0000	0000	0000	4aece
			625		eeee
	61	9058	1740	0625	Biquadrate desired.

Seçt. IV. Of the composition of the fifth Power from a number given for its Root.

1. Let the Root 28 be proposed, and its fifth Power desired: First, let the Root 28 be written in such manner that there may be space enough to set four figures between 2 and 8; then having drawn a line under 28, and down-right lines as in the Examples of the precedent Sections, let 28 be conceived to be divided into 20 and 8, that is, a and e ; now so far as the fifth Power produced from the Binomial Root $a + e$ is $aaaaa + 5aaaae + 10aaaee + 10aaeee + 5aeceee + eeeeee$, (as is manifest by the Table in *Seçt. 4. Chap. 1.*) Therefore the fifth Power of 20 + 8, (that is, of 28) may be composed thus; First, the fifth Power of 20 is 3200000, (that is, $aaaaa$;) secondly, five times the fourth Power of 20 being multiplied by 8 produceth 6400000, (that is, $5aaaae$;) thirdly, ten times the Cube of 20 being multiplied by the Square of 8 produceth 5120000, (that is, $10aaaee$;) fourthly, ten times the Square of 20 multiplied by the Cube of 8 produceth 2048000, (that is, $10aaeee$;) fifthly, five times 20 multiplied by the fourth Power of 8 produceth 409600, (that is, $5aeceee$;) sixthly, the fifth power of 8 is 32768, (that is, $eeeee$;) lastly, the Summ of all those six numbers, *viz.* 3200000, 6400000, 5120000, 2048000, 409600, and 32768 makes 17210368, which is the fifth Power of 28 the Root proposed; as will easily appear by multiplying 28 five times into self.

2. When the given number or Root whose fifth Power is desired consists of three places, as 285; First, the fifth Power of the two foremost figures 28 must be found out in like manner as in Example 1. of this *Seçt.* so there will be produced 17210368 for the fifth Power of 28. Secondly, let 28 be set in a void place and annex a cypher to it, so it makes 280, which must now be esteemed a ; and 5 the next following character of the Root must be taken for e ; then according to these values of a and e , the numbers signified by $aaaaa$, $5aaaae$, $10aaaee$, $10aaeee$, $5aeceee$ and $eeeee$ being added together make 1880287678125, which is the fifth Power of the given Root 285, and the work will stand as you see in the following Example 2. Nor will the Operation be more difficult, (though more laborious,) to find the fifth Power of a number (or Root) consisting of four or more places,

T

Example

Example 2. of Sect. IV.

	2 ¹	8 ¹	5 ¹	Root proposed.
$a = 20$	32	00000		aaaaa
$e = 8$	64	00000		5aaaa
	51	20000		10aaaa
	20	48000		10aaaa
	4	09600		5aaaa
		32768		eeeee
$a = 280$	172	10368	00000	aaaaa
$e = 5$	15	36640	00000	5aaaa
		54880	00000	10aaaa
		980	00000	10aaaa
		875	000	5aaaa
			3125	eeeee
	188	02876	78125	Fifth Power desired.

By the precedent Rules and Examples of this Chapter, the ingenious Reader will easily apprehend, how to compose the sixth, seventh, or any higher Power, from a Root given in number and considered as a Binomial $a + e$, as before hath been directed. The main business consisting in a right understanding of the numbers signified by a and e , and in finding out the numbers answering to the members of the desired Power of $a + e$, according to the Table in Sect. 4. of the precedent Chap. 1.

CHAP. III.

Concerning the resolution of Powers express'd by numbers: or,
The extraction of all kinds of Roots out of Powers given in numbers.

Sect. I. Of the extraction of the Square Root out of a number given.

1. LET it be observed in general, That the Resolution of every Power given in number consists in a regular Subtraction of those numbers which are supposed to be added together in the composition of each Power respectively, according to the Rules of the last preceding Chapter, wherein I presuppose the Reader to be well exercis'd. And for the more ready extraction of any Root, it will be convenient to have in a readiness the respective Powers of the nine single figures; as, if the Square Root be desired, then the Squares of 1, 2, 3, 4, 5, 6, 7, 8, 9 will be useful, which Roots and Squares are express'd in the following Tabulet.

ROOTS.

1	2	3	4	5	6	7	8	9
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SQUARES.

1	4	9	16	25	36	49	64	81
---	---	---	----	----	----	----	----	----

2. When a whole number is proposed and its Square-root desired, the number proposed must be prepared for Extraction, by distributing it into parts or members after this manner; viz. First set a point over the first or Units place of the given number, then passing over the second place set another point over the third, also passing over the fourth place set another point over the fifth; and in that order, if there be more places in the given number, points are to be set, so that between every two points which stand next to one another there will be one place without any point over it. As, for example, if the Square Root of 119025 be desired, I set points as here you see, whereby the said number is distributed into three members, to wit, 11, 90, 25. In like manner if the Square Root of 784 be desired,

the points will stand as here you see, whereby the said 784 is distributed into two members 7 and 84. The points set as aforesaid shew the number of places that will be found in the Root; for if there be two points, 784 there will be two places in the Root; if three points, then the Root will consist of three places, &c. The points also shew what member of the number given belongs to the finding out of every single Character of the Root sought, as is evident by the Rules in Sect. 1. of the precedent Chap. 2. These things being premised as preparatory to the Extraction of the Square Root, I shall proceed to Examples.

Example 1.

3. Let it be required to extract the square Root of 784. By the preceding Rule 2. it is evident that the desired Root consists of two places. viz. of some number of Tens under 100, and of some number of Units under 10; which two numbers, (agreeable to the composition of a Square in Sect. 1. of the precedent Chap. 2.) may be represented by a and e , so that a and e signifies the Root sought; and consequently the Square of $a + e$, that is, $aa + 2ae + ee$ is equal to the proposed number 784: Now to find out the number of Tens, (that is, a), in the Root, (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Division, may be set next after the said crooked line, as also a down-right line next after each of the points, as here you see,) the first work in the Extraction is always to subtract the greatest square whole number contained in the first member towards the left hand, from the said member, and to write the Root of the said square number in the Quotient for the first single figure of the desired Root; so 4 being the greatest square contained in the first member 7, I subscribe 4 under 7, and set 2 the Root of the said 4 in the Quotient; then after a line is drawn under 4, I subtract 4 from 7, or, 400 from 784, and there remains the Resolvend 384, that is, that part of the given number 784, which is yet to be resolved. Now observe, that the said 2 in the Quotient, in respect of the next following unknown character of the Root, is really 20, which is the number signified by a in the Composition, and the Square of 20, to wit, 400, is aa , which being the first number found in the Composition, is the first number to be subtracted in the Resolution. Observe also, that the next single Character of the Root, whether it happen to be a figure or a cypher, is called e , which is yet unknown.

4. Then I proceed to find the value of e , that is, the single Character with this condition, that the sum of the numbers signified by $2ae$ and ee may not exceed the Resolvend 384, for from this number that sum must be subtracted. Now because (for the reason aforesaid) a is 20, therefore $2a$ is 40, which must be esteemed a Divisor; and set under the Resolvend, then I divide the said Resolvend 384 by 40, and find the Quotient 9 for the number e , provided it will answer the condition before-mentioned; and therefore I make trial (in a wast Paper) to see whether 9 will satisfy the said condition or not, in this manner, viz. If e be 9, and $2a$ 40, then consequently $2ae$ is 360, and ee is 81; therefore $2ae + ee = 441$, this ought to be subtracted from the Resolvend 384, but 441 exceeds 384, and therefore cannot be subtracted from it, so as to leave a real Remainder, whence I conclude, that e must be less than 9; and therefore I make trial with 8 in like manner as before with 9, viz. If $e = 8$, and $2a = 40$, then consequently $2ae = 320$, and $ee = 64$, therefore $2ae + ee = 384$, which may be subtracted from the Resolvend 384; wherefore I conclude that e , (that is, the figure which must follow 2 in the Quotient,) is 8, which I set in the Quotient: then I subscribe 320 and 64 (before found) under the Resolvend 384, (in such order that Units may stand under Units, and Tens under Tens,) and adding the said 320 and 64 together, the sum is 384, (which some Authors call the *Grossum*, others, the *Abblatitium*;) which subtracted from the Resolvend 384 leaves 0; so the whole Extraction is finished.

	784 (28
Subtract	400 aa
	384 Resolvend.
	40 $2a$ Divisor
	9
	320 $2ae$
	64 ee
Subtract	384 Abblatitium.
	000

and the Square Root of the given number 784 is found 28, which is the true Root sought, for 28 multiplied by 28 produceth 784.

NOTE 1.

The first Operation in the Extraction of the square Root, is always to subtract the greatest square whole number (that is, aa) contained in the first member (towards the left hand) of the given number, from the said member, and to set the Root of the said Square in the Quotient, (as hath been shewn in the third step,) which Root is the first figure of the Root sought. This work is no more repeated in the whole Extraction, but the work in the fourth step is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

After the first figure of the Root sought is known, and set in the Quotient, let it be written in a void place and multiplied by 10, (by annexing to the said first figure a cypher towards the right hand,) then is the Product to be taken for the value of a , in order to the finding out of the first Divisor. Also, when the first and second Characters of the Root are set in the Quotient, and there be yet another to come forth, then the number consisting of those two Characters with a Cypher annexed to them is to be taken for a new value of a , in order to the finding out of the second Divisor. Likewise, when the first, second and third Characters of the Root are set in the Quotient, and there be yet another to come forth, then the number consisting of those three Characters with a Cypher annexed to them, is to be taken for a new value of a , and so forwards, when there be more Characters in the Root. The reason of which work is manifest from the Composition of Powers in the precedent Chap. 2.

But the letter e represents every single unknown figure or cypher next following that part of the Root which is already discovered and set in the Quotient. This Note concerning the estimation of a and e is to be observed not only in the Extraction of the Square-root, but of any Root whatever.

NOTE 3.

After the number signified by a is found out by Note 2. the Divisor, which shews how to begin the trial in searching out the unknown single Character represented by e , is consequently known; for in the Resolution of every Power produced from the Binomial Root $a + e$, the Divisor consists of such Powers of a as are multiplied into the Powers of e , and because the Square-root of $a + e$ is $aa + 2ae + ee$, therefore in the extraction of the Square-root the Divisor is $2a$; so that when the number a is known, the Divisor $2a$ is consequently known.

NOTE 4.

When the Divisor is found out by Note 3. as also the Ablatium, (that is, the number to be subtracted,) which in the extraction of the Square-root is compos'd of $2ae$ and ee , the two numbers signified by $2ae$ and ee must each of them be set in such order under the present Resolvend, (that is, the number remaining to be resolved,) that Units may stand under Units, Tens under Tens, &c. to the end that the Ablatium may be rightly compos'd and subtracted from the present Resolvend.

NOTE 5.

When the Divisor is not contained once in the particular or present Resolvend, a cypher (to wit, 0,) must be set in the Quotient; and then the Resolvend must be augmented with the next member (towards the right hand) of the Power proposed, for a new particular Resolvend: Also a new Divisor must be found out by Note 3, and the like is to be done as often as the Divisor is not contained once in the particular Resolvend. The practice of these Notes will be shewn in the following Example.

Example

Example 2.

5. If the square Root of 2285126809 be desired, it will be found 47803 by the precedent Rules, and the work will stand as here you see underneath.

	22	85	12	68	09	(47803. Root.
Subtract	16					aa
	68	5				Resolvend.
$a = 40$		80				$2a$ Divisor.
$e = 7$		560				$2ae$
		49				ee
Subtract		609				Ablatium.
		76	12			Resolvend.
$a = 470$			940			$2a$ Divisor.
$e = 8$			7520			$2ae$
			54			ee
Subtract			7584			Ablatium.
			28	68		Resolvend.
$a = 4780$				9560		$2a$ Divisor.
$e = 0$				286809		Resolvend.
$a = 47800$					95600	$2a$ Divisor.
$e = 3$					286000	$2ae$
					9	ee
Subtract					286809	Ablatium.
					000000	

Explication of Example 2.

The first figure of the Root is 4, (by the foregoing Note 1.) whose Square 16, subtracted from 22 the first member towards the left hand of the number proposed leaves 6, to which the second member 85 being annexed, there ariseth 685 for the next Resolvend: Or to cause the same effect, suppose 0 to be annexed to 4 the first figure of the Root, and it makes 40, (that is, a), whose Square 1600 (or aa) subtracted from 2285 the two first Members of the number first proposed, leaves (as before) the Resolvend 685.

Then, the first figure of the Root being found 4, the value of a is 40, (by Note 2.) which doubled gives 80 for a Divisor to the Resolvend 685, (by Note 3.) and then by dividing and making trial as is directed in the precedent fourth step, the number e will be found 7 for the second figure of the Root, and consequently the numbers signified by $2ae$ and ee are 560 and 49, these being set orderly and added together (according to Note 4.) make the Ablatium 609, which subtracted from the said Resolvend 685, there remains 76, to which annexing 12 the third member of the number first proposed, it makes 7612 for a new Resolvend.

Again, the two foremost figures of the Root being found 47, the new value of a is 470, (by Note 2.) which doubled gives 940 for a Divisor to the said Resolvend 7612, (by Note 3.) then by dividing and making trial as is directed in the fourth step, the value of e is found 8 for the third figure of the Root, whence the numbers signified by $2ae$ and ee are 7520 and 64; these being set orderly and added together (according to Note 4.) make the Ablatium 7584, which subtracted from the Resolvend 7612 before-mentioned, leaves 28, to which annexing 68 the fourth member of the number first proposed, it makes 2868 for a new Resolvend.

Again, the three foremost figures of the Root being 478, the value of a is 4780, (by Note 2.) which doubled gives 9560 for a Divisor to the said Resolvend 2868, (by Note 3.) then by dividing as aforesaid the value of e is found 0; therefore, (according to Note 5.) 1 set 0 in the Quotient, and because in this case the Ablatium is also 0, the Resolvend 2868 from which the said Ablatium ought to be subtracted remains the same without alteration; therefore by annexing 09 the last member of the number first proposed,

to the said 2868 it makes 286809 for a new (and the last) *Resolvend*: lastly, by proceeding as before, the last figure of the Root will be found 3; so that the Square-root sought is 47803; for this multiplied by it self produceth 2285126809, the number whose Square-root was desired.

The premises may suffice to shew a perfect Method of extracting the square Root of a whole number having an exact Square Root, which I have explain'd at large, that the reason and certainty of the Rules might be apparent: But this Method may be contracted into more practical and compendious Rules, as I have shewn in the 32. *Chapt.* of Mr. Wingate's common Arithmetick.

6. But when a whole number hath not a Square root exactly expressible by any rational or true number, then to approach infinitely near the exact Root, first, payes of Cyphers, as 00, 0000, 000000, or 00000000, &c. are to be annexed to the number given, then esteeming the number given with the cyphers annexed to be one whole number, let its square Root be extracted according to the precedent (or other practical) Rules; that done, look how many points were set over the number first given, for so many of the formost places in the Quotient are to be taken for the Integers in the Root, and the rest following those Integers express the fractional part of the Root in decimal parts: As, for example, if the square Root of 12 be desired, I annex six cyphers to 12, thus, 12.000000, and then the square Root of 12.000000 being extracted, it will be found 3.464, that is, $3\frac{464}{1000}$; but because after the Extraction is finish'd there happens to be a Remainder, I conclude, that $3\frac{464}{1000}$ is less than the true Root, but $3\frac{464}{1000}$ is greater than it. So that by annexing three pairs of cyphers, you will not miss $\frac{1}{1000}$ part of an Unit of the true Root, and by annexing eight cyphers you will not want $\frac{1}{10000}$ part, and in that order you may approach as near as you please when you cannot obtain the exact square Root of a whole number given.

7. The square Root of a vulgar Fraction is found out thus, *viz.* First, if the Fraction be not in its least terms, let it be reduced to the least terms, then extract the square Root of the Numerator for a new Numerator, and the square Root of the Denominator for a new Denominator, so shall this new Fraction be the square Root of the Fraction proposed. As, for example, the square Root of $\frac{1}{4}$ is $\frac{1}{2}$; likewise, the square Root of $\frac{1}{16}$ is $\frac{1}{4}$.

But when either the Numerator or Denominator of a vulgar Fraction hath not a perfect square Root, then to find the square Root of that Fraction very near, first reduce the Fraction to a decimal Fraction whose Numerator may consist of an even number of places, *viz.* of two, four, or six places, &c. then extract the square Root of that decimal as if it were a whole number, and the Root that comes forth shall be a decimal Fraction expressing nearly the square Root of the Fraction proposed: As, for example, if the square Root of $\frac{1}{16}$ be desired, I first reduce it to this decimal Fraction .81250000, (for, as $\frac{1}{16} = \frac{1}{16} = .0625$, then by extracting the square Root of .81250000, as if it were a whole number, I find .9013, that is $\frac{9013}{10000}$, which is near the square Root of $\frac{1}{16}$ for it wants not $\frac{1}{10000}$ part of an Unit of the exact square Root of $\frac{1}{16}$.

8. Lastly, if the square Root of a mixt number be desired, first reduce it to an improper Fraction, and then extract the square Root of that improper Fraction as before; but if it hath not an exact square Root, then reduce the fractional part of the mixt number first proposed to a decimal Fraction of an even number of places, and after this decimal is annexed to the Integers of the mixt number, extract the square Root out of the whole, then so many points as were set over the Integers, so many of the formost places in the Quotient are to be taken for the Integers in the Root, and the rest express the fractional part of the Root in decimal parts: As, for example, the square Root of $3\frac{4}{16}$, that is, of $3\frac{1}{4}$, will be found $1\frac{7}{8}$, or 1.875 ; and the square Root of $7\frac{1}{2}$, that is, of 7.5 , will be found 2.708 , &c. that is, $2\frac{708}{1000}$, &c.

Sect.

Sect. II. Of the extraction of the Cubick Root out of a number given.

1. For the more ready extraction of the Cubick Root of a number given; the following Tabuler will be useful, which shews at first sight the Cubick Root of any cubical whole number less than 1000.

ROOTS.

1	2	3	4	5	6	7	8	9
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CUBES.

1	8	27	64	125	216	343	512	729
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2. When a whole number is proposed and its cubick Root desired, the number given must be prepared for Extraction, by distributing it into parts or members after this manner, *viz.* First, a point is to be set over the Units place of the given number, then passing over the second and third places towards the left hand, another point is to be set over the fourth place; also passing over the fifth and sixth places another point is to be set over the seventh place; and in that order as many points are to be set as the number propos'd will admit, and consequently between every two adjacent points there will be two places without points. So if the cubick Root of 1331 be desired, after points are set as is above directed, the said 1331 will be distributed into two members, to wit, 1 and 331. In like manner if the cubick Root of 21952 be required, the points will stand as you see in the Example, and the said 21952 will be distributed into two members 21 and 952; likewise this number 941192 being pointed in the same order will be distributed into the two members 941 and 192; and this number 23156436019527 into these five members, 23, 156, 436, 019, 527. The points shew the number of places that will be found in the Root; for so many points as there be, so many places will the Root consist of; they likewise shew what member of the number propos'd belongs to the extraction of every single Character of the Root sought.

3. The given number whose cubick Root is desired may be conceived to be produced from the cubical multiplication of the Binomial Root $a + e$, and then the said number will be compos'd of these four members or solid numbers, *viz.* aaa , $3aee$, $3ace$ and eee , (as appears by the third Power in the Table in *Sect.* 4. *Chap.* 1.) Now because the Resolution of a Cubick number, *viz.* the extraction of the cubick Root, is deducible from the steps of the Composition of a Cubick number from its Root, (for such numbers as are added in the Composition are to be subtracted in the Resolution,) respect must be had to *Sect.* 2. *Chap.* 2. of this Book.

Example 1.

4. Let it be required to extract the cubick Root of 21952. By the precedent second Rule it is evident that the desired Root consists of two places, *viz.* of some number of Tens under 100, and of some number of Unities under 10, which two numbers, (agreeable to the Composition of a Cube in *Sect.* 2. of the precedent *Chap.* 2.) may be represented by a and e , so that $a + e$ signifies the Root sought, and consequently the Cube of $a + e$; that is, $aaa + 3aee + 3ace + eee$ is equal to the given number 21952. Now to find out the number of Tens, (that is, a) in the Root, (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Division, may be set next after the said crooked line, as also a down-right line next after each of the points, as here you see.) The first work in the Extraction is always to subtract the greatest Cubick whole number contained in the first member towards the left hand, from the said member, and to write the Root of the said Cube-number in the Quotient for the first single figure of the desired cubick Root: So 8 being the greatest Cube contained in the first member 21, I subscribe 8 under 21, and set 2 the cubick Root of the said 8 in the Quotient, then after a line is drawn under 8, I subtract 8 from 21, or, 8000 from 21952, and there remains the *Resolvend* 13952, that is, that part of the proposed number 21952 which is yet to be resolv'd. Now observe, that the said 2 in the Quotient,

$$\begin{array}{r} 21952 \quad (2 \\ 8 \quad | \\ \hline 13952 \end{array}$$

in respect of the next following unknown character of the Root, is really 20, which is the number signified by *a* in the Composition, and the Cube of 20, to wit, 8000, is *aaa*, which being the first number found in the Composition, is first to be subtracted in the Resolution. Observe also, that the next single character of the Root whether it happen to be a figure or a cypher is called *e*, which is yet unknown.

5. Then I proceed to find the value of *e*, that is, the single Character with this condition, that the sum of the numbers signified by *3aae*, *3aee* and *eee* may not exceed the remaining *Resolvend* 13952, for from this number that sum must be subtracted. Now because (for the reason aforesaid) *a* is 20, therefore *3aa* = 1200, and *3a* = 60, then subscribing the said 1200 and 60 under the *Resolvend* 13952, (in such order that Units may stand under Units, and Tens under Tens, &c.) and adding them together, the sum is 1260, which must be esteemed a *Divisor*, and set under the *Resolvend*. Then by supposing I were to divide the said *Resolvend* 13952 by 1260, I find the Quotient exceeds 9, but *e* always represents a single figure or a cypher, and therefore it cannot exceed 9, wherefore I make trial with 9 (in a void place) to see whether it will answer the before-mentioned condition to which *e* is subject, in this manner, *viz.* Forasmuch as it was before found that *3aa* = 1200, and *3a* = 60, it will follow, if we suppose *e* = 9, that *3aae* = 10800, also *3aee* = 4860, and *eee* = 729; therefore *3aa* + *3aae* + *eee* = 16389: this ought to be subtracted from the *Resolvend* 13952, but 16389 exceeds 13952, and therefore cannot be really subtracted from it, whence I conclude that *e* must be less than 9; and therefore I make trial with 8 in like manner as before with 9; *viz.* Having before found that *3aa* = 1200, and *3a* = 60, it will follow, if we suppose *e* = 8, that *3aae* = 9600, also *3aee* = 3840, and *eee* = 512; therefore *3aa* + *3aae* + *eee* = 13952, which may be subtracted from the *Resolvend*

	21952	(28
Subtract	8	<i>aaa</i>
	13952	<i>Resolvend.</i>
<i>a</i> = 20	1200	<i>3aa</i>
	60	<i>3a</i>
	1260	<i>Divisor.</i>
<i>e</i> = 8	9600	<i>3aae</i>
	3840	<i>3aee</i>
	512	<i>eee</i>
	13952	<i>Ablatium.</i>
	000	

13952; wherefore I conclude that *e*, (that is, the figure which must follow 2 in the Quotient,) is 8, which I set in the Quotient: then I subscribe the three numbers before found, to wit, 9600, 3840 and 512 under the *Resolvend* 13952, (in such order that Units may stand under Units, Tens under Tens, &c.) and adding together the said three numbers so subscribed, their sum makes 13952, (the *Ablatium*;) which subtracted from the *Resolvend* 13952 leaves 0. So the Extraction is finish'd, and 28 is found to be the cubick Root of the proposed number 21952; for 28 multiplied into it self cubically, *viz.* $28 \times 28 \times 28$ produceth 21952.

NOTE 1.

The first Operation in the extraction of the cubick Root, is always to subtract the greatest Cubick whole number, (that is, *aaa*) contained in the first member (towards the left hand) of the given number, from the said member, and to set the Root of the said Cube-number in the Quotient; which Root is the first figure of the Root sought, as hath been shewn in the fourth step. This work is no more repeated in the whole Extraction, but the work in the fifth step is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The number signified by *a* is to be found out by Note 2. in *Self*. 1. of this *Chapt.* and then the *Divisor* for the finding of the unknown single Character represented by *e* is consequently known: For in the Resolution of every Power produced from the Binomial Root $a + e$, the *Divisor* consists of such Powers of *a* as are multiplied into the Powers of *e*, and because the Cube of $a + e$ is $aaa + 3aae + 3aee + eee$, therefore in the extraction of the cubick Root, the *Divisor* is compos'd of *3aa* and *3a*; so that when the number *a* is known, the *Divisor* $3aa + 3a$ is consequently known.

NOTE

NOTE 3.

When the *Divisor* is found out by the precedent Note 2. as also the *Ablatium*, which in the extraction of the cubick Root is compos'd of *3aae*, *3aee* and *eee*, the numbers signified by the said *3aae*, *3aee* and *eee* must each of them be set in such order under the particular or present *Resolvend*, that Units may stand under Units, Tens under Tens, &c. to the end the *Ablatium* may be rightly compos'd and subtracted from the *Resolvend*.

NOTE 4.

When the *Divisor* is not contained once in the particular or present *Resolvend*, a cypher (to wit, 0,) must be set in the Quotient; and then the *Resolvend* must be augmented with the next member (towards the right hand) of the Power proposed, for a new particular *Resolvend*: Also, a new *Divisor* must be found out by Note 2. of this *Self*. and the like is to be done as often as the *Divisor* is less than the *Resolvend*.

The practice of these Notes will be shewn in the following Example.

Example 2.

6. If the Cubick Root of 23156436019527 be desired, it will be found 28503 by the precedent Rules, and the work will stand as here you see underneath.

	23156436019527	(28503. Root.
Subtract	8	<i>aaa</i>
	15156	<i>Resolvend.</i>
<i>a</i> = 20	1200	<i>3aa</i>
	60	<i>3a</i>
	1260	<i>Divisor.</i>
<i>e</i> = 8	9600	<i>3aae</i>
	3840	<i>3aee</i>
	512	<i>eee</i>
Subtract	13952	<i>Ablatium.</i>
	1204436	<i>Resolvend.</i>
<i>a</i> = 280	235200	<i>3aa</i>
	840	<i>3a</i>
	236040	<i>Divisor.</i>
<i>e</i> = 5	1176000	<i>3aae</i>
	21000	<i>3aee</i>
	125	<i>eee</i>
Subtract	1197125	<i>Ablatium.</i>
	0007311019	<i>Resolvend.</i>
<i>a</i> = 2850	24367500	<i>3aa</i>
<i>e</i> = 0	8550	<i>3a</i>
	24376050	<i>Divisor.</i>
	7311019527	<i>Resolvend.</i>
<i>a</i> = 28500	2436750000	<i>3aa</i>
	8550030	<i>3a</i>
	2436835500	<i>Divisor.</i>
<i>e</i> = 3	7310250000	<i>3aae</i>
	769500300	<i>3aee</i>
	27	<i>eee</i>
Subtract	7311019527	<i>Ablatium.</i>
	0000000000	

Explication of Example 2.

The first figure of the Root is 2, (by Note 1.) whose Cube 8 subtracted from 23 the first member of the number propos'd leaves 15, to which the second member 156 being annexed,

in respect of the next following unknown character of the Root, is really 20, which is the number signified by a in the Composition, and the Cube of 20, to wit, 8000, is aaa , which being the first number found in the Composition, is first to be subtracted in the Resolution. Observe also, that the next single character of the Root whether it happen to be a figure or a cypher is called e , which is yet unknown.

5. Then I proceed to find the value of e , that is, the single Character with this condition, that the sum of the numbers signified by $3aa$, $3ae$ and eee may not exceed the remaining *Resolvend* 13952, for from this number that sum must be subtracted. Now because (for the reason aforesaid) a is 20, therefore $3aa = 1200$, and $3a = 60$; then subscribing the said 1200 and 60 under the *Resolvend* 13952, (in such order that Units may stand under Units, and Tens under Tens, &c.) and adding them together, the sum is 1260, which must be esteemed a *Divisor*, and set under the *Resolvend*. Then by supposing I were to divide the said *Resolvend* 13952 by 1260, I find the Quotient exceeds 9, but e always represents a single figure or a cypher, and therefore it cannot exceed 9; wherefore I make trial with 9 (in a void place) to see whether it will answer the before-mentioned condition to which e is subject, in this manner, *viz.* Forasmuch as it was before found that $3aa = 1200$, and $3a = 60$, it will follow, if we suppose

	21952	(28
Subtract	8	aaa
	13952	<i>Resolvend.</i>
$a = 20$	1200	$3aa$
	60	$3a$
	1260	<i>Divisor.</i>
$e = 8$	9600	$3aae$
	3840	$3aee$
	512	eee
	13952	<i>Ablatium.</i>
	000	

13952; wherefore I conclude that e , (that is, the figure which must follow 2 in the Quotient,) is 8, which I set in the Quotient: then I subscribe the three numbers before found, to wit, 9600, 3840 and 512 under the *Resolvend* 13952, (in such order that Units may stand under Units, Tens under Tens, &c.) and adding together the said three numbers so subscribed, their sum makes 13952, (the *Ablatium*,) which subtracted from the *Resolvend* 13952 leaves 0. So the Extraction is finish'd, and 28 is found to be the cubick Root of the proposed number 21952; for 28 multiplied into it self cubically, *viz.* $28 \times 28 \times 28$ produceth 21952.

NOTE 1.

The first Operation in the extraction of the cubick Root, is always to subtract the greatest Cubick whole number, (that is, aaa) contained in the first member (towards the left hand) of the given number, from the said member, and to set the Root of the said Cube-number in the Quotient; which Root is the first figure of the Root sought, as hath been shewn in the fourth step. This work is no more repeated in the whole Extraction, but the work in the fifth step is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The number signified by a is to be found out by Note 2. in *Self*. 1. of this *Chapt.* and then the *Divisor* for the finding of the unknown single Character represented by e is consequently known: For in the Resolution of every Power produced from the Binomial Root $a + e$, the *Divisor* consists of such Powers of a as are multiplied into the Powers of e ; and because the Cube of $a + e$ is $aaa + 3aae + 3aee + eee$, therefore in the extraction of the cubick Root, the *Divisor* is compos'd of $3aa$ and $3a$; so that when the number a is known, the *Divisor* $3aa + 3a$ is consequently known.

NOTE

NOTE 3.

When the *Divisor* is found out by the precedent Note 2. as also the *Ablatium*, which in the extraction of the cubick Root is compos'd of $3aae$, $3aee$ and eee ; the numbers signified by the said $3aae$, $3aee$ and eee must each of them be set in such order under the particular or present *Resolvend*, that Units may stand under Units, Tens under Tens, &c. to the end the *Ablatium* may be rightly compos'd and subtracted from the *Resolvend*.

NOTE 4.

When the *Divisor* is not contained once in the particular or present *Resolvend*, a cypher (to wit, 0,) must be set in the Quotient; and then the *Resolvend* must be augmented with the next member (towards the right hand) of the Power propos'd, for a new particular *Resolvend*: Also, a new *Divisor* must be found out by Note 2. of this *Self*. and the like is to be done as often as the *Divisor* is less than the *Resolvend*.

The practice of these Notes will be shewn in the following Example.

Example 2.

6. If the Cubick Root of 23156436019527 be desired, it will be found 28503 by the precedent Rules, and the work will stand as here, you see underneath.

	23156436019527	(28503. Root.
Subtract	8	aaa
	15156	<i>Resolvend.</i>
$a = 20$	1200	$3aa$
	60	$3a$
	1260	<i>Divisor.</i>
$e = 8$	9600	$3aae$
	3840	$3aee$
	512	eee
Subtract	13952	<i>Ablatium.</i>
	1204436	<i>Resolvend.</i>
$a = 280$	235200	$3aa$
	840	$3a$
	236040	<i>Divisor.</i>
$e = 5$	1176000	$3aae$
	21000	$3aee$
	125	eee
Subtract	1197125	<i>Ablatium.</i>
	0007311	<i>Resolvend.</i>
$a = 2850$	24367500	$3aa$
$e = 0$	8550	$3a$
	24376050	<i>Divisor.</i>
	7311019527	<i>Resolvend.</i>
$a = 28500$	2436750000	$3aa$
	855000	$3a$
	2436835500	<i>Divisor.</i>
$e = 3$	7311025000	$3aae$
	769500	$3aee$
	27	eee
Subtract	7311019527	<i>Ablatium.</i>
	0000000000	

Explication of Example 2.

The first figure of the Root is 2, (by Note 1.) whose Cube 8 subtracted from 23 the first member of the number propos'd leaves 15; to which the second member 156 being annexed,

annexed, there aritheth 15156 for the next *Resolvend*: Or, to cause the same effect, suppose 0 to be annexed to 2 the first figure of the Root and it makes 20, (that is, 4,) whose Cube 8000 (or *aaa*) subtracted from 23156 the two formost members of the number first propos'd, leaves (as before) the *Resolvend* 15156.

Then, the first figure of the Root being found 2, the value of *a* is 20, and the *Divisor* is 1260, (by Note 2.) and then by dividing and making tryal as is directed in the foregoing fifth step, the number *c* will be found 8 for the second figure of the Root, and consequently the numbers signified by *aaa*, *3aac* and *ccc*, are 9600, 3840 and 512, these being set orderly and added together (according to Note 3.) make the *Abolition* 13952, which subtracted from the *Resolvend* 15156 leaves 1204, to which annexing 436 the third member of the number first propos'd, it makes 1204436 for a new *Resolvend*. The rest of the Operation in Example 2. being but a repetition of what hath been directed for finding out the second figure of the Root, I shall leave it to the Learner's practice.

The precedent Rules and Notes in this *Self*. 2. for extracting the cubick Root of a whole number having an exact cubick Root are exprest at large, that the Reason of the work might be apparent; but this method may be contracted into more practical and compendious Rules, as I have shewn in the 33. *Chapt.* of Mr. Wingate's common Arithmetick.

7. But when a whole number hath not a cubick Root exactly expressible by any rational or true number, then to approach infinitely near the exact Root, first, ternaries of cyphers, *viz.* three, or six, or nine, or twelve, &c. cyphers are to be annexed to the whole number given; then esteeming the number given with the cyphers annexed to be one whole number, let its cubick Root be extracted by the precedent (or other practical) Rules: that done, look how many points were set over the number first given, for so many of the formost places in the Quotient are to be taken for the Integers in the Root, and the rest following those Integers expresse the fractional part of the Root in decimal parts: As, for example, if the cubick Root of 8302348 be desired, I annex six cyphers to 8302348, thus, 8302348.000000, and then the cubick Root of 8302348.000000 being extracted, it will be found 202.48, that is, 202 $\frac{48}{1000}$; but because after the extraction is finish'd there happens to be a Remainder, I conclude that 202 $\frac{48}{1000}$ is less than the true cubick Root sought, but 202 $\frac{48}{1000}$ is greater than it; so that by annexing six cyphers you will not miss $\frac{72}{1000}$ part of an Unit of the true Root, and by annexing nine cyphers you will not want $\frac{72}{1000}$ part; and in that order you may approach as near as you please when you cannot obtain the exact cubick Root of a whole number given.

8. The cubick Root of a vulgar Fraction is found out thus, *viz.* First, if the Fraction be not in its least terms, let it be reduced to the least terms; then extract the cubick Root of the Numerator for a new Numerator, and the cubick Root of the Denominator for a new Denominator, so shall this new Fraction be the cubick Root of the Fraction propos'd; as, for example, the cubick Root of $\frac{1}{27}$ is $\frac{1}{3}$, and the cubick Root of $\frac{8}{27}$ is $\frac{2}{3}$.

9. But when either the Numerator or Denominator of a vulgar Fraction hath not a perfect cubick Root, then to find the cubick Root of that Fraction very near, first reduce the Fraction to a decimal Fraction whose Numerator may consist of ternaries of places, *viz.* either of three, six, nine, or twelve, &c. places, and then extract the cubick Root of that decimal as if it were a whole number, and the Root that comes forth shall be a decimal Fraction expressing nearly the cubick Root of the vulgar Fraction propos'd: As, for example, if the cubick Root of $\frac{1}{2}$ be desired, I first reduce it to this decimal Fraction, .666666666666, and then by extracting the cubick Root of the said decimal as if it were a whole number, I find .8835, that is, $\frac{8835}{10000}$; which is near the cubick Root of $\frac{1}{2}$, for it wants not $\frac{72}{10000}$ part of an Unit of the exact cubick Root of $\frac{1}{2}$.

10. Lastly, if the cubick Root of a mixt number, that is, of a whole number with a Fraction in its least terms, be desired; first reduce it to an improper Fraction, and then extract the cubick Root of that improper Fraction in like manner as before in the eighth step, but if it hath not an exact cubick Root, then reduce the fractional part of the mixt number first propos'd to a decimal Fraction whose Numerator may consist of ternaries of places, and after this decimal is annexed to the Integers of the mixt number, extract the cubick Root out of the whole, then so many points as were set over the Integers, so many of the formost places in the Quotient are to be taken for the Integers in the Root, and the rest expresse the fractional part of the Root in decimal parts: As, for example, the cubick

cubick Root of $12\frac{1}{2}$, that is, of $\frac{25}{2}$, will be found $\frac{5}{2}$ or $2\frac{1}{2}$; and the cubick Root of $2\frac{1}{8}$, that is, of 3.37500000 , &c. will be found 1.334, &c. that is, $1\frac{1}{3}$, &c.

SECT. III. Of the extraction of the Biquadratic Root out of a number given.

1. The briefest way to extract the Root of a Biquadratic number, that is, of a number produced by the multiplication of some number or Root four times into it self, is first to extract the Square Root of the number propos'd, and then to extract the Square Root of that Root; as, for example, if the Root of the Biquadratic number, or fourth Power 256 be desired; First, the Square Root of 256 being extracted is 16; and then the Square Root of 16 is 4, which is the Root of the fourth Power 256: for $4 \times 4 \times 4 \times 4$ produceth 256. But my purpose being to explain the general Method for the extracting of all kinds of Roots, I shall upon that Foundation shew how to extract the Root of a Biquadratic number.

2. For the more ready extraction of the biquadratic Root, the following Tablett will be useful, which shews at first sight the Root of any Biquadratic whole number under 10000.

Roots.	1	2	3	4	5	6	7	8	9
Fourth Powers.	1	16	81	256	625	1296	2401	4096	6561

3. When a whole number is propos'd, and it is desired to extract the Biquadratic Root of that number, set points over the given number in this manner; *viz.* first set a point over the Units place, then passing over the three next places towards the left hand set another point over the fifth place, and in that order as many points are to be set as the given number will admit, that there may be three places between every two adjacent points. So if the biquadratic Root of 614656 be desired, after points are set as is above directed, the said 614656 will be distributed into two members, to wit; 61 and 4656: in like manner this number 6597500625 being pointed in the same order will be distributed into these three members; 65, 9750, and 0625. The points shew the number of places that will be found in the Root, as also what member of the number propos'd belongs to the extraction of every single Character of the Root sought.

4. The given number whose Biquadratic Root is desired may be conceived to be produced from the multiplication of the Binomial Root $a \pm b$ four times into it self, and then the said number will be composed of these five members or numbers, *viz.* *aaaa*, *4aabb*, *6aabb*, *4aabb*, *bbbb*, (as is manifest by the fourth Power in the Table in *Self*. 4. *Chapt.* 1. of this Book.) Now because the Resolution of a Biquadratic number, *viz.* the extraction of the Biquadratic Root is deducible from the steps of the Composition of a Biquadratic number from its Root, (for such numbers as are added in the Composition are to be subtracted in the Resolution,) respect must be had to *Self*. 3. *Chapt.* 1. of this Book.

Example.

5. Let it be required to extract the Biquadratic Root of 614656. After the number given is prepared by punctuations as before is directed, I seek in the Tablett in the precedent second step of this *Self*. 3. for the greatest Biquadratic whole number contained in 61 the first member (towards the left hand) of the number propos'd, and finding it to be 16, I subscribe 16 under 61, and write 2 the Root of the said fourth Power 16 in the Quotient, for the first figure of the Root sought; then after a line is drawn under 16, I subtract 16 from 61, or 160000 from 614656, and there remains to be resolv'd 454656.

$$\begin{array}{r} 614656 \text{ (2)} \\ 16 \overline{) 614656} \\ 454656 \end{array}$$

The *Divisor* for the finding out of e , that is, every Character which is to follow 2 the first figure of the Root, is always in the extraction of the Biquadratick Root composed

Subtract	61	4656	(28. Root.
	16		aaaa
	45	4656	Resolvend.
$a = 20$	3	2000	4aaa
		2400	6aa
		80	4a
	3	4480	Divisor.
$e = 8$	25	6000	4aeee
	15	3600	6aeee
	4	0960	4aeee
		4096	eeee
Subtract	45	4656	Ablatitium.
		00000	

Ablatitium that doth not exceed the *Resolvend* 454656, viz. I suppose $e = 9$; then because a was before found 20, the *Ablatitium* which in the extraction of the Biquadratick Root is always composed of 4aaaa, 6aeee, 4aeee and eeee, will exceed the *Resolvend*, from which it ought to be subtracted: But if $e = 8$, then the *Ablatitium* will be equal to the *Resolvend*, and consequently that being subtracted from this, there will remain 0, wherefore I set 8 in the Quotient, and conclude that the Biquadratick Root of the given number 614656 is 28; for $28 \times 28 \times 28 \times 28$ produceth 614656.

SECT. IV. Of the extraction of the Root of the fifth Power given in number.

1. For the more ready extraction of the Root of any fifth Power given in number, this Tabulet will be useful, which shews at first sight the fifth Powers of every single figure, and consequently any fifth Power in number under 100000 being given, its Root is hereby discovered.

Roots.	5th Powers.
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16807
8	32768
9	59049

2. When a whole number is given for a fifth Power and its Root desired, that is, such a number which being multiplied five times into it self will produce the given number, it must be prepared for extraction by punctuations in this manner; viz. First let a point be set over the Units place of the given number, then passing over the four next places towards the left hand, set another point over the sixth place; and in that order as many points are to be set as the given number will admit, that there may be four places between every two adjacent points. So if the Root of the fifth Power

17210368 be desired, after points are set as is above directed, the said 17210368 will be distributed into two members, to wit, 172 and 10368: in like manner this number 1880287678125 will be distributed into these three members, 188, 02876 and 78125. The points (as before hath been said) shew the number of places that will be found in the Root, as also what member of the number given belongs to the extraction of every single character of the Root sought.

3. Every

3. Every number considered as a fifth Power may be conceived to be produced from the multiplication of the Binomial Root $a - e$ five times into it self, and then the said number will be composed of these six members or numbers, viz. aaaaa, 5aaaae, 10aaaae, 10aaaae, 5aaaae and eeeee; (as is manifest by the fifth Power in the Table in *SECT. 4. Chap. 1.* of this Book.) Now because the Resolution of the fifth Power, viz. the extraction of $\sqrt[5]{}$ out of a given number, is deducible from the steps of the Composition of a fifth Power from its Root given in number 1, (for such numbers as are added in the Composition are to be subtracted in the Resolution,) the Learner must be exercis'd in *SECT. 4. Chap. 2.* of this Book.

Example.

Let it be required to extract $\sqrt[5]{}$ out of 17210368, viz. to find a Root or number which being multiplied five times into it self will produce 17210368: After the given number is prepared by punctuations as before is directed, I seek in the Tabulet in the first step of this Section 4. for the greatest fifth Power contained in 172 the first member (towards the left hand) of the given number, and finding it to be 32, I subscribe 32 under 172, and write 2 the Root of the said fifth Power 32 in the Quotient for the first figure of the Root sought; then after having drawn a line under 32, I subtract 32 from 172, or, 3200000 from 17210368, and there remains to be resolv'd 14010368.

Then to discover the *Divisor*, which shews how to begin the trial in the finding out of e , that is, every Character (whether it be a figure or cypher) which is to follow the first figure of the Root, I take such Powers of a as are multiplied into the Powers of e in the fifth Power produced from $a - e$, viz. 5aaaa, 10aaaa, 10aaa and 5a, so the sum of these four numbers make the *Divisor*: and because the first figure of the Root is found 2, and consequently (by Note 2. in *SECT. 1.* of this *Chapt.*) the number signified by a is 20, therefore the sum of the numbers signified by 5aaaa, 10aaaa, 10aaa and 5a is 884100, which is the *Divisor*; then supposing I were to divide the *Resolvend* 14010368 by the *Divisor* 884100, I find the Quotient exceeds 9, but in regard e always represents a single figure or a cypher it cannot exceed 9, therefore I make trial (in a void place) with 9, to see whether it will constitute an *Ablatitium* that doth not exceed the *Resolvend* 14010368, viz. I suppose $e = 9$, then because a was found 20, the *Ablatitium* 5aaaae + 10aaaae + 10aaaae + 5aaaae exceeds the *Resolvend* from which it ought to be subtracted; But if $e = 8$, then the *Ablatitium* will be equal to the *Resolvend*, and consequently that being subtracted from this, there will remain 0, wherefore I set 8 in the Quotient, so 28 is found to be the $\sqrt[5]{}$ of the given number 17210368; for $28 \times 28 \times 28 \times 28 \times 28$ produceth 17210368. Compare the following work with the precedent Rules of *SECT. 4.*

Subtract	172	10368	(28. Root.
	32	00000	aaaaa
	140	10368	Resolvend.
	8	00000	5aaaa
$a = 20$		80000	10aaaa
		4000	10aaa
		100	5a
		84100	Divisor.
$e = 8$	62	00000	5aaaae
	51	20000	10aaaae
	20	48000	10aaaae
		409600	5aaaae
		32768	eeeeee
	140	10368	Ablatitium.
	000	00000	

By the precedent Rules and Examples of this *Chapt.* the ingenious Reader will easily perceive how to extend this general method to the extraction of the Roots of all kinds

of

of Powers in numbers, viz. of the sixth, seventh, eighth, &c. Powers; as also to find out the Roots infinitely near of such Powers as have not Roots exactly expressible by any rational or true number.

CHAP. IV.

Concerning the extraction of Roots out of Powers expressed by Letters.

I. IN a series or Stale of Powers produced from a Root, suppose from a , as in this series, $a, aa, aaa, aaaa, aaaaa, a^6, a^7, a^8$, &c. those Powers only whose Indices are even numbers are Squares; as $aa, aaaa, a^6, a^8$, &c. (whose Indices are 2, 4, 6, 8, &c.) are Squares: and those Powers only whose Indices are divisible by 3, are Cubes, as $aaa, aaaaa, a^6, a^9$, &c. (whose Indices are 3, 6, 9, &c.) are Cubes. Therefore every Power whose Index is a Prime number greater than 3, as $aaaaa, a^7, a^{11}$, &c. (whose Indices are 5, 7, 11, &c.) is neither a Square nor a Cube. But every Power whose Index is divisible by 6, as a^6, a^{12}, a^{18} , &c. is both a Square and a Cube, because the Index is divisible both by 2 and by 3.

II. If a Simple quantity be expressed by the same letter repeated an even number of times, the square Root thereof is easily extracted; for the Root must be such that its Index may be the half of the Index of the Quantity proposed: As, \sqrt{aa} (that is, the square Root of aa) is a ; for 1, the Index of the Root a , is the half of 2 the Index of the Square aa : in like manner \sqrt{aaaa} is aa , whose Index 2 is the half of 4 the Index of the Square $aaaa$: again, \sqrt{aaaaaa} is aaa , whose Index 3 is the half of 6 the Index of the Square $aaaaaa$.

III. And with the like facility you may extract the Cubick Root of a Simple quantity which is expressed by one and the same letter repeated such a number of times as is divisible by 3; for the Cubick Root must be such that its Index may be $\frac{1}{3}$ of the Index of the Cube proposed: as, $\sqrt[3]{(3)aaa}$ (that is, the cubick Root of the Quantity aaa ,) is a , whose Index 1 is $\frac{1}{3}$ of 3 the Index of aaa : in like manner $\sqrt[3]{(3)a^6}$ is a^2 , whose Index 2 is $\frac{1}{3}$ of 6 the Index of the Cube a^6 .

IV. If the Index of a simple Power expressed by the same letter be some Prime number greater than 3, as 5, 7, 11, &c. then neither $\sqrt{(2)}$, nor $\sqrt{(3)}$, nor any other Root except that denoted by such Index or Prime number can be exactly extracted out of the said Power: so no Root can be exactly extracted out of $aaaaa$ or a^7 , but $\sqrt{(5)}$, which is a ; nor any Root out of a^7 but $\sqrt{(7)}$ which is also a . But when the Root cannot be exactly extracted, the sign of the Root is to be prefixed to the Quantity; as to express the square Root of $aaaaa$ or a^7 , I write \sqrt{aaaaa} or $\sqrt{a^7}$: likewise I express the cubick Root of a^5 , thus, $\sqrt[3]{(3)a^5}$; and $\sqrt[3]{(4)a^7}$; and so of others.

V. When some Power of an unknown simple Root a is found equal to some known number, and the Index of that unknown Power is not a Prime number, then the value of the Root a in number may oftentimes be discovered by two or more extractions more easily than by one single extraction of a Root out of the said unknown number. As, for Example;

If there be proposed or found out $aaaaaa = 729$
Then to find out the value of a , you need not extract the $\sqrt{(6)}$ of 729 by the general method before delivered in Chap. 3. but first by that method extract the square Root of 729, and then by Sect. 2. of this Chap. the square Root of $aaaaaa$, so those two Roots compared give this Equation, viz. $aa = 27$
Lastly, by extracting the cubick Root of each part of the last Equation, the value of a the Root sought is discovered, viz. $a = 3$
Or

Or thus,

First, by extracting the cubick Root of each part of the Equation proposed, there ariseth $aa = 9$
And then by extracting the square Root of each part of the last Equation, the same value of the Root a is found out as before, to wit, $a = 3$
In like manner, if $a^9 = 19683$
First, by extracting the cubick Root, it gives $a^3 = 27$
And again, by extracting the Cubick root of that Root the Root a is made known, viz. $a = 3$

VI. When two or more Squares, Cubes, or other Powers expressed by different letters be multiplied one into another, then if the Root of each Power, viz. the square Root if they be Squares, or the cubick Root if they be Cubes, &c. be extracted, the Product made by the multiplication of these Roots one into another shall be a like Root of the Power or Product first given: As, for example, \sqrt{aabb} is ab , which is the Product of the square Roots of aa and bb ; likewise, $\sqrt[3]{(3)aaabbb}$ is aab , which is the Product of the cubick Roots of aaa and bbb .

Again, \sqrt{aabbcc} is abc , which is the Product of the square Roots of aa , bb and cc ; in like manner, $\sqrt[3]{(3)27aaabbb}$ is $3aab$, which is the Product of the cubick Roots of 27, aaa and bbb ; and $\sqrt[3]{(3)6aabbcc}$ is $4abc$, which is the Product of the square Roots of 16, aa , bb and cc . The like is to be understood of others.

But if the square Root of $5aabb$ be desired, because 5 is not a Square, the said Root is to be expressed either thus, $\sqrt{5aabb}$; or thus, $\sqrt{5} \times \sqrt{aabb}$; or thus, $\sqrt{5} \sqrt{aabb}$. In like manner, to denote the square Root of $aaabbb$ I write $\sqrt{a^3b^3}$; and to signify the cubick Root of $aabb$, I write $\sqrt[3]{(3)aabb}$, but the cubick Root of $5aaabbb$ may be written either thus, $\sqrt[3]{(3)5aabb}$; or thus, $\sqrt[3]{(3)5} \sqrt[3]{aabb}$; or thus, $\sqrt[3]{(3)5} \sqrt[3]{aabb}$.

Concerning the extraction of Roots out of Compound quantities expressed by Letters.

VII. Before the Learner enters upon the Extraction of Roots out of Compound Squares, Cubes or other Powers expressed by letters, he ought to be well exercised in the eighth and ninth Chapters of my first Book of *Algebraical Elements*; as also in the foregoing first, second and third Chapters of this Book; and in the precedent Rules of this Chapter; all which well understood will render the following Rules and Examples of this Chapter very plain and easie.

VIII. Rules for the extraction of Square Roots out of Compound Quantities expressed by Letters.

Rule 1. Set the particular members of the compound Algebraick quantity whose square Root is required, in such order, that one of the simple Squares may stand outermost towards the left hand, and next after the same such other member or members wherein you find the same letter or letters as are in the said simple Square, then the square Root of the said simple Square is to be set in the Quotient for the first member of the compound Root sought, and the Square it self is the first Quantity to be subtracted from the compound Quantity proposed. This is the first work, which is no more to be repeated in the whole Extraction.

Rule 2. Double the Root before set in the Quotient for the first Divisor; likewise, to find every following Divisor, double every simple Quantity that stands in the Quotient, and take the sum of the Products for the Divisor.

Rule 3. When the Divisor is found out, divide out the first simple Quantity (towards the left hand) in the Resolvend, by the first simple Quantity in the Divisor; and set that which comes forth next after the member or members of the Root sought that was before found out.

Rule 4. After the first simple Square is subtracted (according to Rule 1.) then every following Abstrahenda, that is, the sum of the Quantities to be subtracted from the respective Resolvend, must be composed of these two Products, viz. the Product made by the multiplication of the whole Divisor by that particular Quantity which was last set in the Quotient, and the Square of the same simple Quantity.

The practice of these Rules will be apparent in the following Examples.

Example

Example 1.

Let it be required to extract the square Root of $aa - 2ab + bb$.
First, I extract the square Root of aa , and it is a , which I set in the Quotient; then multiplying a by it self, I set the Product aa under, and subtract it from the quantity first proposed, and there remains $2ab + bb$. This is the first work which answers to Rule 1, and is no more to be repeated.

The Square,	$aa - 2ab + bb$	($a + b$. The Root.
Subtract	aa	
Remainder,	$-2ab + bb$	
Divisor,	$+2a$	
Subtract	$-2ab + bb$	
Remainder,	$0 \quad 0$	

Secondly, the Divisor (according to Rule 2.) is $2a$, which I set under $2ab$.

Thirdly, I divide $-2ab$ by the Divisor $+2a$, and the Quotient is $-b$, which I set next after a , (the particular Root before found out,) according to Rule 3.

Fourthly, I multiply the Divisor $+2a$ by $-b$, (that was last set in the Quotient,) and the Product is $-2ab$, to which adding $+bb$, (the Square of $-b$), the sum is $-2ab + bb$, which (according to Rule 4.) I set under and subtract from the Resolvend $-2ab + bb$, and there remains 0: so the Extraction being finish'd, the Root sought is found $a - b$; for if it be multiplied by it self it produceth $aa - 2ab + bb$ the quantity first proposed.

Note. By what I have said in the eighth and ninth Chapters of my first Book of *Algebraical Elements*, 'tis easie to discover at first sight whether a Compound Algebraick Quantity consisting of three Terms be a perfect Square or not, and if a Square what its Root is. Nevertheless, in this first Example I have express'd the work at large according to the four Rules before given, that the like Operation may the more easily be perceived in the following Examples.

Example 2.

If the square Root of $aa - 2ab + 2ac - 2bc + bb + cc$ be desired, it will be found $a - b + c$, by the precedent Rules, and the work stands as here you see underneath.

The Square,	$aa - 2ab + 2ac - 2bc + bb + cc$	($a - b + c$. The Root.
Subtract	aa	
Remainder,	$-2ab + 2ac - 2bc + bb + cc$	
Divisor,	$+2a$	
Subtract	$-2ab + 2ac$	
Remainder,	$-2bc + bb + cc$	
Divisor,	$+2a - 2b$	
Subtract	$-2ac + 2bc + cc$	
Remainder	$0 \quad 0 \quad 0$	

Example 3.

In like manner the square Root of $64abb + 32abc - 144ab + 4cc - 36c + 81$ will be found $8ab + 2c - 9$; as is manifest by the following Operation.

The Square,	$64abb + 32abc - 144ab + 4cc - 36c + 81$	($8ab + 2c - 9$
Subtract	$64abb$	
Remainder,	$+32abc - 144ab + 4cc - 36c + 81$	
Divisor,	$+16ab$	
Subtract	$+32abc$	
Remainder,	$-144ab - 36c + 81$	
Divisor,	$+16ab + 4c$	
Subtract	$-144ab - 36c + 81$	
Remainder,	$0 \quad 0 \quad 0$	

Example 4.

Example 4.

Again, the square Root of $ddd + 2addb + 3adbb + 2dbbb + bbbb$ will be found $dd + db + bb$, and the Extraction stands thus;

The Square,	$d^4 + 2d^3b + 3d^2b^2 + 2db^3 + b^4$	($dd + db + bb$.
Subtract	d^4	
Remainder,	$+2d^3b + 3d^2b^2 + 2db^3 + b^4$	
Divisor,	$+2d^2$	
Subtract	$+2d^3b + d^4$	
Remainder,	$+2d^2b^2 + 2db^3 + b^4$	
Divisor,	$+2d^2 + 2db$	
Subtract	$+2d^2b^2 + 2db^3 + b^4$	
Remainder,	$0 \quad 0 \quad 0$	

IX. Rules for the extraction of Cubick Roots out of Compound Quantities express'd by Letters.

Rule 1. Set the particular members or parts of the Compound Algebraick Quantity whose cubick Root is required, in such order, that one of the simple Cubes may stand outermost towards the left hand, and next after the same such other members wherein you find the same letter or letters as are in the said simple Cube; then the cubick Root of the said simple Cube is to be set in the Quotient for the first member of the Root sought, and the simple Cube it self is the first Quantity to be subtracted from the compound Quantity proposed. This is the first work, and no more to be repeated in the whole Extraction.

Rule 2. The first Divisor must be composed of the triple of the Square of the Root before set in the Quotient, (which triple Square I call the first part of the Divisor,) and the triple of the same Root, (which triple Root I call the latter part of the Divisor;) likewise, every following Divisor must be composed of the triple of the Square of the sum of all the simple Quantities or parts of the Root already found out and set in the Quotient, and of the triple of the same sum.

Rule 3. When the Divisor is found out, divide only the first simple Quantity (towards the left hand) in the Resolvend, by the first simple Quantity in the Divisor, and set that which comes forth in the Quotient, next after the member or members of the Root sought before found out.

Rule 4. After the first simple Cube is subtracted, (according to Rule 1.) then every following *Ablatissimum*, that is, the sum of the quantities to be subtracted from the Resolvend, must be composed of these three Products, viz. First, the Product made by the multiplication of the first part of the Divisor, (to wit, the triple Square mentioned in Rule 2.) by the simple Quantity last set in the Quotient; secondly, the Product made by the multiplication of the latter part of the Divisor (to wit, the triple Root or sum mentioned in Rule 2.) by the Square of the same simple Quantity; and thirdly, the Cube of the said simple Quantity last set in the Quotient.

The practice of these Rules will appear in the following Examples.

Example 1.

Let it be required to extract the Cubick Root out of $aaa + 3aae + 3aee + eee$.
First, beginning at the left hand, I extract the cubick Root of aaa , and it is a , which I set in the Quotient, then multiplying the said Root a cubically it makes aaa , which I subtract from the Compound quantity first proposed for Extraction, and there remains to be resolv'd $+3aae + 3aee + eee$. This is the first work, which answers to Rule 1. and is no more to be repeated in the whole Extraction.

The Cube,	$aaa + 3aae + 3aee + eee$	($a + e$. The Root.
Subtract	aaa	
Remainder,	$+3aae + 3aee + eee$	
Divisor,	$+3aa + 3a$	
Subtract	$+3aae + 3aee + eee$	
Remainder,	$0 \quad 0 \quad 0$	

Secondly,

Secondly, I seek a Divisor thus, viz. to $\div 3aa$, which is the triple of aa the Square of the Root a , I add $\div 3a$ the triple of the said Root a , and the sum $3aa + 3a$ is the Divisor, which I set underneath the remaining *Resolvend*, according to Rule 2.

Thirdly, according to Rule 3. I divide $\div 3aa$ by $\div 3aa$, and it gives $\div e$, which I set in the Quotient next after a .

Fourthly, to find out the *Abblatium* (or quantity next to be subtracted) I make a threefold Multiplication. viz. First, I multiply $\div 3aa$ (the first part of the Divisor) by $\div e$ the Root last set in the Quotient, and the Product is $\div 3aee$; secondly, I multiply $\div 3a$ the latter part of the Divisor by $\div ee$ the Square of the said Root e , and the Product is $\div 3aee$; thirdly, I multiply the said Root e cubically, and the Product is $\div eee$; lastly, I subtract the sum of the said three Products from the *Resolvend*, and there remains 0. So the Extraction is finish'd, and $a + e$ is the true Cubick Root sought; for if it be multiplied cubically, it will produce $aaa + 3aae + 3aee + eee$ first proposed.

Example 2.

In like manner, the cubick Root extracted out of $125aaa + 225aae + 135aee + 27eee$ is $5a + 3e$, and the work stands thus:

The Cube,	$125aaa + 225aae + 135aee + 27eee$	($5a + 3e$ Root,
Subtract	$125aaa$	
Remainder,	$+ 225aae + 135aee + 27eee$	
Divisor,	$+ 75aa + 15a$	
Subtract	$+ 225aae + 135aee + 27eee$	
Remainder,	0 0 0	

Example 3.

So the cubick Root of $27a^3 - 54a^2 + 171a - 188a^3 + 285aa - 150a + 125$ will be found $3aa - 2a + 5$, and the Operation stands thus:

Cube,	$27a^3 - 54a^2 + 171a - 188a^3 + 285aa - 150a + 125$	The Root.
Subtr.	$27a^3$	($3aa - 2a + 5$)
Rem.	$- 54a^2 + 171a - 188a^3 + 285aa - 150a + 125$	
Divisor,	$+ 27a^2 + 9a^2$	
Subtr.	$- 54a^2 + 36a^2 - 8a^3$	
Rem.	$+ 135a^2 - 180a^3 + 285aa - 150a + 125$	
Divisor,	$\left\{ \begin{array}{l} + 27a^2 + 36a^2 + 12aa \\ + 9aa - 6a \end{array} \right.$	
Add these	$\left\{ \begin{array}{l} + 135a^2 - 180a^3 + 60aa \\ + 225aa - 150a \end{array} \right.$	
Subtract	$+ 135a^2 - 180a^3 + 285aa - 150a + 125$	
	0 0 0 0 0	

If there be occasion to extract the Root of the fourth, fifth, or other higher Compound Power, the Divisors and Ablatious quantities may be drawn out of the Table in *Self. 4 Chap. 1.* of this Book.

X. Concerning the extraction of Roots out of Algebraical Fractions:

1. Forasmuch as in the extraction of Roots out of Fractions, the Root of the Numerator and Denominator being severally extracted gives the Root sought; therefore if the square Root of $\frac{aabb}{cc}$ be to be extracted, I write $\frac{ab}{c}$ for the Root sought; for the square Root of the Numerator $aabb$ is ab , and the square Root of the Denominator cc is c .

In like manner if the square Root of $\frac{aaaa - 2aabb + bbbb}{aa + 4ab + 4bb}$ be desired; by extracting the square Root out of the Numerator and Denominator, there ariseth $\frac{aa - bb}{a + 2b}$ for the Root sought.

And for the same reason the cubick Root of this Fraction, $\frac{27a^3 - 54a^2 + 171a - 188a^3 + 285aa - 150a + 125}{aaa - 9aa + 27a - 27}$ will be $\frac{3aa - 2a + 5}{a - 3}$, which is found by extracting the cubick Root out of the Numerator and Denominator of the Fraction proposed.

2. But if the Root sought cannot be extracted out of the Numerator and Denominator, then the radical sign $\sqrt{\quad}$ with the Index of the Power, if it exceed a Square, is to be prefixt to the Fraction; as, to denote the square Root of $\frac{ccxx}{4bb}$, that is, of $\frac{ccxx - 4abb}{4bb}$,

I write $\sqrt{\frac{ccxx - 4abb}{4bb}}$, or, (because the square Root of the Denominator is $2b$),

the square Root of the quantity proposed may be exprest thus, $\frac{\sqrt{ccxx - 4abb}}{2b}$; likewise,

the cubick Root of $\frac{a^3b^3}{aa + bb}$ may be designed either thus, $\sqrt[3]{\frac{a^3b^3}{aa + bb}}$, or (be-

cause the Numerator is a Cube) thus, $\frac{ab}{\sqrt[3]{(3aa + bb)}}$. The like is to be understood in expresting the irrational Roots of higher Powers.

CHAP. V.

Concerning Geometrical Proportion.

1. THE Difference of two numbers is found out by Subtraction, but the *Ratio*, Reason or *Habitude* of one number to another is discovered by dividing the Antecedent (or first number) by the Consequent, (or second number;) for the Quotient denominates the *Ratio*, Reason, or (as some call it) the Proportion, which the Antecedent hath to the Consequent: As if 6 be compared to 2, then $\frac{6}{2}$, that is $\frac{3}{1}$, or 3, shews that 6 hath triple Reason to 2; viz. 6 contains 2 thrice, or 6 is in proportion to 2 as 3 to 1: but if 2 be compared to 6, then $\frac{2}{6}$ or $\frac{1}{3}$ shews that 2 hath subtriple Reason to 6; viz. 2 is $\frac{1}{3}$ part of 6, or 2 is in proportion to 6 as 1 to 3.

In like manner if the quantity a be compared to the quantity b , then $\frac{a}{b}$ expresteth the

Ratio or Reason of a to b ; and $\frac{b}{a}$ shews the Reason of b to a .

Note, that the Reason of two numbers or quantities ought to be exprest by the smallest Terms or Quantities than can possibly be found to exprest that Reason: So the Denominator of the Reason of 16 to 12 is $\frac{4}{3}$, where 16 and 12 are first reduced to the smallest Terms 4 and 3, (by dividing the said 16 and 12 severally by their greatest common Divisor 4,) and then dividing the Antecedent 4 by the Consequent 3, the Quotient $\frac{4}{3}$ expresteth the Reason or Proportion of 16 to 12; viz. 16 is to 12 as 4 to 3. In like manner the Reason of bb to ba , or of bbb to bba is $\frac{b}{a}$.

II. Quantities which proceed by equal Differences are said to be in a continued Arithmetical Progression, (as hath been shewn in *Chap. 17. Book 1.* of my *Algebraical Elements*;) but quantities which proceed by equal Reasons, (or Proportions,) are said to be in a continued Geometrical Progression or Proportion: So these numbers 2, 6, 18,

54, 16: are continually proportional, because the Reason (or Proportion) of the first to the second is equal to the Reason of the second to the third, also of the third to the fourth, and so forward; viz. $\frac{5}{2}$ (or $\frac{1}{2}$) = $\frac{16}{5}$ = $\frac{5}{2}$ = $\frac{16}{5}$; or backward, $\frac{2}{5}$ = $\frac{5}{16}$ = $\frac{2}{5}$ = $\frac{5}{16}$; In like manner if these quantities a, b, c, d, e be such, that $\frac{a}{b} = \frac{b}{c}$

= $\frac{c}{d}$ = $\frac{d}{e}$; or backwards, if $\frac{e}{d} = \frac{d}{c} = \frac{c}{b} = \frac{b}{a}$, then those quantities are continually proportional; viz. as the first is in proportion to the second, so is the second to the third, the third to the fourth, &c.

But if there be four such quantities that the Reason (or Proportion) of the first to the second, is equal to the Reason of the third to the fourth; but the Reason of the second to the third, is not equal to the Reason of the first to the second, then those quantities are said to be in Geometrical Proportion discontinued or interrupted; such are these four numbers, 2 . 6 :: 12 . 36; for $\frac{2}{6}$ (or $\frac{1}{3}$) = $\frac{12}{36}$, but $\frac{6}{12}$ (or $\frac{1}{2}$) is not equal to $\frac{2}{6}$ or $\frac{1}{3}$. In like manner if a, b, c, d be such quantities that $\frac{a}{b} = \frac{c}{d}$, but $\frac{b}{c}$ is not

equal to $\frac{a}{b}$, (or $\frac{c}{d}$;) then are a, b, c, d discontinued Proportionals.

III. If three quantities be Proportionals, the Product made by the mutual multiplication of the Extremes is equal to the Square of the Mean; as,

If there be proposed $\left. \begin{array}{l} 18, 6, 2 \\ a, b, c \end{array} \right\} \frac{a}{b} = \frac{b}{c} = 3$
Then this Equation ensueth, $ac = bb = 36$
For since by supposition $a : b :: b : c$
It follows (by Sect. 1. and 2.) that $\frac{a}{b} = \frac{b}{c} = 3$

Whence by multiplying each part by c , $\frac{ac}{b} = b = 6$

And by multiplying each part of the last Equation by b , it produceth $ac = bb = 36$
Which was to be proved.

IV. If four quantities be Proportionals, whether they be continual or discontinued, the Product made by the mutual multiplication of the extremes is equal to the Product of the means; and consequently if the Product of the means be divided by either of the extremes, the Quotient is the other extreme. As, for example,

Let four discontinued Proportionals be proposed $\left. \begin{array}{l} d : c :: b : a \\ 12 : 4 :: 15 : 5 \end{array} \right\} \frac{d}{c} = \frac{b}{a} = 3$
Then by the foregoing Sect. 2. $\frac{d}{c} = \frac{b}{a} = 3$
And by multiplying each part of that Equation by a , this $\frac{da}{c} = b = 15$
is produced, viz. $\frac{da}{c} = b = 15$
And by multiplying each part of the last Equation by c , the $da = cb = 60$
first part of the Proposition is manifest, viz. $da = cb = 60$
And, by dividing each part by d , there ariseth $a = \frac{cb}{d} = 5$

Which last Equation being compared with the four Proportionals first proposed, doth shew, that if three quantities d, c, b be given, to find such a fourth as shall have the same proportion to b as c hath to d , then the Product of the second and third terms, to wit, cb , being divided by the first term d will give the fourth Proportional sought, which is the very Operation in the Rule of Three direct.

V. If three quantities a, b, c be Proportionals, and the first and second, to wit, a and b be given severally, the third is also given; for by Sect. 3. of this Chap. $ac = bb$, whence by dividing each part by a there ariseth $c = \frac{bb}{a}$, which shews, that if the Square of the mean or second term be divided by the first, the Quotient is the third Proportional; hence a, b , and $\frac{bb}{a}$ are continual Proportionals. In like manner if three quantities in continual proportion be given severally, and a fourth Proportional be defined,

the Square of the third term divided by the second gives the fourth: as if there be given these three, $a, b, \frac{bb}{a}$; then by dividing the Square of $\frac{bb}{a}$ to wit, $\frac{bbbb}{aa}$ by b , the Quo-

tient $\frac{bbb}{aa}$ shall be the fourth continual proportional: hence $a, b, \frac{bb}{a}, \frac{bbb}{aa}$ are continual proportionals. Likewise if the Square of the fourth continual proportional be divided by the third, the Quotient will be the fifth; so to those four continual proportionals, this fifth will be found, to wit, $\frac{bbbb}{aaa}$; and so forwards infinitely. Therefore,

VI. If numbers, how many soever, be continually proportionals, and the least term be esteemed the first, that next greater than the least the second; and so forwards; then the second term is produced by the multiplication of the first into the Reason of the second term to the first, the third term is produced by the multiplication of the first into the Square of the same Reason, the fourth term is produced by the multiplication of the first into the Cube of the same Reason; and in like manner every following term is produced by the multiplication of the first into such a Power of the Reason of the second term to the first as hath fewer dimensions by one than the number of terms hath unities: as in these following six continual proportionals, to wit,

$$\begin{array}{l} a, b, \frac{bb}{a}, \frac{bbb}{aa}, \frac{bbbb}{aaa}, \frac{bbbbb}{aaaa} \quad \div \\ 2, 6, 18, 54, 162, 486 \quad \div \end{array}$$

Supposing a to be the first and least term, the second term b is equal to the Product of the first term a into $\frac{b}{a}$, to wit, the Reason of the second term to the first; also the

third term $\frac{bb}{a}$ is produced by the multiplication of the first term a into the Square of the same Reason, that is into $\frac{bb}{aa}$; and the fourth term $\frac{bbb}{aa}$ is produced by the

multiplication of the first term a into the Cube of the same Reason, that is, into $\frac{bbb}{aaa}$;

and the fifth term $\frac{bbbb}{aaa}$ is produced by the multiplication of the first term a into the fourth

Power of the same Reason, that is into $\frac{bbbb}{aaaa}$; and so forwards.

But if the greatest term be esteemed the first, that next less than the greatest the second, and so downwards; then the second term is equal to the Quotient that ariseth by dividing the first (or greatest) term by the Reason of the first to the second; the third is equal to the Quotient that ariseth by dividing the first term by the Square of the same Reason; the fourth term is equal to the Quotient that ariseth by dividing the first term by the Cube of the same Reason; and in like manner every term beneath the greatest is equal to the Quotient that ariseth by dividing the first (or greatest term) by such a Power of the Reason of the greatest to the greatest but one, (or second term,) as hath fewer dimensions by one than the number of terms: as in these following six continual proportionals, to wit,

$$\begin{array}{l} \frac{bbbbb}{aaaa}, \frac{bbbb}{aaa}, \frac{bbb}{aa}, \frac{bb}{a}, b, a \quad \div \\ 486, 162, 54, 18, 6, 2 \quad \div \end{array}$$

If we suppose $\frac{bbbbb}{aaaa}$ to be the first and greatest term, then the second term $\frac{bbbb}{aaa}$ is equal to the Quotient of the first term $\frac{bbbbb}{aaaa}$ divided by $\frac{b}{a}$, to wit, by the Reason of the first term to the second; also the third term $\frac{bbb}{aa}$ is equal to the Quotient of the first term $\frac{bbbbb}{aaaa}$ divided by $\frac{bb}{aa}$, that is, by the Square of the Reason $\frac{b}{a}$; and the fourth

fourth term $\frac{bb}{a}$ is equal to the Quotient of the first term $\frac{bbbb}{aaaa}$ divided by $\frac{bbb}{aaa}$ the Cube of the same Reason: and so of the rest.

VII. From the last preceding Section it follows, that if in a Series or Rank of numbers which are in continual proportion; the first term, the second term and the number of terms be given severally, the last term shall be also given by this Rule; viz. First, (according to the Note in Sect. 1. of this Chap.) find out the smallest numbers that may shew the Reason of the greater of the two given terms to the less; then esteeming the said Reason as a Root, find such a Power thereof whose Index may be equal to the given multitude of terms less by unity, which Power multiplied by the first term, when the first term is less than the second, gives the last, to wit, the greatest term. But when the first term is greater than the second, then the first term divided by the said Power gives the last term; as if there be given a and b the first and second of six numbers in continual proportion, and that b is greater than a ; then the Reason of b to a is $\frac{b}{a}$, (by Sect. 1. of this

Chapt.) and the fifth Power of $\frac{b}{a}$ is $\frac{bbbb}{aaaa}$, this multiplied by the first term a produceth $\frac{bbbbb}{aaaa}$ which is the sixth Proportional sought, (as is evident by Sect. 6.) but if the first term a be greater than the second term b , then the Reason of a to b is $\frac{a}{b}$, whose fifth Power is $\frac{aaaa}{bbbbb}$, by which if you divide the first term a , the Quotient is the sixth term $\frac{bbbbb}{aaaa}$.

This Rule may be exemplified by these four following Ranks of numbers in continual proportion.

2	6	18	54	162	468	÷÷
3072	768	192	48	12	3	÷÷
2	3	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	÷÷
$\frac{16}{81}$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	2	÷÷

VIII. If there be given two Integers expressing a Reason in the least terms, and it be desired to find out a given multitude of continual Proportionals in the same Reason, and that all the terms may be Integers; First, to those two Integers, or first and second Proportionals given, find out (by Sect. 5. or 6. of this Chap.) so many Proportionals as with those given may make the desired multitude; then multiply every term by the Denominator of the last term, so shall the Products be continual Proportionals in Integers in the same Reason as the two terms first given. As, for example, if a and b be given, and it be desired to find three Proportionals in Integers in the Reason of a to b ; first, to a and b I find a third Proportional, which (by Sect. 5.) is $\frac{bb}{a}$, then a , b , $\frac{bb}{a}$ being multiplied severally by the Denominator a , the Products aa , ab , bb are Proportionals express'd by Integers, and in the Reason of a to b , as was desired.

Hence if $a = 2$, and $b = 3$; then aa , ab and bb will give 4, 6 and 9, which are continual Proportionals in Integers in the given Reason of 2 to 3.

So if four continual Proportionals in the Reason of a to b be desired; first (by Sect. 5. or 6.) these will be found continual Proportionals, to wit, a , b , $\frac{bb}{a}$, $\frac{bbb}{aa}$, which multiplied severally by aa , (the Denominator of the last term,) will produce aaa , aab , abb , bbb , which are four continual Proportionals in Integers in the given Reason of a to b . Hence if $a = 2$, and $b = 3$; then aaa , aab , abb and bbb will give 8, 12, 18 and 27, which are continual Proportionals in Integers in the given Reason of 2 to 3.

In like manner these five quantities $aaaa$, $aaab$, $aabb$, $abbb$ and $bbbb$ will be found continual Proportionals in the Reason of a to b ; so that if $a = 2$, and $b = 3$, then these five Proportionals will give these five, to wit, 16, 24, 36, 54 and 81 ÷÷ in the Reason of 2 to 3: after the same manner you may proceed infinitely.

IX. If

IX. If there be quantities in continual proportion, how many soever, the Product made by the multiplication of the extremes is equal to the Product of any two means equally distant from the extremes; and also to the Square of the mean term when the number of terms is odd: as, for example, If a, b, c, d, e, f be continual Proportionals, I say the Product of the extremes a and f , to wit, af is equal to the Product of any two terms equally distant from the extremes, viz. to the Product cd and to the Product be , For,

1. By supposition, (and by Sect. 1, and 2.) $\frac{a}{b} = \frac{c}{d}$
2. Therefore by multiplying each part by f , it produceth $\frac{af}{b} = \frac{cf}{d}$
3. And by multiplying each part of the last Equation by b , it gives $\frac{af}{b} = \frac{bf}{d}$
4. Again, by supposition $\frac{b}{c} = \frac{d}{e}$
5. Therefore (by multiplying in like manner as before.) $\frac{cd}{b} = \frac{be}{e}$
6. Therefore from the third and fifth Equations (per 1. Axiom.) $af = cd = be$

1. Elem. Euclid.) Which was to be proved. And if more continual Proportionals even in multitude were proposed, the Demonstration would not be otherwise.

But if the multitude of terms be odd; as in these seven quantities which we may suppose to be continually proportional, a, b, c, d, e, f, g ÷÷; then the Product made by the multiplication of the two extremes a and g is equal to the Square of the middle term d , viz. $ag = dd$. For,

1. By supposition, (and by Sect. 1, and 2.) $\frac{a}{b} = \frac{c}{d}$
2. Therefore by multiplying each part of that Equation by d , it makes $\frac{ad}{b} = \frac{cd}{d}$
3. And by multiplying each part of the last Equation by e , it produceth $\frac{ce}{b} = \frac{de}{d}$
4. And by what hath been already proved in the first part of this Proposition, $ce = ag$
5. Therefore from the two last Equations (per 1. Axiom. Elem. Euclid.) $ag = dd$

Which was to be proved. Therefore the Proposition is every way manifest. But for further illustration,

Let there be proposed these six continual Proportionals in numbers, to wit, $2, 6, 18, 54, 162, 486$ ÷÷

Then according to the first part of the Proposition, $2 \times 486 = 6 \times 162 = 18 \times 54 = 972$

Again, let there be proposed these seven continual Proportionals, to wit, $2, 6, 18, 54, 162, 486, 1458$

Then according to the latter part of the Proposition, $2 \times 1458 = 54 \times 54 = 2916$.

X. If four quantities be Proportionals, $a : b :: c : d$, they shall be also alternly; and inversly, and compoſedly, and dividedly, and convertly, Proportionals; viz.

$$\text{If } \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right\}$$

Then alternly, $\left\{ \begin{array}{l} a : c :: b : d \\ 6 : 12 :: 4 : 8 \end{array} \right\}$ per 16. prop. 5. Elem. Eucl.

And inversly, $\left\{ \begin{array}{l} c : a :: d : b \\ 12 : 6 :: 8 : 4 \end{array} \right\}$ per Cor. of prop. 4. Elem. 5.

And compoſedly, $\left\{ \begin{array}{l} a+b : b :: c+d : d \\ 10 : 4 :: 20 : 8 \end{array} \right\}$ per 18. Prop. 5. Elem.

And dividedly, $\left\{ \begin{array}{l} a-b : b :: c-d : d \\ 2 : 4 :: 4 : 8 \end{array} \right\}$ per 17. prop. 5. Elem.

And convertly, $\left\{ \begin{array}{l} a : a+b :: c : c+d \\ 6 : 10 :: 12 : 20 \end{array} \right\}$ per Cor. of prop. 19. Elem. 5.

But

But that the Learner may the better perceive the meaning and use of these ways of arguing about Proportionals, I shall apply some of them to the Resolution of this following

QUEST.

The difference (b) between the greater extreme and mean of three quantities continually proportional being given, as also the difference (c) between the mean and lesser extreme, to find the Proportionals; but the first difference must be greater than the latter.

RESOLUTION.

1. For the mean Proportional sought put a
2. To which adding the given difference (b) the sum is the greater extreme, to wit, $a+b$
3. But if from the mean (a) the given difference (c) be subtracted, the Remainder is the lesser extreme, to wit, $a-c$
4. Then (according to the Question) these three quantities $a+b$, a , and $a-c$ must be in continual proportion, viz. $a+b : a :: a : a-c$
5. Therefore by division of Reason, $b : a :: c : a-c$
6. And alternately, (or by permutation,) $b : c :: a : a-c$
7. And by division of Reason, $b-c : c :: c : a-c$
8. Wherefore by conversion of Reason, $b-c : b :: c : a$

Which last Analogy if it be express'd by words gives this

CANON.

As the difference between the two given Differences is to either of them, so is the other to the mean Proportional sought.

Therefore if $36=b$, and $12=c$; the Canon will discover 18 for the mean Proportional sought, (to wit, a in the Resolution,) which increased with 36, and lessened by 12, gives 54 and 6 for the extremes. Therefore the three Proportionals sought are manifestly 54, 18 and 6.

Note. If the Analogy in the fourth step of the Resolution be converted into an Equation, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, that Equation after due Reduction will give the same Canon as above; so that the argumentation in the four last steps of the Resolution is not of necessity, but only to shew how without the help of any Equation, the number sought may sometimes be made the fourth Term of an Analogy whose three first Terms are known, whence by the Rule of Three the number sought is also known. Which ways of inferring one Analogy out of another are more proper when the nature of a Question will admit the same, than the common way of proceeding by Equations; especially in the Resolution of Geometrical Problems, where every step ought to be express'd in the most simple Terms, to the end the Composition of the Problem may the more easily be formed by the steps of the Resolution, but in a retrograde or backward order, as I shall hereafter shew in the Fourth Book of my *Algebraical Elements*.

XI. If Proportionals be multiplied or divided by Proportionals, the Products also or Quotients shall be Proportionals; as,

If these four proportional numbers, $a : b :: ca : cb$
 to wit, $2 : 4 :: 3 \times 2 : 3 \times 4$
 be multiplied by these four proportional numbers, $d : f :: gd : gf$
 $5 : 6 :: 7 \times 5 : 7 \times 6$
 there will be produced these four proportional numbers, to wit, $ad : bf :: cgad : cgbf$
 $2 \times 5 : 4 \times 6 :: 3 \times 7 \times 2 \times 5 : 3 \times 7 \times 4 \times 6$

Whereby the first part of the Proposition is manifest.

And if these four proportional numbers, to wit, $ad : bf :: cgad : cgbf$
 be divided by these four Proportionals, to wit, $d : f :: gd : gf$
 the Quotients will be these four Proportionals, to wit, $a : b :: ca : cb$

Whereby the latter part of the Proposition is manifest.

Hence

Hence it may easily be proved, that the Squares, Cubes, fourth Powers, fifth Powers, &c. of proportional numbers shall be also Proportionals; as,

If $a, b, c, d, e, f ::$
 Then their Squares also shall be Proportionals, viz. $a^2 : b^2 :: ca^2 : cb^2$
 And the Cubes of the first four Proportionals shall be Proportionals, viz. $aaa : bbb :: cccaa : cccbb$
 And so of higher Powers.

XII. In every Series or Rank of Quantities continually proportional, all the mean Terms between the first and the last are both Antecedents and Consequents of Reasons; as,

If $a, b, c, d, e, f ::$

That is, $a : b :: b : c :: c : d :: d : e :: e : f$

It is evident that every Term except the last (f) is an Antecedent of a Reason, and every Term except the first (a) is a Consequent; wherefore if (s) be put for the sum of all the Terms in the Series, then $s-f$ shall be the sum of all the Antecedents; and $s-a$ the sum of all the Consequents; Therefore,

From the premises (per 12. prop. 5. Elem. Euclid.) this Analogy ariseth, viz. $a : b :: s-f : s-a$

Whence by comparing the Product of the extremes to the Product of the means $as - aa = bs - bf$

Therefore, by due Transposition in that Equation, when b is greater than a , $bf - aa = bs - as$

And by dividing each part of the last Equation by $b-a$, there ariseth $\frac{bf-aa}{b-a} = s$

But if a exceed b , then there will arise $\frac{aa-bf}{a-b} = s$

Which two last Equations give a Canon to find the sum of all the Terms of a Geometrical Progression, the first, second and last Terms being severally given.

CANON.

Divide the difference between the Square of the first Term and the Product made by the multiplication of the second Term into the last, by the difference of the first and second Terms, so the Quotient shall be the sum of all the Terms of the Geometrical Progression proposed.

Examples in numbers.

Let the values of these, $a, b, c, d, e, f ::$
 be express'd by these numbers, $32, 48, 72, 108, 162, 243 ::$

Then by the Canon, $\frac{bf-aa}{b-a} = 665$ the Sum of all.

But if the values of the same Proportionals, $a, b, c, d, e, f ::$
 be expounded by these numbers, $243, 162, 108, 72, 48, 32 ::$

Then by the Canon, $\frac{aa-bf}{a-b} = 665$ the Sum of all.

XIII. If what hath been said in the eighth Self. of this Chap. be compared with the Table in Self. 4. Chap. 1. of this Book, it will be manifest that if we cast away the numbers of multitude which are prefix'd to all the mean Terms or Members belonging to any Compound Power produced from a Binomial Root, suppose from $a+b$, then all the Members or Simple quantities whereof the said Compound Power is compos'd are in continual proportion: As, for example, the Members whereof the Square of $a+b$ is compos'd are $aa, 2ae$ and ee ; now if 2 which is prefix'd to ae be cast away, then aa, ae and ee are continual Proportionals, (as is evident by the preceding eighth Self. of this Chap.)

Again, it appears by the said Table, that the Members whereof the Cube of $a+b$ is compos'd are $aaa, 3aae, 3aee$ and eee ; here if 3 and 3 which are prefix'd to the mean Terms be cast away, then these four quantities $aaa, aaee, aee$ and eee will be in continual proportion.

Y

Like.

Likewise, forasmuch as the fourth Power of $a + e$ is composed of these Members, $aaaa, 4aaae, 6aaee, 4aece$ and $eeee$; by casting away the numbers of multitude 4, 6 and 4, these five quantities $aaaa, aaee, aece, aece$ and $eeee$ shall be continual Proportionals; and so of higher Powers infinitely.

XIV. Forasmuch as by the last preceding *Self*. these quantities are in continual proportion, to wit,

Therefore their Square Roots also shall be in continual proportion, (per 22. prop. 6. Elem. Euclid.) to wit,

Hence, if a mean Proportional between any two given numbers a and e be desired, it shall be \sqrt{ae} ; as, if $a = 12$, and $e = 3$, then $ae = 36$, and \sqrt{ae} or $\sqrt{36}$, that is, 6, is a mean Proportional between 12 and 3; for as 12 is to 6, so is 6 to 3.

Again, forasmuch as these quantities are in continual proportion, to wit,

Therefore their cubick Roots also shall be continual Proportionals, (per 37. prop. 11. Elem. Euclid.) to wit,

Hence, if two mean Proportionals between any two given numbers a the greater and e the lesser be desired, then $\sqrt[3]{(3)aae}$ shall be the greater mean, and $\sqrt[3]{(3)ae}$ the lesser; as if $a = 54$, and $e = 2$, then $aae = 5832$, and $\sqrt[3]{(3)aae} = \sqrt[3]{(3)5832}$, therefore $\sqrt[3]{(3)5832}$, that is, 18 is the greater mean sought; also $ae = 108$, and therefore $\sqrt[3]{(3)108}$, that is, 6, is the lesser mean: so that 18 and 6 are the two desired mean Proportionals between 54 and 2; for 54, 18, 6 and 2 are in continual proportion. But when one mean next to either of the extremes is found out, the other mean may be found out by *Self*. 5. of this *Chapt.* without extracting any Root.

After the same manner, by the help of the said Table in *Self*. 4. *Chap.* 1. of this Book, continued to higher Powers if need be, you may find out as many mean proportional numbers as shall be desired between any two given numbers: As, if you would find five mean proportional numbers between 1458 (or a ;) and 2 (or e ;) look into the said Table for the sixth Power, (to wit a Power whose Index exceeds by unity the number of means sought,) and you will find $aaaaaa, 6aaaaa, 15aaaae, 20aaee, 15aece, 6aece$ and $eeeeee$; then casting away 6, 15, 20, 15 and 6 which are prefix'd to the mean terms, and extracting $\sqrt[6]{(6)}$ out of every one of those six terms after the said numbers prefix'd are cast away, there will arise $a, \sqrt[6]{(6)aaaaa}, \sqrt[6]{(6)aaaae}, \sqrt[6]{(6)aaee}, \sqrt[6]{(6)aece}$ and e ; now to find the five mean proportional numbers answering to those five proportional Roots express'd by letters which fall between a and e , it will be convenient to find the smallest mean first, viz. forasmuch as a was put for 1458, and e for 2, therefore $aeceee = 46656$, and $\sqrt[6]{(6)aeceee} = \sqrt[6]{(6)46656}$, that is, 6, shall be the least mean sought: then 2 being the first Proportional, or lesser extreme, and 6 the second, the third will (by *Self*. 5. of this *Chapt.*) be found 18, the fourth 54, the fifth 162, the sixth 486, and the seventh, to wit, the greater extreme, was first given 1458: so that between 2 and 1458, five mean Proportionals are found out as was desired; and the seven continual Proportionals are these, to wit, 2, 6, 18, 54, 162, 486 and 1458.

Many other admirable properties adherent to numbers in Geometrical Proportion continued, are deducible from the said Table of Powers in *Self*. 4. *Chap.* 1. of this Book, as will partly appear by the Theorems in the following sixth Chapter, which I find dispersed in several Algebraical Treatises.

CHAP.

CHAP. VI.

Various Theorems about Quantities in Continual proportion.

Theorem 1.

IF three numbers be Proportionals, the Solid number made by the continual multiplication of all the three is equal to the Cube of the mean.

Let three Proportionals be expos'd in Integers, according to *Self*. 8, or 13. of the preceding *Chap.* 5.

Thence it is evident, that $aaaaee$ the Product made by the multiplication of all the three Proportionals one into another, is equal to the Cube of the mean ae , as is affirmed by the Theorem.

Theor. 2.

If three numbers be Proportionals, the Product made by the multiplication of the Square of the first by the third, is equal to the Product of the Square of the second by the first:

As in these three, $\frac{aa}{9}, \frac{ae}{6}, \frac{ee}{4}$

It is evident that $aaaa \times ee = aaee \times aa = aaaaae$.

Theor. 3.

If three numbers be Proportionals, the Square of the sum of the extremes is equal to both the Squares of the extremes, together with twice the Square of the mean:

As in these three, $\frac{aa}{9}, \frac{ae}{6}, \frac{ee}{4}$

The Square of $aa + ee$ is $aaaa + 2aaee + eeee$, which is manifestly equal to the Squares of aa and ee , together with twice the Square of ae .

Theor. 4.

If three numbers be Proportionals, the Product of the lesser extreme multiplied by the difference of the extremes, is equal to the difference of the Squares of the mean and lesser extreme:

As in these three, $\frac{aa}{9}, \frac{ae}{6}, \frac{ee}{4}$

It is evident that $ee \times aa - ee = aece - eeee$.

Theor. 5.

If three numbers be Proportionals, the Product of the greater extreme multiplied by the difference of the extremes, is equal to the difference of the Squares of the greater extreme and the mean:

As in these three, $\frac{aa}{9}, \frac{ae}{6}, \frac{ee}{4}$

It is evident that $aa \times aa - ee = aaaa - aece$.

Theor. 6.

If three numbers be Proportionals, the difference of the Squares of the extremes is equal to the Square of the difference of the extremes, together with twice the difference of the Squares of the mean and lesser extreme:

As in these three, $\frac{aa}{9}, \frac{ae}{6}, \frac{ee}{4}$

1. The difference of the Squares of the extremes is $aaaa - eeee$
2. The Square of $aa - ee$ (the difference of the extremes) is $aaaa - 2aaee + eeee$
3. The double of the difference of the Squares of the mean and lesser extreme is $+2aece - 2eece$

Now the sum of the two latter of those three Quantities is manifestly equal to the first, as the Theorem affirms.

Y 2

Theor.

Theor. 7.

If three numbers be Proportionals, the difference of the Squares of the greater extreme and the mean is equal to the Square of the difference of the extremes, and to the difference of the Squares of the mean and the lesser extreme:

As in these three, $\dots \dots \dots \left\{ \begin{array}{l} aa, ae, ee \\ 9, 6, 4 \end{array} \right. \div \div$

1. The difference of the Squares of the greater extreme and the mean is $aaaa - aacc$
2. The Square of $aa - ee$ (the difference of the extremes) is $aaaa - 2aacc + eeee$
3. The difference of the Squares of the mean and lesser extreme is $\dots \dots \dots + aacc - eeee$

Now the sum of the two latter of those three Quantities is manifestly equal to the first, as the Theorem affirms.

Theor. 8.

If three numbers be Proportionals, then as the first is to the third, so is the Square of the first to the Square of the second; and so is the Square of the second to the Square of the third:

As in these three, $\dots \dots \dots \left\{ \begin{array}{l} aa, ae, ee \\ 9, 6, 4 \end{array} \right. \div \div$

1. It is evident that $aa : ee :: aa : ee$
2. Therefore by drawing aa as a common Factor into the two latter terms of that Analogy, this aritheth, $aa : ee :: aaaa : aacc$
3. And by drawing ee as a common Factor into the two latter terms of the first Analogy, this aritheth, $aa : ee :: aacc : eeee$

By which two last Analogies the truth of the Theorem is manifest.

Theor. 9.

If three numbers be Proportionals, then as the first is to the second, (or as the second is to the third,) so is the difference of the first and second, to the difference of the second and third:

As in these three, $\dots \dots \dots \left\{ \begin{array}{l} aa, ae, ee \\ 9, 6, 4 \end{array} \right. \div \div$

1. It is evident (as before hath been shewn in Theor. 4.) that $ee \times aa - ee = aacc - eeee$
2. And by Multiplication it will appear that $ae + ee \times ae - ee = aacc - eeee$
3. Therefore from the two last Equations, (per 1. Ax. 1. Elem. Euclid.) $ee \times aa - ee = ae + ee \times ae - ee$
4. Therefore, by resolving the last Equation into Proportionals, $aa - ee : ae - ee :: ae + ee : ee$
5. Therefore by Division of Reason, $aa - ae : ae - ee :: ae : ee$

Which was to be demonstrated.

Theor. 10.

If four numbers be continually proportional, the sum of the means is a mean Proportional between the sum of the first and second and the sum of the third and fourth.

Let four continual Proportionals be $\left\{ \begin{array}{l} aaa, aae, aee, eee \\ 8, 4, 2, 1 \end{array} \right. \div \div$
expos'd in Integers, to wit,

Then according to the import of the Theorem, it must be proved that these three Quantities are Proportionals, viz.

$$aaa + aae : aae + aee : aee + eee \div \div$$

But that they are Proportionals it will be evident by Multiplication, for the Product of the extremes is equal to the Square of the mean: therefore the truth of the Theorem is manifest.

Theor. 11.

Theor. 11.

If four numbers be continual Proportionals, the sum of all is to the sum of the means, as the sum of the first and third to the second:

As in these four, $\dots \dots \dots \left\{ \begin{array}{l} aaa, aae, aee, eee \\ 8, 4, 2, 1 \end{array} \right. \div \div$

1. The sum of all four is $aaa + aae + aee + eee$
2. The sum of the means is $\dots \dots \dots + aae + aee$
3. The sum of the first and third is $\dots \dots \dots + aaa + aee$
4. And the second is $\dots \dots \dots + aae$

I say those four quantities are Proportionals, in such order as they are above written; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifest.

Theor. 12.

If four numbers be in continual proportion, the sum of all is to the sum of the means, as the sum of the Squares of the means is to the Product of the means or extremes:

As in these four, $\dots \dots \dots \left\{ \begin{array}{l} aaa, aae, aee, eee \\ 8, 4, 2, 1 \end{array} \right. \div \div$

1. The sum of all is $a^3 + a^2e + ae^2 + e^3$
2. The sum of the means is $\dots \dots \dots + a^2e + ae^2$
3. The sum of the Squares of the means is $\dots \dots \dots + a^4e^2 + a^2e^4$
4. The Product of the means or extremes is $\dots \dots \dots + a^2e^3$

I say those four quantities are Proportionals, in such order as they are above written; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifest.

Theor. 13.

If four numbers be continual Proportionals, the sum of the Squares of the means is a mean Proportional between the sum of the Squares of the first and second, and the sum of the Squares of the third and fourth:

As in these four, $\dots \dots \dots \left\{ \begin{array}{l} aaa, aae, aee, eee \\ 8, 4, 2, 1 \end{array} \right. \div \div$

1. The sum of the Squares of the first and second is $a^6 + a^4e^2$
2. The sum of the Squares of the means is $a^4e^2 + a^2e^4$
3. The sum of the Squares of the third and fourth is $a^2e^4 + e^6$

I say those three quantities are Proportionals, in such order as they are above written; for it will appear by multiplication that the Square of the mean (or second quantity) is equal to the Product of the extremes; therefore the Theorem is manifest.

Theor. 14.

If four numbers be continual Proportionals, the Square of the sum of the means is equal to the Square of their difference, together with four times the Product of the extremes or means:

As in these four, $\dots \dots \dots \left\{ \begin{array}{l} aaa, aae, aee, eee \\ 8, 4, 2, 1 \end{array} \right. \div \div$

1. The Square of $a^2e - ae^2$ (the sum of the means) is $a^4e^2 + 2a^3e^3 + a^2e^4$
2. The Square of $a^2e + ae^2$ (the difference of the means) is $a^4e^2 - 2a^3e^3 + a^2e^4$
3. The quadruple of the Product of the extremes or means is $\dots \dots \dots + 4a^3e^3$

Now it is evident that the first of those three Quantities is equal to the sum of the second and third: therefore the Theorem is manifest.

Theor. 15.

Theor. 15.

If four numbers be continual Proportionals, the sum of their Squares shall be to the sum of the Products of the first into the second, and the third into the fourth; as the sum of all the four Proportionals to the sum of the means:

As in these four, $\frac{aaa, aae, aee, eee}{8, 4, 2, 1} \div \div \div$

1. The sum of the Squares of the four Proportionals is $a^6 + a^4e^2 + a^2e^4 + e^6$
2. The sum of the Products of the first into the second, and the third into the fourth is $a^4e + ae^4$
3. The sum of all the four Proportionals is $a^3 + a^2e + ae^2 + e^3$
4. The sum of the means is $a^2e + ae^2$

I say those four quantities are Proportionals in such order as they are above seated, for it will appear by multiplication that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifest.

Theor. 16.

If from the Square of the sum of four numbers in continual proportion the sum of their Squares be subtracted, and from half the Remainder there be also subtracted the Square of the sum of the two means, this latter Remainder shall be the sum of the Products of the first Proportional into the second, and of the third into the fourth, and shall be to the sum of the Squares of those four Proportionals, as the sum of the two means is to the sum of all the Proportionals:

As in these four, $\frac{aaa, aae, aee, eee}{8, 4, 2, 1} \div \div \div$

1. The Square of the sum of the four Proportionals will by multiplication be found $a^6 + 2a^4e + 3a^2e^2 + 4a^2e^2 + 3a^2e^2 + 2ae^4 + e^6$
2. The sum of the Squares of the four Proportionals is $a^6 + a^4e^2 + a^2e^4 + e^6$
3. Which sum of the Squares being subtracted from the said Square of the sum, the half of the Remainder will be $a^4e + a^2e^2 + 2a^2e^2 + a^2e^2 + ae^4$
4. The Square of the sum of the two means, to wit, of $a^2e + ae^2$ is $a^4e^2 + 2a^2e^3 + a^4e^4$
5. Which last mentioned Square being subtracted from the half Remainder in the third step, there will remain the sum of the Products of the first Proportional into the second, and of the third into the fourth, to wit, $a^4e + ae^4$

6. Now according to the import and meaning of the Theorem it remains to prove, that the Remainder in the last step is to the sum of the Squares in the second step, as the sum of the two mean Proportionals is to the sum of all four, viz. that

These four quantities are Proportionals, $\frac{a^4e + ae^4}{a^6 + a^4e^2 + a^2e^4 + e^6} :: \frac{a^4e^2 + 2a^2e^3 + a^4e^4}{a^3 + a^2e + ae^2 + e^3}$

7. But that they are Proportionals will be evident by multiplication, for the Product of the extremes is equal to the Product of the means, each Product being $a^4e + ae^4$

Therefore the Theorem is manifest.

Theor. 17.

If four numbers be continual Proportionals, the sum of all their Squares shall be to the sum of the Squares of the means; as the sum of the Products of the first into the second and the third into the fourth, to the Product of the means or extremes.

This is infer'd from Theor. 12, and 15. by exchange of equal Reasons.

Theor. 18.

If four numbers be continual Proportionals, the sum of the Squares of the extremes shall be to the sum of the Squares of the means; as the excess whereby the sum of the

the Products of the first into the second and third into the fourth exceeds the Product of the means is to the Product of the means or extremes.

This is infer'd from Theor. 17. by Division of Reason.

Theor. 19.

If four numbers be continual Proportionals, the sum of the first and third shall be to the second; as the sum of the Squares of the means, is to the Product of the means or extremes.

This is deduced from Theor. 11, and 12. by exchange of equal Reasons.

Theor. 20.

If four numbers be continual Proportionals, the sum of all their Squares shall be to the sum of the Products of the first into the second, and the third into the fourth; as the sum of the first and third is to the second.

This is deduced from Theor. 17, and 19. by exchange of equal Reasons.

Theor. 21.

If four numbers be continual Proportionals, the sum of the Cubes of the means is equal to the Product made by the multiplication of the sum of the extremes into the Product of the means or extremes:

As in these four, $\frac{aaa, aae, aee, eee}{8, 4, 2, 1} \div \div \div$

1. The sum of the Cubes of the means is $a^6e^3 + a^3e^6$
2. The sum of the extremes is $a^3 + e^3$
3. The Product of the means or extremes is a^3e^3

Now it is evident that the first of those three Quantities is equal to the Product of the second Quantity multiplied by the third; as is affirmed by the Theorem.

Theor. 22.

If four numbers be continual Proportionals, the Cube of the sum of the extremes is equal to the Cubes of the extremes, together with the triple sum of the Cubes of the means:

As in these four, $\frac{aaa, aae, aee, eee}{8, 4, 2, 1} \div \div \div$

1. The Cube of $a^3 + e^3$ (the sum of the extremes) is $a^9 + 3a^6e^3 + 3a^3e^6 + e^9$
2. The Cubes of the extremes is $a^9 + e^9$
3. The triple sum of the Cubes of the means is $3a^6e^3 + 3a^3e^6$

Now it is manifest that the first of those three Quantities is equal to the sum of the other two, as the Theorem affirms.

Theor. 23.

If four numbers be continual Proportionals; the difference of the Cubes of the extremes is equal to the triple of the difference of the Cubes of the means, together with the Cube of the difference of the extremes:

As in these four, $\frac{aaa, aae, aee, eee}{8, 4, 2, 1} \div \div \div$

1. The difference of the Cubes of the extremes is $a^9 - e^9$
2. The triple of the difference of the Cubes of the means is $3a^6e^3 - 3a^3e^6$
3. The Cube of $a^3 - e^3$ (the difference of the extremes) is $a^9 - 3a^6e^3 + 3a^3e^6 - e^9$

Now it is manifest that the first of those three Quantities is equal to the sum of the other two. Which was to be proved.

Thib. 24.

Theor. 24.

If four numbers be continual Proportionals, the Cube of the summ of the first and second is equal to the Product made by the multiplication of the Square of the first by the Aggregate of the summ of the extremes and the triple summ of the means:

As in these four, $\sum \frac{aaa}{8}, \frac{aac}{4}, \frac{acc}{2}, \frac{eee}{1} \div \div \div$

1. The Cube of the summ of the first and second, to wit, of $a^3 + a^2e$ is $\sum a^3 + 3a^2e + 3a^2e^2 + a^3e^3$
2. The Square of the first is $\sum a^6$
3. The Aggregate of the extremes and the triple of the summ of the means is $\sum a^3 + e^3 + 3a^2e + 3ae^2$

Now it is evident that the first of those three Quantities is equal to the Product made by the multiplication of the third by the second. Which was to be proved.

Theor. 25.

If four numbers be continual Proportionals, the Cube of the summ of the means is equal to the Product made by the multiplication of the Product of the extremes or means into the Aggregate of the extremes and the triple summ of the means:

As in these four, $\sum \frac{aaa}{8}, \frac{aac}{4}, \frac{acc}{2}, \frac{eee}{1} \div \div \div$

1. The Cube of the summ of the means, to wit, of $a^2e + ae^2$, is $\sum a^6e^3 + 3a^4e^2 + 3a^4e^2 + a^4e^3$
2. The Product of the extremes or means is $\sum a^2e^3$
3. The Aggregate of the extremes and the triple summ of the means is $\sum a^3 + e^3 + 3a^2e + 3ae^2$

Now it is evident that the first of those three Quantities is equal to the Product of the two latter. Which was to be proved.

Theor. 26.

If four numbers be continual Proportionals, the Product made by the multiplication of the summ of the extremes by the summ of the Squares of the extremes, is equal to the Cubes of the four Proportionals:

As in these four, $\sum \frac{aaa}{8}, \frac{aac}{4}, \frac{acc}{2}, \frac{eee}{1} \div \div \div$

1. The summ of the extremes is $\sum a^3 + e^3$
2. The summ of the Squares of the extremes is $\sum a^6 + e^6$
3. The Product of those two summs is $\sum a^9 + a^6e^3 + a^3e^6 + e^9$
4. The summ of the Cubes of the four Proportionals is $\sum a^9 + a^6e^3 + a^3e^6 + e^9$

But the Product in the third step is manifestly equal to the summ in the fourth; as the Theorem affirms.

Theor. 27.

If five numbers be continual Proportionals, the Product of the mean (or third Proportional) into the summ of the extremes, is equal to the Squares of the second and fourth:

As in these five, $\sum \frac{aaaa}{16}, \frac{aaac}{8}, \frac{aacc}{4}, \frac{aace}{2}, \frac{eeee}{1}$

1. The Product of the mean into the summ of the extremes is $\sum a^4e^2 + a^2e^4$
2. And the summ of the Squares of the second and fourth is also $\sum a^4e^2 + a^2e^4$

Therefore the Theorem is manifest.

Theor. 28.

Theor. 28.

If five numbers be continual Proportionals, the summ of the first, third and fifth, shall be to the third; as the summ of the Squares of the second, third and fourth is to the Square of the third:

As in these five, $\sum \frac{aaaa}{16}, \frac{aaac}{8}, \frac{aacc}{4}, \frac{aace}{2}, \frac{eeee}{1}$

1. The summ of the first, third and fifth is $\sum a^4 + a^2e^2 + e^4$
2. The third is $\sum a^2e^2$
3. The summ of the Squares of the second, third and fourth is $\sum a^4e^2 + a^4e^2 + a^2e^4$
4. The Square of the third is $\sum a^4e^4$

I say those four quantities are Proportionals, in such order as they are above seated; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means, each Product being $a^2e^4 + a^2e^4 + a^2e^4$: therefore the Theorem is manifest.

Theor. 29.

If five numbers be continual Proportionals, the summ of the extremes more by the double of the mean, the summ of the second and fourth, and the mean, are also continual Proportionals:

As in these five, $\sum \frac{aaaa}{16}, \frac{aaac}{8}, \frac{aacc}{4}, \frac{aace}{2}, \frac{eeee}{1}$

1. The summ of the extremes more by the double of the mean is $\sum a^4 + e^4 + 2a^2e^2$
2. The summ of the second and fourth is $\sum a^2e^2 + a^2e^2$
3. The mean is $\sum a^2e^2$

I say those three quantities are Proportionals; for it will be evident by multiplication that the Product of the first and third is equal to the Square of the second: therefore the Theorem is manifest.

Theor. 30.

If five numbers be continual Proportionals, the summ of the extremes is to the mean, as the difference of the Squares of the extremes, to the difference of the Squares of the second and fourth:

As in these five, $\sum \frac{aaaa}{16}, \frac{aaac}{8}, \frac{aacc}{4}, \frac{aace}{2}, \frac{eeee}{1}$

1. The summ of the extremes is $\sum a^4 + e^4$
2. The mean is $\sum a^2e^2$
3. The difference of the Squares of the extremes is $\sum a^4 - e^4$
4. The difference of the Squares of the second and fourth is $\sum a^4e^2 - a^2e^4$

I say those four quantities are Proportionals in such order as they are above placed; for it will be evident by multiplication, that the Product of the extremes is equal to the Product of the means, each Product being $a^4e^2 - a^2e^4$: therefore the Theorem is manifest.

Theor. 31.

If five numbers be continual Proportionals, the summ of the Squares of the second and fourth, shall be to the Square of the mean; as the difference of the Squares of the extremes, to the difference of the Squares of the second and fourth:

As in these five, $\sum \frac{aaaa}{16}, \frac{aaac}{8}, \frac{aacc}{4}, \frac{aace}{2}, \frac{eeee}{1}$

1. The summ of the Squares of the second and fourth is $\sum a^4e^2 + a^2e^4$
2. The Square of the mean is $\sum a^4e^4$
3. The difference of the Squares of the extremes is $\sum a^4 - e^4$
4. The difference of the Squares of the second and fourth is $\sum a^4e^2 - a^2e^4$

Z

I say

I say those four quantities are Proportionals in such order as they are above seated, for it will be evident by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifest.

Theor. 32.

If five numbers be continual Proportionals, the sum of the extremes shall be to the mean, as the sum of the Squares of the second and fourth is to the Square of the mean. This is evident from the two last preceding Theorems, by exchange of equal Reasons.

Theor. 33.

If five numbers be continual Proportionals, the sum of the Squares of the second and fourth shall be equal to the Product made by the multiplication of the third into the sum of the first and fifth:

As in these five, . . . $\begin{matrix} aaaa, aaac, aace, acce, cccc \\ 16, 8, 4, 2, 1 \end{matrix}$

1. The sum of the Squares of the second and fourth is . . . $a^2e^2 + a^2c^2$
 2. The mean or third is . . . a^2e^2
 3. The sum of the first and fifth is . . . $a^4 + e^4$
- But the Product of the second and third of those three Quantities above-written is equal to the first: therefore the Theorem is manifest.

CHAP. VII.

Questions about Quantities in Continual proportion, resolved by Literal Algebra.

QUEST. 1.

THE sum of three proportional Quantities being given, as also (c) the sum of their Squares, to find the Proportionals.

RESOLUTION.

1. For the mean Proportional sought put . . . a
2. Then subtracting the said mean from (b) the given sum of all the three Proportionals, there will remain the sum of the extremes, to wit, . . . $b - a$
3. Therefore the Square of the sum of the extremes is . . . $bb - 2ba + aa$
4. From which Square, if there be subtracted the double of the Square of the mean, to wit, . . . $2aa$
5. There will remain (as is manifest by Theor. 3. of the preceding Chap. 6.) the sum of the Squares of the extremes, to wit, . . . $bb - 2ba - aa$
6. To which sum of the Squares of the extremes if you add (aa) the Square of the mean, the aggregate shall be the sum of the Squares of the three Proportionals sought, to wit, . . . $bb - 2ba$
7. Which sum in the last step must be equal to (c) the given sum of the Squares: Hence this Equation, viz. . . . $bb - 2ba = c$
8. Which Equation after due Reduction gives . . . $\frac{bb - c}{2b} = a$

And the last Equation in words is this

CANON.

From the Square of the given sum of the three Proportionals sought subtract the given sum of their Squares; then divide the Remainder by the double of the sum of the three Proportionals, and the Quotient is the mean Proportional.

Therefore if 14 be given for the sum of three numbers in continual proportion, and 84 for the sum of their Squares, the mean Proportional will be found 4 by the said Canon. Then the mean being given 4, as also 10 the sum of the extremes, the extremes

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about Continual Proportionals.

extremes will be found 2 and 8, (by the Canon of Quest. 4. Chap. 16. of my First Book of Algebraical Elements;) and therefore the three Proportionals sought are 2, 4 and 8.

QUEST. 2.

The sum of three proportional Quantities being given; as also (c) the sum of the Squares of the extremes; to find the Proportionals.

RESOLUTION.

1. For the mean Proportional sought put . . . a
2. Then subtracting the said mean from (b) the given sum of all the three Proportionals, there will remain the sum of the extremes, to wit, . . . $b - a$
3. Therefore the Square of the sum of the extremes is . . . $bb - 2ba + aa$
4. From which Square if you subtract the double of the Square of the mean, to wit, . . . $2aa$
5. There will remain (as is manifest by the third Theorem of the preceding sixth Chapter,) the sum of the Squares of the extremes, to wit, . . . $bb - 2ba - aa$
6. Which sum of the Squares of the extremes must be equal to the given sum (c,) hence this Equation, viz. . . . $bb - 2ba - aa = c$
7. From which Equation after due Reduction, this will arise, . . . $bb - c = aa + 2ba$
8. Therefore by resolving the last Equation, (according to the Canon in Sect. 6. Chap. 15. of my First Book of Algebraical Elements;) the value of (a) the mean Proportional will be made known, viz. . . . $\sqrt{bb - c} - b = a$

Which last Equation in words is this

CANON.

From the double of the Square of the given sum of all the three Proportionals sought subtract the given sum of the Squares of the extremes; then from the square Root of the Remainder subtract the sum of the three Proportionals; so shall this last Remainder be the mean Proportional sought.

Therefore, if 14 be given for the sum of three continual Proportionals, and 68 for the sum of the Squares of the extremes, the mean Proportional will be found 4 by the said Canon: Then the mean being given 4, as also 10 the sum of the extremes, the extremes will be found 2 and 8, (by the Canon of Quest. 4. Chap. 15. of my First Book of Algebraical Elements;) and therefore the three Proportionals sought are 2, 4 and 8.

QUEST. 3.

The difference (b) of the extremes of three proportional Quantities being given, as also (c) the sum of the Squares of the three Proportionals, to find the Proportionals.

RESOLUTION.

1. For the sum of the extremes, (to wit, of the first and third Proportionals sought,) put . . . a
2. Then, inasmuch as the difference of the extremes is given (b,) and their sum is assumed to be (a,) therefore (by the Theorem in Quest. 1. Chap. 14. of my First Book of Algebraical Elements,) the greater extreme shall be . . . $\frac{1}{2}a + \frac{1}{2}b$
3. And by the same Theorem the lesser extreme is . . . $\frac{1}{2}a - \frac{1}{2}b$
4. Then the Product made by the multiplication of the extremes in the second and third steps will give the Square of the mean, to wit, . . . $\frac{1}{4}aa - \frac{1}{4}bb$
5. And from the second step the Square of the greater extreme is . . . $\frac{1}{4}aa + \frac{1}{4}ab + \frac{1}{4}bb$
6. And from the third step the Square of the lesser extreme is . . . $\frac{1}{4}aa - \frac{1}{4}ab + \frac{1}{4}bb$
7. Therefore from the fourth, fifth and sixth steps, the sum of the Squares of all the three Proportionals is . . . $\frac{1}{2}aa + \frac{1}{2}bb$
8. Which

8. Which summ in the last step must be equal to (c) the summ of the Squares given in the Question; hence this Equation ariseth, to wit, $\frac{1}{2}aa + \frac{1}{2}bb = c$
9. Which Equation after due Reduction will give $aa = \frac{4c - bb}{3}$
10. Therefore by extracting the Square Root out of each part of the last Equation the summ of the extreme Proportionals is discovered, to wit, $a = \sqrt{\frac{4c - bb}{3}}$
- Which last Equation gives this

CANON.

From four times the given summ of the Squares of the three Proportionals fought, subtract the Square of the given difference of the extremes, then the Square Root of one third part of that Remainder shall be the summ of the extreme Proportionals.

Then half the summ of the extremes increased with half their difference gives the greater extreme, and half the said summ lessened by half the said difference leaves the lesser extreme. Lastly, the square Root of the Product made by the mutual multiplication of the extremes is the mean Proportional.

Therefore if 16 be given for the difference of the extremes of three Proportionals, and 264 for the summ of the Squares of all the three Proportionals, the Proportionals are also given severally, to wit, 2, 6, 18.

QUEST. 4.

One extreme (b) of three proportional Quantities being given, as also (c) the summ of the Squares of the other extreme and the mean; to find out that other extreme and mean.

RESOLUTION.

1. For the extreme Proportional fought put a
2. Which multiplied by the given extreme (b) produceth the Square of the mean, to wit, ba
3. But from the first step the Square of the extreme Proportional fought is aa
4. Therefore from the second and third steps the summ of the Squares of the two Proportionals fought is $aa + ba$
5. Which summ in the last step must be equal to (c) the summ given in the Question, hence this Equation ariseth, viz. $aa + ba = c$
6. Which Equation being resolved by the Canon in Sect. 6. Chap. 15. of my first Book of *Algebraick Elements*, will discover the extreme Proportional fought, to wit, $a = \sqrt{c + \frac{1}{2}bb} - \frac{1}{2}b$
- The last Equation in words is this

CANON.

To the given summ add the Square of half the extreme Proportional given, and out of this summ extract the square Root; then this square Root lessened by half the given extreme will give the other extreme.

Therefore if 18 be given for one of the extremes of three Proportionals, and 40 for the summ of the Squares of the other two Proportionals, the Canon will discover 2 for the extreme fought. Lastly, the square Root of the Product of the extremes, to wit, 6 is the mean fought. Therefore the three Proportionals are 18, 6 and 2.

QUEST. 5.

The difference (b) between the extremes of three proportional Quantities being given, as also the Proportion which the difference of the Squares of the extremes hath to the summ of the Squares of all the three Proportionals, suppose the difference be to the summ as (r) to (s); to find the Proportionals. But (r) must be less than (s).

RESOLUTION.

1. For the summ of the extremes put a
2. Then for as much as their difference is given b
3. Therefore the difference of the Squares of the extremes shall be ba ; (for the Product of the multiplication of the summ of any two numbers into their difference is equal to the difference of their Squares;)

4. Then

4. Then from the first and second steps, (by the Theor. of Quest. 1. Chap. 14. of my first Book of *Algebraick Elements*), the greater extreme shall be $\frac{1}{2}a + \frac{1}{2}b$
5. And (by the same Theor.) the lesser extreme shall be $\frac{1}{2}a - \frac{1}{2}b$
6. Therefore from the fourth step the Square of the greater extreme is $\frac{1}{4}aa + \frac{1}{4}bb + \frac{1}{2}ba$
7. And from the fifth step the Square of the lesser extreme is $\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$
8. And because the Product made by the mutual multiplication of the extremes is equal to the Square of the mean, therefore the extremes in the fourth and fifth steps being multiplied one by the other, will give the Square of the mean, to wit, $\frac{1}{4}aa - \frac{1}{4}bb$
9. Therefore by adding together the Squares in the three last steps, the summ of the Squares of the three Proportionals shall be $\frac{1}{2}aa + \frac{1}{2}bb$
10. Then according to the Question, As r is to s, so must the difference in the third step be to the summ in the ninth step; hence this Analogy ariseth, viz. $r : s :: ba : aa + \frac{1}{2}bb$
11. Whence, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, this Equation comes forth, viz. $ba = \frac{r}{s}aa + \frac{r}{s}\frac{1}{2}bb$
12. From which Equation, after due Reduction, there will arise $\frac{r}{s}a - a = \frac{r}{s}\frac{1}{2}b$
13. Therefore (per Canon in Sect. 16. Chap. 15. Book 1.) the two Roots or values of a in the last Equation are these, to wit, $a = \frac{2sb + \sqrt{4s^2bb - 3r^2bb}}{3r}$ the greater. $a = \frac{2sb - \sqrt{4s^2bb - 3r^2bb}}{3r}$ the lesser.
14. But the greater of those two values of (a) is the desired summ of the extreme Proportionals fought; for if we should suppose the lesser value to be the summ of the extremes, it ought to exceed (b) the difference of the extremes, but from that supposition it will follow, that (r) is greater than (s), and consequently that the difference of the Squares of the extremes is greater than the summ of the Squares of all the three Proportionals, which is impossible. Now to prove the said consequence:
15. Suppose $\frac{2sb - \sqrt{4s^2bb - 3r^2bb}}{3r} = b$
16. Then by multiplying each part by 3r, it follows, that $2sb - \sqrt{4s^2bb - 3r^2bb} = 3rb$
17. And by adding $\sqrt{4s^2bb - 3r^2bb}$ to each part in the sixteenth step, $2sb = 3rb + \sqrt{4s^2bb - 3r^2bb}$
18. And by subtracting 3rb from each part in the seventeenth step, $2sb - 3rb = \sqrt{4s^2bb - 3r^2bb}$
19. And by squaring each part in the eighteenth step, $4s^2b^2 - 12r^2sb + 9r^2b^2 = 4s^2bb - 3r^2bb$
20. And by adding 3r^2bb to each part in the nineteenth step, $4s^2b^2 - 12r^2sb + 12r^2b^2 = 4s^2bb$
21. And by adding 12r^2sb to each part in the twentieth step, $4s^2b^2 + 12r^2sb + 12r^2b^2 = 4s^2bb + 12r^2sb$
22. And by subtracting 4s^2bb from each part in the twenty-first step, $12r^2sb + 12r^2b^2 = 12r^2sb$
23. Wherefore by dividing each part in the twenty-second step by 12r^2b, $r = s$
24. Thus, from a supposition that the lesser value of (a) in the thirteenth step is greater than (b) the given difference of the extremes, it follows by just consequence that (r) is greater than (s), which is impossible; for in regard the difference of the Squares of the extremes is less than the summ of the Squares of all three Proportionals, and that according to the Question the said difference is to the said summ as (r) to (s), therefore (r) is less than (s), and because the series of Inferences drawn from the said supposition ends in an impossibility, therefore that which was supposed cannot be true; viz. The lesser value

value of (*a*) is not greater than (*b*) the given difference of the extremes, and consequently it cannot be equal to the sum of the extremes. Which was to be proved. But by the like argumentation it may be proved that the greater value of (*a*) in the thirteenth step exceeds (*b*) the given difference of the extremes; and if it be expressed by Words, it will give the following Canon to find out the sum of the extreme Proportionals sought; whence by the help of the given difference of the extremes, the extremes are severally given.

CANON.

From four times the Square of the latter or greater term (*s*) of the given Reason subtract thrice the Square of the first term (*r*), and multiply the Remainder by the Square of the given difference of the extreme Proportionals sought; then add the Square Root of that Product to the double of the Product made by the multiplication of the latter term (*s*) into the difference of the extremes, and divide the sum of that addition by the triple of the first term (*r*), so shall the Quotient be the sum of the extreme Proportionals; lastly, half the sum of the extremes increased with half their difference gives the greater extreme, but the said half sum lessened by the said half difference leaves the lesser extreme. As, for example, if 6 be given for the difference of the extremes of three continual Proportionals, and the difference of the Squares of the extremes hath such proportion to the sum of the Squares of all the three Proportionals as 5 to 7, then by the Canon, the three Proportionals will be found 2, 4 and 8.

Again, if $2\frac{1}{2}$ be given for the difference of the extremes, and the difference of the Squares of the extremes be to the sum of the Squares of all the three Proportionals as 123 to 427, the Proportionals will be found 4, 5 and $6\frac{1}{2}$.

QUEST. 6.

The sum (*b*) of the extremes, and the sum (*c*) of the means of four Quantities in continual proportion being given; to find out the Proportionals; but (*b*) must exceed (*c*).

RESOLUTION.

1. For one of the means put a .
2. Then by subtracting that mean from (*c*) the given sum of the means, the Remainder is the other mean, to wit, $c - a$.
3. And by dividing the Square of the latter mean by the former, the Quotient gives one of the extremes, to wit, $\frac{cc - 2ca + aa}{a}$.
4. In like manner the Square of the first mean (*a*) being divided by the other mean ($c - a$) gives the other extreme, to wit, $\frac{aa}{c - a}$.
5. Therefore from the third and fourth steps the sum of the two extremes is $\frac{cc - 3ca + 3aa}{ca - aa}$.
6. Which sum must be equal to (*b*) the given sum of the extremes; hence this Equation ariseth, to wit, $\frac{cc - 3ca + 3aa}{ca - aa} = b$.
7. From which Equation after due Reduction this ariseth, $\frac{ccc}{3c - b} = ca - aa$ to wit,
8. Wherefore by resolving the last Equation by the Canon in *Sect. 10. Chap. 15. Book 1.* the two values of (*a*), to wit, the mean Proportionals sought will be made known, viz.

$$a = \frac{1}{2}c + \sqrt{\frac{cc}{4} - \frac{ccc}{3c - b}} \quad \text{the greater mean;}$$

$$a = \frac{1}{2}c - \sqrt{\frac{cc}{4} - \frac{ccc}{3c - b}} \quad \text{the lesser mean;}$$

Which values of (*a*) give this

CANON.

Divide the Cube of the sum of the means by the aggregate of the triple sum of the means and the sum of the extremes; subtract the Quotient from the Square of half the sum of the means, and extract the Square Root of the Remainder; then the said Square Root being added to and subtracted from half the sum of the means, the Sum and Remainder shall be the means sought.

The

Then the Square of the lesser mean being divided by the greater will give the lesser extreme; and the Square of the greater mean divided by the lesser gives the greater extreme.

Therefore if 18 be given for the sum of the extremes, and 12 for the sum of the means of four continual Proportionals, the Proportionals are given severally by the said Canon, to wit, 2, 4, 8 and 16.

QUEST. 7.

The difference (*b*) of the extremes, and the difference (*c*) of the means of four Quantities continually proportional being given; to find out the four Proportionals.

RESOLUTION.

1. For the lesser mean Proportional put a .
2. Which added to (*c*) the given difference of the means gives the greater mean, to wit, $c + a$.
3. Then the Square of the said greater mean being divided by the lesser, gives for the greater extreme $\frac{cc + 2ca + aa}{a}$.
4. Likewise by dividing (*aa*) the Square of the lesser mean by the greater, there ariseth for the lesser extreme $\frac{aa}{c + a}$.
5. Therefore the difference of the two extremes in the third and fourth steps is $\frac{ccc + 3cca + 3caa}{ca + aa}$.
6. Which difference must be equal to (*b*) the given difference of the extremes, hence this Equation ariseth, viz. $\frac{ccc + 3cca + 3caa}{ca + aa} = b$.
7. From which Equation, after due Reduction, this ariseth, $\frac{ccc}{b - 3c} = ca + aa$ to wit,
8. Wherefore by resolving the last Equation by the Canon in *Sect. 6. Chap. 15. Book 1.* the value of (*a*), to wit, the lesser mean Proportional sought will be made known, viz.

$$a = \sqrt{\frac{cc}{4} + \frac{ccc}{b - 3c}} - \frac{1}{2}c.$$

Which Equation in words is this

CANON.

Divide the Cube of the given difference of the means by the excess of the given difference of the extremes above the triple of the difference of the means, add the Quotient to the Square of half the difference of the means; then from the Square Root of that sum subtract half the difference of the means, so shall this Remainder be the lesser mean.

Then to the lesser mean add the difference of the means, and the sum is the greater. Lastly, the Square of the greater mean divided by the lesser gives the greater extreme, and the Square of the lesser mean divided by the greater gives the lesser extreme.

Therefore if 52 be given for the difference of the extremes of four continual Proportionals, and 12 for the difference of the means, the Proportionals will be found 2, 6, 18, 54.

QUEST. 8.

The sum (*b*) of four Quantities in continual proportion being given, as also (*c*) the sum of their Squares; to find the Proportionals.

RESOLUTION.

1. For the sum of the means put a .
2. Which subtracted from (*b*) the given sum of all the four Proportionals, leaves the sum of the extremes; to wit, $b - a$.
3. The Square of (*b*) the given sum of all the four Proportionals is bb .
4. Now (according to *Theor. 16. of the preceding Chap. 6.*) from the said Square (bb) I subtract (*c*) the given sum of the Squares of the four Proportionals, and from the half of the Remainder I also subtract (*aa*) the Square of the sum of the means, so this Quantity remains, to wit, $\frac{1}{2}bb - \frac{1}{2}c - aa$.
5. Which Remainder, to wit, $\frac{1}{2}bb - \frac{1}{2}c - aa$, (by the said *Theor. 16.*) shall be to the given sum of the Squares of the four Proportionals, as the sum of the means is to the sum of all the four Proportionals; hence this Analogy ariseth, viz. $\frac{1}{2}bb - \frac{1}{2}c - aa}{c} :: a : b$.

6. Which

6. Which Analogy, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, will be converted into this Equation, viz.

$$\frac{1}{2}bbb - \frac{1}{2}bc - baa = ca$$

7. Whence after due Reduction this Equation arifeth, to wit,

$$\frac{1}{2}bb - \frac{1}{2}c = aa + \frac{c}{b}a$$

Which Equation being resolved (*per Canon in Sect. 6. Chap. 15. Book 1.*) gives this following

CANON.

From the Square of the given sum of the four Proportionals subtract the given sum of their Squares, and to the half of the Remainder add the Square of half the Quotient that arifeth by dividing the sum of the Squares of the four Proportionals by the sum of the four Proportionals. Then extract the square Root of the sum of that addition, and from the said square Root subtract half the Quotient aforesaid, so shall the Remainder be the sum of the two desired mean Proportionals.

Then the sum of the means of four continual Proportionals being given, as also the sum of the extremes, the Proportionals shall be given severally by the Canon of the preceding *Quest. 6.* of this *Chap.*

So if 30 be given for the sum of four Proportionals, and 340 for the sum of their Squares, first, by the Canon above exprest, the sum of the means will be found 12, which subtracted from 30 the given sum of the four Proportionals, leaves 18 for the sum of the extremes: then the sum of the means being given 12, and the sum of the extremes 18, the four Proportionals (by the Canon of the preceding sixth Question) will be found 2, 4, 8, 16.

QUEST. 9.

The sum of (*b*) of four Quantities in continual proportion being given, as also (*c*) the sum of the Squares of the means, to find the Proportionals.

RESOLUTION.

1. For the sum of the means put
2. Then, because (by *Theor. 12.* of the preceding *Chap. 6.*) the sum of four Quantities continually proportional is to the sum of the means, as the sum of the Squares of the means is to the Product made by the mutual multiplication of the means or extremes, say,

$$\text{If } b : a :: c : \frac{ca}{b}$$

Whence the Product of the means or extremes is found

3. And because if from the Square of the sum of the means there be subtracted the sum of the Squares of the means, there will remain the double Product of the means or extremes; therefore if from (*aa*) you subtract (*c*), the half of the Remainder shall be the Product of the means or extremes, to wit,

4. Which Product, to wit, $\frac{1}{2}aa - \frac{1}{2}c$ must be equal to $\frac{ca}{b}$ the Product in the second step; hence this Equation arifeth, to wit,

5. From which Equation after due Reduction there arifeth $aa - \frac{2c}{b}a = c$

Which last Equation being resolved (by the Canon in *Sect. 8. Chap. 15. Book 1.*) gives this following

CANON.

To the given sum of the Squares of the means add the Square of the Quotient that arifeth by dividing the said sum by the given sum of the four Proportionals, and out of the sum made by that addition extract the square Root; then this square Root added to the aforesaid Quotient gives the sum of the mean Proportionals sought.

Then the sum of the means being given, as also the sum of the extremes, (for the sum of the means found out being subtracted from the given sum of all the four Proportionals leaves the sum of the extremes,) the four Proportionals will be discovered by the Canon of the sixth Question of this Chapter.

Therefore

Therefore, If 30 be given for the sum of four continual Proportionals, and 80 for the sum of the Squares of the means, the four Proportionals are also severally given, to wit, 2, 4, 8, 16; by the Canon above-exprest.

QUEST. 10.

The sum of (*b*) of four Quantities continually proportional being given, as also (*c*) the sum of the Squares of the extremes, to find out the Proportionals.

RESOLUTION.

1. For the sum of the means put
2. Which subtracted from (*b*) the given sum of the four Proportionals leaves the sum of the extremes, to wit,
3. Therefore the Square of the sum of the extremes is
4. From which Square, if (*c*) the given sum of the Squares of the extremes be subtracted, there will remain the double Product made by the mutual multiplication of the extremes or means, therefore the Product of the means is
5. And, because if from *aa* the Square of the sum of the means there be subtracted $bb - 2ba + aa - c$ the double Product of the means, there will remain the sum of the Squares of the means, therefore the sum of the Squares of the means is
6. And because by *Theor. 12.* in the preceding *Chap. 6.* the sum of the Squares of the means is to the Product of the means, as the sum of all the four Proportionals is to the sum of the means; therefore from the premises this following Analogy arifeth, viz.

$$2ba - bb + c : \frac{bb - 2ba + aa - c}{2} :: b : a$$

7. From which Analogy, by comparing the Product of the extremes to the Product of the means, this Equation arifeth, viz.

$$2ba - bb + ca = \frac{bb - 2ba + aa - c}{2}$$

8. Which Equation, after due Reduction, gives this following Equation, viz.

$$aa + \frac{2c}{3}a = \frac{bb - c}{3}$$

Whence (*per Canon in Sect. 6. Chap. 15. Book 1.*) there arifeth this following

CANON.

Divide the given sum of the Squares of the extremes by the triple of the given sum of all the four Proportionals, and to the Square of the Quotient add one third part of the excess of the Square of the sum of the four Proportionals above the sum of the Squares of the extremes, then from the square Root of the sum made by that Addition subtract the Quotient first found out: so shall the Remainder be the desired sum of the mean Proportionals.

Then the sum of the means being given, as also the sum of the extremes, (for the sum of the means being subtracted from the given sum of the four Proportionals leaves the sum of the extremes,) the four Proportionals will be discovered by the Canon of the sixth Question of this Chapter.

Therefore, If 80 be given for the sum of four continual Proportionals, and 2920 for the sum of the Squares of the extremes, the four Proportionals will be found 2, 6, 18, 34.

QUEST. 11.

The sum of (*b*) of the Squares of the extremes of four Quantities in continual proportion being given, as also (*c*) the sum of the Squares of the means, to find out the Proportionals.

RESOLUTION.

1. Add the two given sums into one, that you may have the sum of the Squares of the four Proportionals sought, for which last mentioned sum put
2. Then for the sum of the Squares of the first and second Proportionals put
3. Therefore the sum of the Squares of the third and fourth Proportionals is
4. Then,

4. Then,

4. Then, because (by *Theor.* 13. of the preceding *Chap.* 6.) the sum of the Squares of the two means is a mean Proportional between the sum of the Squares of the first and second, and the sum of the Squares of the third and fourth, this Analogy is manifest, *viz.* $a : c :: c : d$
5. Therefore by comparing the Product made by the multiplication of the extremes of that Analogy to the Product of the means, this Equation ariseth, *viz.* $da - aa = cc$
6. Which Equation being resolved by the Canon in *Self.* 10. *Chap.* 16. *Book* 1. gives this following

CANON.

Add the given sum of the Squares of the extremes to the given sum of the Squares of the means, and reserve half of the sum: from the Square of this half sum subtract the Square of the sum of the Squares of the means and extract the Square Root of the Remainder: add this Square Root to the half sum before reserved, and also subtract it from the same half sum; so the Summ shall be the sum of the Squares of the first and second Proportionals, and the Remainder shall be the sum of the Squares of the third and fourth.

Then (according to *Theor.* 3. of the preceding *Chap.* 6.) add severally the sum of the Squares of the first and second Proportionals, and the sum of the Squares of the third and fourth, to the sum of the Squares of the means, and out of each sum extract the square Root, so shall one of these Roots be the sum of the first and third Proportionals, and the other shall be the sum of the second and fourth: which two last mentioned summs being added together give the sum of the four Proportionals sought.

Lastly, the sum of four Proportionals being given, as also the sum of the Squares of the means, the Proportionals shall be given severally by the ninth Question of this *Chapt.*

Therefore if 260 be given for the sum of the Squares of the extremes of four continual Proportionals, and 80 for the sum of the Squares of the means, the Proportionals will be found 16, 8, 4, 2.

QUEST. 12.

The sum (*b*) of the extremes of four Quantities in continual proportion being given, as also (*c*) the sum of the Cubes of the means; to find out the Proportionals.

RESOLUTION.

1. For one of the extreme Proportionals put a
2. Then the other extreme, by subtracting (*a*) from (*b*) the given sum of the extremes, shall be $b - a$
3. Therefore the Product made by the mutual multiplication of the extremes is $ba - aa$
4. And because (per *Theor.* 21. of the preceding *Chap.* 6.) the Product made by the multiplication of the means or extremes into the sum of the extremes, is equal to the sum of the Cubes of the means; therefore if you multiply $ba - aa$ by *b*, this Product shall be equal to (*c*) the given sum of the Cubes of the means; hence ariseth this Equation, *viz.* $bba - baa = c$
5. And by dividing every term of that Equation by (*b*), there ariseth $ba - aa = \frac{c}{b}$

Which last Equation being resolved (by the Canon in *Self.* 10. *Chap.* 15. *Book* 1.) gives this following

CANON.

6. From the Square of half the given sum of the extremes subtract the Quotient that ariseth by dividing the given sum of the Cubes of the means by the sum of the extremes, and extract the square Root of the Remainder, then half the sum of the extremes being increased & also lessened by the said square Root, gives the extremes severally.

Then you may find out the means by a new work, thus,

7. Let the greater extreme found out as above be f
8. And the lesser extreme g
9. Then for the greater mean put h
10. Therefore by dividing (aa) the square of the greater mean by the greater extreme (f) the Quotient shall be the lesser mean, to wit, $\frac{aa}{f}$

11. But

11. But the Square of the lesser mean is equal to the Product of the lesser extreme multiplied by the greater mean; therefore from the three last preceding steps this Equation ariseth, *viz.* $\frac{aaa}{ff} = ga$
12. Which Equation, after due Reduction, gives $aaa = ffg$
13. Therefore by extracting the cubick Root out of each part of the last Equation the greater mean is made known, *viz.* $a = \sqrt[3]{(3)ffg}$
- Which last Equation, together with that in the tenth step, will give this

CANON.

14. Multiply the Square of the greater extreme by the lesser, then the cubick Root of the Product shall be the greater mean. Lastly, the Square of the greater mean divided by the greater extreme gives the lesser mean.

Therefore if 18 be given for the sum of the extremes of four numbers in continual proportion, and 576 for the sum of the Cubes of the means, then by the first Canon of this Question the extremes will be found 16 and 2; and lastly, by the latter Canon, the means will be found 8 and 4: wherefore the four continual Proportionals sought are 16, 8, 4, 2.

QUEST. 13.

The sum (*b*) of the Cubes of the extremes of four Quantities in continual proportion being given, as also (*c*) the sum of the Cubes of the means; to find the four Proportionals.

RESOLUTION.

1. For the sum of the extremes put a
2. Therefore the Cube of that sum is aaa
3. Then because by *Theor.* 22. of the preceding *Chap.* 6. if four Quantities be continually proportional, the sum of the Cubes of the extremes more by the triple of the Cubes of the means is equal to the Cube of the sum of the extremes, therefore if to b you add $3c$, it gives the Cube of the sum of the extremes, which Cube must be equal to aaa ; hence this Equation, $b + 3c = \sqrt[3]{aaa}$
4. Therefore by extracting the cubick Root out of each part of that Equation, the sum of the extremes is made known, *viz.* $\sqrt[3]{(3)b + 3c} = a$
- Which last Equation in words is thus following

CANON.

Add the triple of the given sum of the Cubes of the means to the given sum of the Cubes of the extremes, and out of the sum made by that Addition extract the cubick Root, which shall be the sum of the extremes sought.

Then the sum of the extremes being given, as also the sum of the Cubes of the means, the four Proportionals shall be given severally by the Canon of the preceding twelfth Question. As, for example, if 157472 be given for the sum of the Cubes of the extremes of four numbers in continual proportion, and 6048 for the sum of the Cubes of the means; first, by the Canon of this Question the sum of the extremes will be found 56, and then by the Canon of the preceding twelfth Question, the four Proportionals will be found 2, 6, 18, 54.

QUEST. 14.

The sum of the extremes (*b*) of five Quantities in continual proportion being given, as also (*c*) the sum of the three means; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle term of all the five, put a
2. Then subtract that middle term (*a*) from (*c*) the given sum of the three means, and there will remain the sum of the second and fourth, *viz.* $c - a$
3. And because (by *Theor.* 29. of the preceding *Chap.* 6.) the sum of the extremes of five continual Proportionals together with the double of the mean, the sum of the second and fourth, and the mean, are also in continual proportion; therefore this Analogy is manifest, *viz.* $b + 2a : c - a :: c - a : a$

A a 2

4. From

4. From which Analogy, by comparing the Product made by the multiplication of the extremes to the Product of the means, this Equation is produced, viz. $ba + 2aa = cc - 2ca + aa$
5. Which Equation, after due Reduction, gives $aa + ba + 2ca = cc$.
- Lastly, by resolving the last Equation according to the Canon in *Self. 6. Chap. 15; Book 1.* there will arise this following

CANON.

Add the sum of the extremes to the double of the sum of the three means, and take the half of the sum made by such Addition; then to the Square of the said half sum add the Square of the sum of the three means, and out of this sum extract the Square Root; from which Root subtract the half sum first taken, and the Remainder shall be the middle (or third) Proportional of the five sought.

Then by subtracting the said third Proportional from the sum of the three means, the Remainder is the sum of the second and fourth; by which Summ and the third Proportional, the second and fourth shall be given severally, (by the Canon of *Quest. 4. Chap. 16. Book 1.*) Then the Square of the second Proportional being divided by the third gives the first, and the Square of the fourth being divided by the third gives the fifth.

Therefore, if 34 be given for the sum of the first and fifth of five continual Proportionals, and 28 for the sum of the three means, the five Proportionals shall be given severally, viz. 2, 4, 8, 16, 32 \div .

QUEST. 15.

The sum (b) of the first, third and fifth of five Quantities in continual proportion being given, as also (c) the sum of the second and fourth; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle term of the five, put a
2. Then subtract that middle term (a) from the given sum (b), and the Remainder is the sum of the first and fifth, viz. $b - a$
3. And because (by *Theor. 27.* of the preceding *Chap. 6.*) the Product made by the multiplication of the third or middle term of five continual Proportionals into the sum of the first and fifth is equal to the Squares of the second and fourth; therefore (from the first and second steps) the sum of the Squares of the second and fourth Proportionals is $ba - aa$
4. The Square of the third Proportional (a) is equal to the Product of the second multiplied into the fourth; therefore the double of that Product is $2aa$
5. Therefore, from the two last steps, the Aggregate of the Squares and the double Product of the second and fourth Proportionals is $aa + ba$
6. But the Aggregate of the Squares and the double Product of the second and fourth Proportionals is equal to the Square of their sum, therefore the Aggregate in the fifth step must be equal to the Square of the given sum (c), viz. $aa + ba = cc$

Which Equation being resolved by the Canon in *Self. 6. Chap. 15. Book 1.* will give this following

CANON.

Add the Square of half the given sum of the first third, and fifth Proportionals to the Square of the given sum of the second and fourth; then from the square Root of the sum made by that Addition subtract the said half sum, and the Remainder shall be the third Proportional.

Then by subtracting the said third Proportional from the given sum of the first, third and fifth, the Remainder is the sum of the first and fifth; by which sum and the third (or mean) Proportional, the first and fifth, (to wit, the extremes) shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.* Then the third Proportional being multiplied into the first and fifth severally, and the square Root being extracted out of each Product, these Roots shall be the second and fourth Proportionals.

Therefore, if 42 be given for the sum of the first, third and fifth of five numbers in continual Proportion, and 20 for the sum of the second and fourth, the five Proportionals will be found these, to wit, 2, 4, 8, 16, 32.

Quest. 16.

QUEST. 16.

The third Proportional (b) of five Quantities in continual proportion being given, as also (c) the sum of the other four; to find out the five Proportionals.

RESOLUTION.

1. For the sum of the second and fourth Proportionals put a
2. Then subtract that sum (a) from (c) the given sum of the first, second, fourth and fifth Proportionals, and there will remain the sum of the first and fifth, to wit, $c - a$
3. The Square of the third (that is, of the mean) Proportional (b) is equal to the Product of the second multiplied into the fourth; therefore the double of that Product is $2bb$
4. Which double Product ($2bb$) subtracted from (aa) the Square of the sum of the second and fourth Proportionals, leaves for the sum of the Squares of the second and fourth, $aa - 2bb$
5. And because (by *Theor. 33.* of the preceding *Chap. 6.*) the sum of the Squares of the second and fourth of five continual Proportionals is equal to the Product of the third (or mean) multiplied by the sum of the first and fifth; therefore, if ($aa - 2bb$) the sum of the Squares of the second and fourth be divided by the mean (b) the Quotient shall be the sum of the first and fifth, viz. $\frac{aa - 2bb}{b}$
6. Which sum found out in the last step, must be equal to the sum of the first and fifth Proportionals found out in the second step; hence this Equation ariseth, viz. $\frac{aa - 2bb}{b} = c - a$
7. Which Equation, after due Reduction, gives $aa - ba = 2bb - bc$. Wherefore by resolving the last Equation (according to the Canon in *Self. 6. Chap. 15; Book 1.*) there will come forth this following

CANON.

To the Square of the half of the given third (or mean) Proportional add the double of the Square of the said mean, as also the Product of the said mean multiplied into the given sum of the other four Proportionals, and out of the sum of that Addition extract the square Root; this Root lessened by half the given mean, gives the sum of the second and fourth Proportionals.

Then from the given sum of the first, second, fourth and fifth Proportionals subtract the sum of the second and fourth (found out as above,) and the Remainder is the sum of the first and fifth; by which sum and the third (or mean) Proportional, the said first and fifth shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the square Roots of the Product of the first multiplied into the third, and of the Product of the third into the fifth, shall be the second and fourth Proportionals.

Therefore, if 8 be given for the third of five numbers in continual proportion; and 54 for the sum of the other four; the five Proportionals will be found these, to wit, 2, 4, 8, 16, 32.

QUEST. 17.

The sum (b) of the extremes of five Quantities in continual proportion being given, as also (c) the sum of the Squares of the three means; to find the five Proportionals.

RESOLUTION.

1. For the mean (or third) Proportional put a
2. Then (by *Theor. 33.* of the preceding *Chap. 6.*) the mean (a) multiplied by (b) the given sum of the extremes, produceth the sum of the Squares of the second and fourth Proportionals, viz. ba
3. Therefore if to (aa) the Square of the mean, you add (ba) the sum of the Squares of the second and fourth, there will come forth the sum of the Squares of the second, third and fourth Proportionals, viz. $aa + ba$
4. Which sum found out in the last step must be equal to the given sum (c); hence this Equation ariseth, viz. $aa + ba = c$

Wherefore,

Wherefore by resolving that Equation (according to the Canon in *Self. 6. Chap. 15. Book 1.*) there will arise this following

CANON.

Add the Square of half the given sum of the extremes to the given sum of the Squares of the three means, and out of the sum of that Addition extract the Square Root, this Root lessened by half the sum of the extremes, will give the mean (or third) Proportional.

Then the mean (or third) Proportional being given, and the sum of the extremes, (*viz.* of the first and fifth,) the said extremes shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the Square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore, if 34 be given for the sum of the extremes of five numbers in continual Proportion, and 336 for the sum of the Squares of the three means, the five Proportionals shall be also given, to wit, 2, 4, 8, 16, 32.

QUEST. 18.

The sum of the extremes of five Quantities in continual proportion being given, as also (*c*) the sum of the Squares of the second and fourth, to find the five Proportionals.

RESOLUTION.

1. For the mean Proportional put
2. Then (by *Theor. 33.* of the preceding *Chap. 6.*) the mean (*a*) multiplied by (*b*) the sum of the extremes, produceth the sum of the Squares of the second and fourth, *viz.* $ba = c$
3. Which sum must be equal to the given sum (*c*), therefore $ba = c$
4. Wherefore, by dividing each part of that Equation by (*b*), the mean Proportional will be made known, *viz.* $a = \frac{c}{b}$

Which last Equation, in words, is this following

CANON.

Divide the given sum of the Squares of the second and fourth Proportionals by the given sum of the first and fifth, so shall the Quotient be the mean or third Proportional.

Then the mean (or third) Proportional being given, as also the sum of the first and fifth, these shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the Square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore, if 34 be given for the sum of the extremes of five numbers in continual proportion, and 272 for the sum of the Squares of the second and fourth, the Proportionals will be discovered severally, *viz.* 2, 4, 8, 16, 32.

QUEST. 19.

A Vintner having a vessel full of Wine containing 16 (or *b*) Gallons, draws out 4 (or *c*) Gallons, and then pours into the vessel as much Water as he drew out Wine; then out of that mix'd quantity of Wine and Water he draws out the same number of Gallons as before, and pours in the same quantity of Water; again he makes a third draught of the same quantity as at first: The Question is, to find how much pure Wine remained in the vessel after the third draught.

RESOLUTION.

1. The number of Gallons of Wine in the vessel at first was b
2. Out of which quantity, (*c*) Gallons being drawn, there remained of pure Wine in the vessel $b - c$
3. To which remaining quantity of pure Wine, (*c*) Gallons of Water being added, the vessel is again full, and contains (*b*) Gallons of Wine and Water together; out of which drawing again (*c*) Gallons, we must seek how much pure Wine was in this second draught, saying by the Rule of Three,

$$\text{If } b : b - c :: c : \left(\frac{bc - cc}{b} \right)$$

Whence it is found, that the quantity of pure Wine in the second draught was

4. Which

4. Which quantity $\frac{bc - cc}{b}$ being subtracted from $b - c$ the

quantity of pure Wine in the vessel before the second draught was made, there remains for the quantity of pure Wine in the vessel after the second draught,

$$bb - 2bc + cc$$

5. To which remaining quantity of pure Wine add (*c*) Gallons of Water, so the vessel is again full, and contains (*b*) Gallons of Wine and Water together; out of which drawing again (*c*) Gallons, we must seek how much pure Wine was in this third draught, saying,

$$bb - 2bc + cc$$

$$\text{As } b : \frac{bb - 2bc + cc}{b} :: c : \text{to a fourth}$$

Proportional or quantity of pure Wine in the third draught, which will be found

6. Then by subtracting the said fourth Proportional or quantity of pure Wine in the third draught, from $\frac{bb - 2bc + cc}{b}$

$$bbb - 3bbc + 3bcc - ccc$$

the quantity of pure Wine in the vessel when the third draught was made, there remains for the desired quantity of pure Wine in the vessel after the third draught

Which Quantity last found out is the Answer of the Question; and if it be resolved into numbers it gives $6\frac{1}{2}$ for the number of Gallons of pure Wine that remained in the vessel after the third draught. Moreover, if the first, second, fourth and sixth steps of the Resolution be well examined and compared with *Self. 2, 5, and 6. Chap. 5.* of this Second Book, it will be manifest that the quantity of pure Wine in the vessel at first, and the several quantities of Wine remaining in the vessel after each draught are in Continual Proportion:

$$\text{Viz. } \begin{matrix} b & b - c & \frac{bb - 2bc + cc}{b} & \frac{bbb - 3bbc + 3bcc - ccc}{bb} & \dots \\ 16 & 12 & 9 & 6\frac{1}{2} & \dots \end{matrix}$$

Of which continual Proportionals the first is the given quantity of Wine in the vessel at first, the second is the excess of the same quantity above the given quantity drawn out at each draught, and then the fourth continual Proportional is the quantity of pure Wine remaining in the vessel when three draughts have been made, according to the import of the Question; but the fifth continual Proportional when four draughts, the sixth when five draughts, the seventh when six draughts shall be the remaining quantity of pure Wine sought by the Question. Lastly, the first and the second Terms of a Rank of numbers in continual proportion being given, any of the following Terms shall be given by the Rule in *Self. 5, and 6. Chap. 5.* of this Second Book.

QUEST. 20.

A Vintner having a vessel full of Wine containing 16 (or *b*) Gallons, draws out a certain quantity, and then pours into the vessel as much Water as he drew out Wine; again, out of that mix'd quantity of Wine and Water he draws out the same quantity as before, and pours in the same quantity of Water: then he makes a third draught of the same quantity as at first, and after this third draught there remained $6\frac{1}{2}$ (or *d*) Gallons of pure Wine. The Question is, to find what quantity of pure Wine was drawn out at the first draught, or what quantity of Wine and Water together at the second or third draught, (for the three draughts were Equal quantities.)

RESOLUTION.

1. The number of Gallons of Wine in the vessel at first was b
2. For the number of Gallons of Wine drawn out at the first draught put a
3. Then the quantity of Wine remaining in the vessel after the first draught was $b - a$
4. By prosecuting the search as in the preceding nineteenth Question, saying that (*a*) is to be

be used here instead of (c) there, you will find this quantity, viz. $\frac{bbb - 3bba + 3baa - aaa}{bb}$ to be the number of Gallons of pure Wine remaining in the vessel after the third draught, and therefore it must be equal to the given quantity $6\frac{1}{2}$, (or d ;) hence aritheth this Equation, viz.

$$\frac{bbb - 3bba + 3baa - aaa}{bb} = d,$$

5. Therefore by multiplying each part of that Equation by the Denominator bb , there will come forth this Equation in Integers, viz.

$$bbb - 3bba + 3baa - aaa = bbd,$$

6. And by extracting the Cubick Root out of each part of the last Equation, there aritheth

$$b - a = \sqrt[3]{(3)bbd},$$

7. Wherefore from the last Equation after due Transposition, the value of (a) will be made known, viz.

$$a = b - \sqrt[3]{(3)bbd} = 4.$$

Whence it is manifest that four Gallons were drawn out at every one of the three draughts. But if the Resolution had been wrought out at large, as in the preceding nineteenth Question, then it would appear, that if between (b) and (d,) viz. the quantity of Wine first given and the quantity of Wine remaining after the last draught, there be found the greater of two mean Proportionals when three draughts are proposed, or the greatest of three means when four draughts, and so forwards; then the mean so found out being subtracted from the greater extreme (b) leaves the Quantity drawn out at each draught. The manner of finding out mean Proportional numbers between any two numbers given for Extremes hath already been shewn in *Self*. 14. Chap. 5. of this Second Book.

If the Reader desires more variety of Questions about Quantities in continual Proportion, he may consult the *Algebra of Jac. de Billy*, entituled *Nova Geometria Clavi* and the First Part of our Learned Dr. Wallis his Mathematical Works.

CHAP. VIII.

The manner of finding out all the Aliquot parts both of Numbers and Algebraical Quantities, as also the smallest numbers that shall have given multipliers of Aliquot parts.

IN the Resolution of knotty Questions about Quantity, there is oftentimes great use of finding out all the *Aliquot parts*, or *just Divisors*, as well of Numbers, as of Quantities represented by Letters; and therefore in this Chapter I shall shew how that work may be done; as also, how to find out the least number that shall have a given multitude of *Aliquot parts*, according to the method of *Fran. van Schooten* in *Self*. 2; 3; and 4. of his *Miscellanies*, and in his *Principia Mathes. universal*.

II. A *Prime* or *Incomposit* number is that which can only be measured or divided by it self, or by Unity, and leave no Remainder: as, 2, 3, 5, 7, 11, 13, &c. are *Prime* numbers.

III. A *Composite* number is that which may be divided by some number less than the *Composite* it self, but greater than Unity: as, 4, 6, 8, 9, 10, &c. are *Composites*.

IV. *Just Divisors* are such numbers or quantities as will divide a given number or quantity and leave no Remainder; every one of which Divisors, except that which is equal to the given Quantity, is called an *Aliquot part*, because if it be taken (*Aliquoties*, that is, certain times, it will precisely constitute the given Quantity: As, if 6 be a number proposed, its just Divisors are 1, 2, 3, and 6; but the Aliquot parts of 6 are only 1, 2, and 3: for 6 cannot be a part of 6, but it may be a Divisor to it self, that is, 6 may be divided by 6, and the Quotient is Unity. Hence it is manifest, that the just Divisor of a number are more in multitude by one than the number of its *Aliquot parts*.

V. The Aliquot parts of a whole number may be found out in this manner, viz. First, if the number proposed be even, divide it by 2, and reserve the Divisor; again if the Quotient be even divide it by 2 and reserve the Divisor; and continue the Division

of every

of every following Quotient by 2 until the Quotient be an odd number: But if either the number first proposed, or the Quotient resulting from such Division by 2, be odd, divide it by 3, if it will give an Integer Quotient, and continue the Division by 3 in like manner as before by 2, so long as the Quotient is an Integer without any Fraction; likewise, when the Division by 3 ceaseth, divide by 5, 7, 11, 13, 17, 19, &c. that is, by every Prime number, until you find a Quotient less than the Divisor; and if no such Divisor will give an Integer Quotient before the Quotient is less than the Divisor, you may conclude the number first proposed to be Incomposit, (viz. such as hath no Divisor but it self or Unity,) and that last Divisor to be greater than the Square Root of the proposed number: then by the help of those Prime Divisors to the given number, all the rest may be found out by the Operation directed in the following Examples.

Example 1.

Suppose it be desired, to find out all the Aliquot parts and Divisors of 360: First, I divide 360 by 2, and the Quotient is 180, this divided by 2 gives 90, which divided by 2 gives 45, this being an odd number, the Division by 2 ceaseth: then I divide the said 45 by 3 and the Quotient is 15, this divided by 3 gives the Quotient 5, and so the Division by 3 ceaseth; then I divide 5 by it self, and the Quotient is Unity. Now by the help of those Divisors or Prime numbers, which (as may easily be proved,) are such, that if they be continually multiplied will produce the given number 360, all the rest of the just Divisors of the said 360 may be found out thus:

First, I set every one of the said Prime Divisors, 2, 2, 2, 3, 3 and 5 at the Head of a Columel, as you see in this Table; then I multiply the first Divisor 2 by the second

Divisor 2, and set the Product 4 under 2 in the second Columel; again, I multiply the said 4 by 2, (which stands at the Head of the third Columel,) and set the Product 8 under 2 in the third Columel. Then I multiply every one of the numbers in the first, second and third Columels, by 3, which stands at the Head of the fourth Columel, and write the Products under 3 in the said fourth Columel; except such Products which happen to be the same with any of those before written, (for one and the same Product must not be written twice,) so multiplying 2, 4 and 8 by 3, I set the Products 6, 12 and 24 under 3 in the fourth Columel. Again, I multiply every one of the numbers in the first, second, third and fourth Columels by 5, (which stands at the top of the fifth Columel,) and set the Products under the said 5; except (as before) such Products which happen to be the same with any of those before written in any of the precedent Columels: so the Products written under 5 in the fifth Columel are 9, 18, 36 and 72. Lastly, I multiply every one of the numbers in the first, second, third, fourth and fifth Columels by 5, (which stands at the Head of the last Columel,) and write the several Products, (except as is before excepted,) under the said 5: So at length all the just Divisors to the given number 360 are found these, to wit, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180 and 360; every one of which Divisors except the greatest, (which is always equal to the number first proposed,) is an *Aliquot part* of 360, which (as you see) hath 23 *Aliquot parts*, and 24 *Divisors*.

2	2	2	3	3	5
	4	8	6	9	10
			12	18	20
			24	36	40
				72	15
					30
					60
					120
					45
					90
					180
					360

B b

Example 2.

Example 2.

Again, if it be required to find out all the Aliquot parts and Divisors of 2310, the Operation will be like that in Example 1. For, first the Prime Divisors will be found these, to wit, 2, 3, 5, 7, 11; then after the said Prime Divisors are set at the heads of so many Columns, as you see in the Table in the Margin, the rest of the Divisors will be found out by Multiplication according to the foregoing directions; which in sum amount to this, viz. Each Prime Divisor standing at the head of every Column following the first, is to be multiplied by every one of the numbers in the foregoing Columns, (except such which make the same Products as were before produced,) and the Products are to be set under each Prime Divisor respectively by which they were produced: So all the Divisors to the given number 2310 are discovered to be these, to wit, 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, &c. as you see in this Table; every one of which Divisors except the greatest, to wit, 2310, (which is the same with the number proposed,) is an Aliquot part of the said 2310, which hath 31 Aliquot parts, but 32 Divisors.

2	3	5	7	11
6	10	14	22	
15	21	33		
30	42	66		
	35	55		
	70	110		
	105	165		
	210	330		
		77		
		154		
		231		
		462		
		385		
		770		
		1155		
		2310		

Upon the same Foundation the Divisors of Quantities express'd by Letters may be found out; as will appear by the following Examples. But this work requires that the Analyst be well exercis'd in the Rules of Algebraical Multiplication, Division, and the Extraction of Roots; for the finding out of the Primitive or Incomposit Divisors, when the given Quantity is compos'd of many large Members connected by different Signs, is oftentimes both difficult and laborious.

Example 3.

Let it be required to find out all the Divisors and Aliquot Parts of this Quantity $aaabbc$. First, I divide the said $aaabbc$ by a , and the Quotient is $aabc$, which divided by a gives abc , this divided by a gives bc , and so the Division by a ceaseth. Then I divide abc by b , and the Quotient is c , this divided by b gives c , which being 1 Primitive or Incomposit quantity I divide by it self, and the Quotient is 1: So all the Primitive Divisors of the proposed Quantity $aaabbc$ are found a, a, a, b, b and c ; which are manifestly such as being multiplied continually will produce the given quantity $aaabbc$.

Now out of those Divisors, after they are set at the heads of so many Columns as you see in this Table, I search out the rest of the Divisors by Algebraical multiplication; in

a	a	a	b	b	c
aa	aa	ab	bb	ac	
aaa	aab	abb	aac		
	aaab	aaab	aaac		
		aaab	aaab	bc	
			abc	aabc	
				aaabc	
				bbc	
				abbc	
				aaabc	
				aaabc	

like manner as in Example 1.) So all the different Divisors to the given quantity $aaabbc$ are found these, to wit, 1, $a, a, a, a, b, ab, aab, aabb, bb, abb, aabb, aaabb, c, ac, aac, aaaa, bc, abc, aabc, aaabc, bbc, abbc, aabbc, aaabbc$; every one of which Division except the last and greatest is an Aliquot part of the given Quantity $aaabbc$, which hath 23 Parts, and 24 Divisors.

Note, That this third Example differs not from Example 1. Inving the Algebraical Division and Multiplication is used here, in stead of vulgar Division and Multiplication in numbers there.

Exam

Example 4.

After the same manner, 31 Aliquot parts and 32 Divisors will be found to this quantity $abcde$, viz. 1, $a, b, ab, c, ac, bc, abc, d, ad, bd$, &c. as you see them express'd in the following Table.

Primitive Divisors;				
a	b	c	d	e
ab	ac	ad	ae	
bc	bd	be		
abc	abd	abe		
	acd	ace		
	bcd	bce		
	abcd	abce		
		ade		
		ade		
		bde		
		abde		
		cde		
		acde		
		bced		
		abcde		

Compare this Example with the precedent Example 2.

Example 5.

Again, to find all the Divisors of this Compound quantity $aaabc - abbc$: First, I search out all its Prime Divisors thus, viz. I divide the said Compound quantity by a , and the Quotient is $aabc - bbcc$; this divided by b gives $aac - bcc$, which divided by c gives the Quotient $aa - bb$: This divided by $a - b$ gives the Quotient $a + b$, which being a Primitive quantity I divide it by it self and the Quotient is 1. So the prime Divisors are found $a, b, c, a - b$ and $a + b$, which are to be reserved.

aaabc - abbc	aabc - bbcc	aac - bcc	aa - bb	a + b	1
a	b	c	a - b	a + b	

Then, (as in the foregoing Examples,) I set the said Primitive Divisors at the heads of so many Columns, and from those Divisors, (according to the directions in Example 1.) I find out all the rest by Multiplication: so at length it appears that $aaabc - abbc$ the Compound quantity proposed hath 31 Aliquot parts and 32 Divisors; to wit, 1, $a, b, ab, c, ac, bc, abc, a - b, aa - ab, ab - bb$, &c. as you see them express'd in the following Table.

a	b	c	a - b	a + b
ab	ac	ad - ab	aa - ab	
bc	bc	ab - bb	ab - bb	
abc	abc	aab - abb	aab - abb	
	ac - bc	ac - bc	ac - bc	
	aac - abc	aac - abc	aac - abc	
	abc - bcc	abc - bcc	abc - bcc	
	aaabc - abbc	aaabc - abbc	aaabc - abbc	
		aa - bb	aa - bb	
		aa - abb	aa - abb	
		aab - bbb	aab - bbb	
		aaab - abbb	aaab - abbb	
		aac - bbc	aac - bbc	
		aaac - abbc	aaac - abbc	
		aaab - bbcc	aaab - bbcc	
		aaabc - abbb	aaabc - abbb	

Bb 2

Example 6:

Example 6.

Again, to find out all the Divisors of this Quantity $aaabbc - 2aabbc + abbbb$; First, (as before,) I search out the Primitive Divisors, viz. I divide the Quantity proposed by a , and the Quotient is $aabbc - 2abbc + bbbb$, which divided by b gives the Quotient $aa - 2ab + bb$; this divided again by b gives $aa - 2ab + bb$, which divided by b gives $aa - 2ab + bb$; this last Quotient being a Square whose side is either $a - b$ or $b - a$, according as a is greater or less than b , I shall suppose a to be greater than b , and then dividing the said Square $aa - 2ab + bb$ by its side $a - b$ the Quotient is also $a - b$; and lastly, by dividing $a - b$ by itself, (because 'tis a Primitive quantity,) the Quotient is 1. Thus the Primitive Divisors of the quantity proposed are found $a, b, b, a - b$ and $a - b$. Then every one of them being set at the head of a Column, and multiplication made according to the Operation in the precedent Examples, the rest of the desired Divisors to the quantity $aaabbc - 2aabbc + abbbb$ will be found out; and at length all the Divisors to the said quantity are discovered to be these, viz. 1, $a, b, ab, bb, abb, c, ac, bc, abc, bbc, abbc, a - b, aa - ab, ab - bb$, &c. as you see them exprest in the following Table.

a	b	b	c	$a - b$	$a - b$
	ab	bb	ac	$aa - ab$	$aa - 2ab + bb$
		abb	bc	$ab - bb$	$aaa - 2aab + abb$
			abc	$aab - abb$	$aab - 2abb + bbb$
			bbc	$abb - bbb$	$aaab - 2aabb + abbb$
			$abbc$	$aabb - abbb$	$aabb - 2abb + bbbb$
				$ac - bc$	$aaab - 2aabb + abbb$
				$aac - abc$	$aac - 2abc + bbc$
				$abc - bbc$	$aaac - 2aac + abc$
				$aabc - abbc$	$aabc - 2abc + bbb$
				$abbc - bbb$	$aaabc - 2aaab + abbb$
				$aabbc - abbbc$	$aabbc - 2abbc + bbbb$
					$aaabbc - 2aaabbc + abbbb$

Example 7.

In like manner, if it be desired to find out all the Divisors of this Quantity $aaaaa + 2aaacc + aaccc$, that is, $a^5 + 2a^4cc + aac^3$, I divide it first by a and the Quotient is $a^4 + 2a^3cc + ac^3$, this divided again by a gives $a^3 + 2a^2cc + c^3$. Now 'tis evident that this last Quotient cannot be divided by a or by c , or the like quantity; but because (by Sect. 4. Chap. 8. Book 1.) the said $a^3 + 2a^2cc + c^3$ is a Square, whose Root is $aa + cc$, I divide the said Square by its Root $aa + cc$, and the Quotient is also the same Root $aa + cc$, which being a Primitive quantity, I divide it by itself, and the Quotient is 1. So the Divisors to be reserved are $a, a, aa + cc$ and $aa + cc$.

$a^5 + 2a^4cc + aac^3$	$a^3 + 2a^2cc + ac^3$	$a^3 + 2a^2cc + ac^3$	$aa + cc$	1
a	a	$aa + cc$	$aa + cc$	

Then after those Divisors are set at the heads of so many Columns, (as you see in the following Table,) I proceed to find out the rest of the Divisors by Multiplication according to the directions in Example 1. viz. I multiply each primitive Divisor standing at the head of every Column following the first by every one of the Quantities in the preceding Columns, and set the Products under the respective primitive Divisor, with this Caution, that one and the same Product be not written down twice: So at length I find all the different Divisors to be these, viz. 1; a ; aa ; $aa + cc$; $a^3 + acc$; $a^4 + aacc$; $a^5 + 2a^4cc + ac^3$; $a^3 + 2a^2cc + ac^3$; and $a^5 + 2a^4cc + aac^3$: all which Divisors except the last are Aliquot parts of the proposed Quantity $a^5 + 2a^4cc + aac^3$.

a	a	$aa + cc$	$aa + cc$
	aa	$a^3 + acc$	$a^4 + 2aacc + c^3$
		$a^4 + aacc$	$a^5 + 2a^4cc + aac^3$
			$a^5 + 2a^4cc + aac^3$

V.I. By

V.I. By this skill of finding out all the Divisors of Quantities, we may reduce two or more given Quantities, when they are not Prime between themselves, to others in the same Reason (or Proportion) with those given, and in the smallest Terms: As, to reduce these three quantities, $aaa - abb$; $aab - bbb$; and $aaa + aab - abb - bbb$ to the smallest quantities in the same Proportion with those proposed; First, I seek (by the Method before delivered) all the different Divisors to every one of those three given Quantities, so I find the Divisors of the first quantity $aaa - abb$ to be these, viz. 1; a ; $a + b$; $a - b$; $aa - ab$; $aa - bb$; $aaa - abb$; and the Divisors of the second quantity $aab - bbb$ to be these, viz. 1; b ; $a - b$; $ab - bb$; $a + b$; $ab + bb$; $aa + bb$; and $aab - bbb$: also the Divisors of the third quantity $aaa + aab - abb - bbb$ to be these, to wit, 1; $a - b$; $a + b$; $aa - bb$; $aa + 2ab + bb$ and $aaa + aab - abb - bbb$. Now because among those three Companies of Divisors, these three $a - b$, $a + b$ and $aa - bb$ are found in each Company, we may by the help of any one of those three Divisors reduce the given Quantities, to others more simple and in the same Proportion with those given. But to find out the smallest Terms, I divide the proposed Quantities $aaa - abb$; $aab - bbb$ and $aaa + aab - abb - bbb$ severally by $aa - bb$, (to wit,) such of the said three Divisors which hath most Dimensions,) and there arise $a - b$ and $a + b$; which three Quantities are the smallest Terms that can be found in the same Proportion with the three Quantities first proposed.

Note. The Quantities propos'd to be reduced are said to be Prime the one to the other when they have no common Divisor besides 1, (to wit, Unity,) in which case the Quantities proposed are already in their smallest Terms.

V.II. The finding out of Divisors may very fitly be applied to the reducing of Fractions to their smallest Terms: As, to abbreviate this Fraction,

$$\frac{aaa + aab - abb - bbb}{aaa - abb}$$

First, the Divisors of the Numerator (by the precedent Method) are found 1; $a - b$; $a + b$; $aa - bb$; $aa + 2ab + bb$; and $aaa + aab - abb - bbb$: likewise, the Divisors of the Denominator are 1; a ; $a + b$; $a - b$; $aa + ab$; $aa - ab$; $aa - bb$; and $aaa - abb$. Then because among those Divisors, these three, to wit, $a + b$, $a - b$ and $aa - bb$ are common both to the Numerator and Denominator, I divide the Numerator and Denominator severally by $aa - bb$, (to wit, that common Divisor which hath most Dimensions;) so there ariseth $a + b$ for a new Numerator, and a for a new Denominator, which gives this Fraction $\frac{a + b}{a}$, (or $1 + \frac{b}{a}$) equal to that proposed, and in the smallest Terms; as was desired.

In like manner to abbreviate $\frac{aaa - abb}{aa + 2ab + bb}$, because the greatest Divisor common to the Numerator and Denominator is $a + b$, I divide the Numerator and Denominator severally by $a + b$, and there ariseth $\frac{aa - ab}{a + b}$; which is equal to the Fraction proposed, and in the smallest Terms.

VIII. Observations upon the Examples in the foregoing Sect. V.

First, When two, three, or more of the formost Letters (towards the left hand) of a Simple quantity are equal to one another, (viz. exprest by one and the same Letter,) then mark well how many equal Letters stand formost together, for so many Aliquot parts they will give: As, in Example 3. in Sect. 5. where the Quantity proposed is $aaabbb$, the three first letters a, a, a , (that is, aaa) give three Aliquot parts, to wit, 1, a , aa ; but four Divisors, 1, a , aa , aaa . In like manner, if four equal Letters stand formost together, as a, a, a, a , or $aaaa$, they will afford these four Parts, 1; a , aa , aaa ; but five Divisors, to wit, 1, a , aa , aaa , $aaaa$. The like property ensues, when five or more equal Letters stand formost together.

Hence it is evident that every Power hath so many Aliquot Parts as there be Dimensions in the Power; As, the Square aa whose Index (or number of Dimensions) is 2, hath two Parts, to wit, 1 and a ; likewise the Cube aaa , or a^3 , hath three Parts; the fourth Power $aaaa$ (or a^4) hath four Parts; and so forwards.

Secondly,

Secondly, It is evident from all the precedent Examples in *Señt. 5.* that when among the Primitive Divisors, (which are set at the tops of the Columns,) a following Divisor differs from the next precedent Primitive Divisor, then the multitude of Divisors in the Column of the said following Divisor is more by one than the multitude of all the different Divisors in the precedent Columns: As, in Example 3. in *Señt. 5.* where the Quantity proposed is *aaabbc*, the letter (or Primitive Divisor) *b* which follows and is different from the next foregoing Primitive Divisor *a*, gives four Divisors, to wit, *b, ab, ab², and aab*, which are more in multitude by one than all the foregoing different Divisors *a, aa, and aaa*.

Again, in Example 4. in *Señt. 5.* where the Quantity proposed is *abcde*, the Divisors *b* and *ab* in the second Column are more in number by one than *a* in the first; likewise the Divisors *c, ac, bc, and abc* in the third Column are more in multitude by one than *a, b* and *ab*, to wit, all the Divisors in the first and second Columns: also *d, ad, bd, abd, cd, acd, bcd* and *abcd* in the fourth Column, are more in multitude by one than all the Divisors in the first, second and third Columns, and so forward. The Reason is manifest; for every Primitive Divisor which stands at the top of a following Column is multiplied into all the different Divisors severally in all the foregoing Columns; and therefore if that multiplying Primitive Divisor be added to the number of those Products, the total multitude must necessarily be more by one than the multitude of different Divisors in all the foregoing Columns.

Thirdly, It is also evident, that when the said Primitive Divisors are all different, then the numbers which express the multitude of Divisors in every Column are in continual Proportion increasing from Unity in a Duple Reason: As, in the fourth Example in *Señt. 5.* where the Primitive Divisors *a, b, c, d, e* are all different, there is one Divisor in the first Column, two in the second, four in the third, eight in the fourth, and sixteen in the fifth, which numbers of multitude, to wit, 1, 2, 4, 8 and 16 are manifestly in Duple Proportion. Therefore when all the Primitive Divisors of a Quantity proposed are different, or unlike, then if so many of the formost Terms of the said continual Proportionals 1, 2, 4, 8, 16, &c. be added together, as there be Primitive Divisors, (to wit, those Incomplete quantities, which being continually multiplied will produce the Quantity proposed,) the sum shall be the number of Aliquot Parts contained in that Quantity; and the number of Divisors shall be more by one than that sum.

As, for Example, if the number of Aliquot Parts in the quantity *ab* be desired, I add 1 and 2 together, (to wit, the two first Terms of the said Geometrical Progression 1, 2, 4, 8, 16, &c.) and the sum 3 shews that *ab* contains three Aliquot Parts, and 4 (that is, 3 + 1) Divisors. Likewise if there be proposed the Quantity *abc*, (which consists of three different letters,) the sum of 1, 2, 4, (to wit, of the three first Terms of the said Geometrical Progression,) is 7; which shews that *abc* contains seven Parts, but eight (or 7 + 1) Divisors. Again, if *abcd* (which consists of four different letters) be proposed, the sum of 1, 2, 4, 8, (the four formost Terms of the said Progression,) is 15; which shews that the quantity *abcd* contains fifteen Aliquot Parts, and sixteen (or 15 + 1) Divisors, and so forward. But because the said Proportionals proceed in a Duple Reason from Unity, the sum of any number of Terms may be found out by this brief Rule, *viz.* The third Term (or Proportional) lessened by Unity, (the first Term) gives the sum of the first and second Terms; likewise the fourth Term lessened by 1, gives the sum of the first, second and third Terms; and the fifth Term lessened by 1, gives the sum of the first, second, third and fourth Terms; and so forward, infinitely. All which may be further illustrated by the ten Quantities, and their respective multitudes of Aliquot Parts, express in the following Table.

Quantities given.	Multitude of Parts.	Sums of Terms in continual Proportion, proceeding from 1 in Duple Reason.
<i>a</i>	hath 1 =	1
<i>ab</i>	3 =	1 + 2
<i>abc</i>	7 =	1 + 2 + 4
<i>abcd</i>	15 =	1 + 2 + 4 + 8
<i>abcde</i>	31 =	1 + 2 + 4 + 8 + 16
<i>abcdef</i>	63 =	1 + 2 + 4 + 8 + 16 + 32
<i>abcdefg</i>	127 =	1 + 2 + 4 + 8 + 16 + 32 + 64
<i>abcdefgh</i>	255 =	1 + 2 + 4 + 8 + 16 + 32 + 64 + 128
<i>abcdefghi</i>	511 =	1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256
<i>abcdefghik</i>	1023 =	1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512

Zetwih

Fourthly, When two, three or more equal letters in a Simple quantity stand together, and follow some different foregoing letter or letters, then as many Aliquot Parts as the first of those following equal letters produceth, (according to *Observat. 2.*) so many Parts every one of the rest of the said following letters will produce. As, in Example 3. in *Señt. 5.* where this quantity *aaabbc* is proposed, the three first letters, *a, a, a*, (or *aaa*) give three Parts; (by *Observat. 1.*) and the first following letter *b*, in regard it differs from the next preceding letter *a*, gives four Parts, (by *Observat. 2.*) now I say the second *b* shall also give four Parts, and if there had been a third *b*, or a fourth *b*, &c. every one of them would give four parts, to wit, as many as the first *b* produced.

In like manner, if this quantity *abbbb* or *ab⁵* be proposed, the first letter *a* gives one Part; then (by *Observat. 2.*) the next following letter *b* (in regard it differs from *a*) gives two Parts: now I say every *b* following the first *b* will also give two Parts, and so *bbbb* will give ten (to wit, five times two) Parts; which added to one Part noted for *a* makes 11 Parts; whence I conclude that the quantity *abbbb* contains 11 Aliquot Parts, and 12 Divisors. All which may be produced particularly by the Rule in the foregoing *Señt. 5.*

Again, if this quantity *abcccc* be proposed, first, (by *Observat. 3.*) *abc* will give seven Parts, and (by *Observat. 2.*) the next following letter *d* gives eight Parts; therefore (by this fourth *Observat.*) every *d* following the first *d* gives also eight Parts, and consequently *ddda* gives 24 Parts, which added to the seven Parts before noted for *abc*, makes 31 Parts. So that the quantity *abcccc* hath 31 Aliquot Parts; and 32 Divisors; and the same number of Parts and Divisors will be found in the number produced by the continual Multiplication of these five Prime numbers 2, 3, 5, 7, 7, 7.

Fifthly, From what hath been said in the precedent Observations 'tis easie to discover how many Aliquot Parts are contained in any Simple quantity design'd by letters, without producing the particular Parts: As, if *aaabbc* be proposed, first, three Parts are to be noted for *aaa*, (according to *Observat. 1.*) and eight Parts more for *bb*, (by *Observat. 4.*) which eight Parts added to the three Parts before noted make eleven Parts; then for *c*, twelve Parts are to be noted, (to wit, 11 + 1, according to *Observat. 2.*) which added to the said eleven Parts makes 23 Parts: whence I conclude that the quantity *aaabbc* hath 23 Aliquot Parts, and 24 Divisors; which are particularly express in Example 3. *Señt. 5.*

In like manner, we may discover that this quantity *aaaabbbccedd* or *a⁴b³c²d²* hath 359 Aliquot Parts, and 360 Divisors; for first, I note 5 Parts for *a⁴*, (according to *Observat. 1.*) then (by *Observat. 4.*) *bbb* or *b³* gives 24 Parts, which added to the five Parts before noted makes 29 Parts; and because one single *c* gives 30 Parts, to wit 29 + 1, (by *Observat. 2.*) *cc* or *c²* will give 90, to wit, 3 times 30 Parts, (by *Observat. 4.*) which added to 29 Parts before noted makes 119 Parts; lastly, because the letter *d* is written twice, and one single *d* gives 120, to wit, 119 + 1 Parts, (by *Observat. 2.*) *dd* will give 240 Parts; (by *Observat. 4.*) which added to 119 Parts before noted, makes 359 Parts; which is the multitude of Aliquot Parts the proposed Quantity hath, but its number of Divisors is 360.

And with the like facility we may discover the multitude of Parts and Divisors of a given number, after its Primitive Divisors are found out: As, for example, to find how many Parts and Divisors 15876000 hath, I search out by Division (in like manner as in the Examples in *Señt. 5.*) all the Primitive Divisors which being continually multiplied will produce the said given number, and find them to be these, to wit, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 7, 7, 7, which may be noted by *a⁴b³c³dd*; but this Quantity (as before hath been shewn) hath 359 Aliquot Parts and 360 Divisors; and therefore the said 15876000 hath the same number of Parts and Divisors; which may be particularly found out by the method in the precedent Examples in *Señt. 5.*

Sixthly, If a Quantity be composed of different letters or Powers, and Unity be added severally to the Indices of those Powers, that is, to the numbers expressing how oft each letter is found in that Quantity, then the numbers resulting by those Additions being multiplied one into the other continually, will produce a number greater by Unity than the number of Aliquot Parts that Quantity hath: As, for example, if *aaaabbb* or *a⁴b³* be proposed, I add 1 to 4 and 3 severally, (because the Indices of *aaa* and *bbb* are 4 and 3,) and it makes 5 and 4; these multiplied one into the other make 20, which is greater by 1 than 19 the number of Aliquot Parts that the proposed quantity *a⁴b³* hath. The reason of this property is not difficult to be conceived; for since (by *Observat. 1.*),

aaaa

aaaa hath four Parts, that is, five Parts wanting one Part; and *bbb* following *aaaa* hath thrice five Parts, (by *Observat. 4.*) therefore the whole quantity *aaaabbb* (or *a⁵b³*) hath 4×5 Parts wanting one Part, *viz.* 19 Parts; which numbers 4 and 5 exceed 3 and 4 the Indices of *bbb* and *aaaa*, severally by Unity.

Again, if *aaaabbbcc* be proposed, the Indices of *aaaa*, *bbb* and *cc* are 4, 3 and 2; which increased severally by 1, make 5, 4 and 3; these multiplied continually produce 60, which is greater by Unity than 59; the number of Aliquot Parts which the proposed quantity *aaaabbbcc* hath. For since (for the reason in the last preceding Example) *aaaabbb* hath 4×5 Parts wanting one Part, and *cc* following *aaaabbb* hath (by *Observat. 4.*) $2 \times 4 \times 5$ Parts, the proposed quantity *aaaabbbcc* hath consequently $3 \times 4 \times 5$ Parts wanting one Part, that is, 59 Parts; which numbers 3, 4 and 5 do severally exceed the Indices of *cc*, *bbb* and *aaaa* by Unity.

Seventhly, From the preceding *Observat. 6.* it follows, That if a Composite number be resolved into any two or more of such of its Factors, the least of which exceeds Unity, and if from every one of those Factors Unity be subtracted, the Remainders shall be Indices of so many several Powers expressible by different letters that being joyned together, (that is, multiplied one into the other,) will give a Quantity having a number of Aliquot Parts less by Unity than the Composite number proposed: As, for example, if 20 be proposed; for as much as 5 and 4 multiplied one by the other produce 20, I subtract 1 from 5 and 4 severally; so the Remainders 4 and 3 do shew, that if the fourth Power of some quantity *a*, as *aaaa*, be multiplied into the third Power of some other quantity *b*, as into *bbb*, the Quantity produced, to wit, *aaaabbb* hath 19 Aliquot Parts, which 19 is less by Unity than 20 the number proposed. Again, because the Product of 10 into 2 doth also make 20, I subtract 1 from 10 and 2 severally; so the Remainders 9 and 1 do shew, that if the ninth Power of some quantity *a*, as *a⁹*, be multiplied by some other different quantity *b*, the Quantity produced, to wit, *a⁹b* hath also 19 Aliquot Parts. Hence it is manifest, that often times many Quantities may be found out, every one of which shall have a given multitude of Aliquot Parts; as will appear in the next following Section.

IX. The manner of finding out all such Quantities as shall have a given multitude of Aliquot Parts.

If the multitude of Aliquot Parts desired be any of the numbers of the second Column of the Table in *Observat. 3. Sect. 8.* the Quantity there standing on the left hand of the number, and on the same line with it; hath the number of Parts desired. As, if it be desired to find a Quantity that hath 63 Aliquot Parts, that Table shews that *abedf* hath 63 Parts; and therefore if six Prime numbers, suppose 2, 3, 5, 7, 11, 13 be taken for the values of those six letters, *a, b, e, d, f*, the Product made by the continual multiplication of the said Prime numbers, to wit, 30030, shall have 63 Aliquot Parts, and 64 Divisors.

But without respect to that Table, by the help of the Observations in the foregoing *Sect. 8.* many Quantities for the most part, and alwayes one Quantity may easily be found out that shall have a given multitude of Aliquot Parts; as will be made manifest by the following Examples.

Example 1.

Let it be required to find out all such simple Quantities expressible by letters, that may every one of them have 15 Aliquot Parts, and 16 Divisors.

1. To the said 15 I add 1 and it makes 16, this I divide by 2 and the Quotient is 8, which divided by it self gives 1; then from each of the Divisors 2 and 8, (the Product of whose multiplication makes the first Dividend 16,) I subtract 1; so the Remainders 1 and 7 do shew that if some letter, as *a*, be written once, and next after it another different letter *b* seven times, the Quantity so composed, to wit, *abbbbbb*

or *a¹b⁷* shall have 15 Aliquot Parts, and 16 Divisors; as was desired.

2. Again, I divide the said 16 (to wit, $15 + 1$) by 2, and the Quotient is 8; this divided again by 2 gives 4, which divided again by 2 gives 2, which divided by it self gives 1; then from every one of the Divisors 2, 2, 2, 2, 2 I subtract 1; so the Remainders 1, 1, 1, 1, 1 do shew that if 4 different single letters be set together, as *abcd*, this quantity shall have 15 Parts, and 16 Divisors, as before.

3. Again,

$$\begin{array}{r} 16 \ 8 \ 4 \ 2 \ 1 \\ 2 \ 2 \ 2 \end{array}$$

$$\begin{array}{r} 16 \ 8 \ 4 \ 2 \ 1 \\ 2 \ 2 \ 2 \ 2 \end{array}$$

3. Again, I divide 16 by 2, and the Quotient is 8; this divided by 2 gives 4, which divided by it self gives 1; then from every one of the Divisors 2, 2, 4 I subtract 1, and the Remainders 1, 1, and 3 do shew, that if two different letters *a* and *b* be joyned together, and next after them a third different from each of them, as *c* be written thrice, the Quantity so composed, to wit *abccc*, shall have 15 Aliquot Parts, and 16 Divisors; as before.

4. Again, I divide 16 by 4, and the Quotient is 4, this divided by it self gives 1; then from each of the Divisors 4 and 4, I subtract 1, and the Remainders 3 and 3 do shew, that if some letter *a* be written thrice, as *aaa*, and next after the same another letter different from *a*, as *b*, be likewise written thrice, the Quantity so composed, to wit, *aaaabbb*, or *a³b³* shall have 15 Aliquot Parts, and 16 Divisors; as before.

5. Lastly, I divide 16 by it self and the Quotient is 1, then from 16 I subtract 1, and the Remainder 15 shews that if some letter *a* be written 15 times, as *aaaaaaaaaaaaa*, or *a¹⁵*, this Quantity shall have 15 Parts, and 16 Divisors; as before.

Hence, because 16 cannot be divided by any other ways than those five before express'd, we may conclude that the five Quantities found out, and those only, to wit, *a¹b⁷*, *abccc*, *aaaabbb*, *a³b³* and *a¹⁵*, have each of them 15 Aliquot Parts, and 16 Divisors. All which Operations do clearly result from *Observat. 6.* and 7. in the precedent *Sect. 8.*

Example 2.

Let it be required to find out all such Quantities expressible by letters, which may every one of them have 23 Aliquot Parts, and 24 Divisors.

First, (as before) I add 1 to 23, and it makes 24; this may be divided by its Factors in a seven-fold manner before the Quotient be Unity, as here you see.

$$\begin{array}{r} 24 \ 8 \ 4 \ 2 \ 1 \\ 3 \ 2 \ 2 \ 2 \end{array} \quad ; \quad \begin{array}{r} 24 \ 6 \ 2 \ 1 \\ 4 \ 3 \ 2 \end{array} \quad ; \quad \begin{array}{r} 24 \ 4 \ 2 \ 1 \\ 6 \ 2 \ 2 \end{array} \\ \hline \begin{array}{r} 24 \ 4 \ 1 \\ 6 \ 4 \end{array} \quad ; \quad \begin{array}{r} 24 \ 3 \ 1 \\ 8 \ 3 \end{array} \quad ; \quad \begin{array}{r} 24 \ 2 \ 1 \\ 12 \ 2 \end{array} \quad ; \quad \begin{array}{r} 24 \ 1 \\ 24 \end{array}$$

Whence I conclude that seven different Quantities may be produced, every one of which shall have 23 Aliquot Parts, and 24 Divisors; now to find out the said Quantities, I subtract 1, (to wit Unity,) from every one of the Divisors of the foregoing seven-fold Division, so the Divisors, 3, 2, 2, 2 of the first Division being severally lessened by Unity give 2, 1, 1, 1; whence, according to the precedent directions in Example 1. of this *Sect. 9.* this Quantity may be composed, to wit, *abccc*; and by proceeding in like manner with the rest of the Divisors, seven different Quantities, every one of which hath 23 Aliquot Parts and 24 Divisors, are discovered; and may be express'd either

$$\text{Thus, } \left\{ \begin{array}{l} abccc; \\ aaabcc; \\ aaaaabcc; \\ aaaaaabcc; \\ aaaaaaabcc; \\ aaaaaaaaaabcc; \\ aaaaaaaaaaaaaabcc; \end{array} \right\} \quad \text{Or thus, } \left\{ \begin{array}{l} a^2bcd; \\ a^3b^2c; \\ a^4b^3c; \\ a^5b^4c; \\ a^6b^5c; \\ a^7b^6c; \\ a^8b^7c; \end{array} \right\}$$

Example 3.

Let it be required to find out a Quantity which hath 42 Aliquot Parts.

First, (as before) I add 1 to 42 and it makes 43, which being a Prime number, (that is, such as cannot be divided by any number but by it self or Unity,) doth shew, that there is only one Quantity can be found that hath 42 Aliquot Parts; *viz.* some letter, as *a* being written 42 times one after another, or a single *a* with its Index 42, as *a⁴²*, doth express a Quantity (to wit, the forty-second Power of *a*) which hath 42 Aliquot Parts, and 43 Divisors. The like is to be understood of other Quantities, when the multitude of Aliquot Parts desired being increased with Unity makes a Prime number.

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For further illustration of the premises, the Learner may view the following Table, which shews all the various Quantities exprest by Letters, that have a given multitude of Aliquot Parts not exceeding 50; and upon the grounds before explained, the Table may be continued as far as you please.

Quantities.

Aliquot Parts.

a	hath	1
aa	hath	2
ab, a^2	have each	3
a^3	&c.	4
aab, a^4		5
a^5		6
a^3b, abc, a^7		7
$aabb, a^8$		8
a^4b, a^9		9
a^{10}		10
$a^2bc, a^3b^2, a^4b, a^{11}$		11
a^{12}		12
a^4b, a^{13}		13
a^5b^2, a^{14}		14
$a^6bc, abcd, a^3b^3, a^2b, a^{15}$		15
a^{16}		16
$a^3b^2c, a^2b^2, a^2b, a^{17}$		17
a^{18}		18
$a^4bc, a^4b^3, a^2b, a^{19}$		19
a^5b^2, a^{20}		20
$a^{10}b, a^{21}$		21
a^{22}		22
$a^3b^2c, a^2bcd, a^3bc, a^4b^2, a^2b^2, a^{11}b, a^{23}$		23
a^{24}		24
$a^{12}b, a^{25}$		25
$a^2b^2c^2, a^3b^2, a^{26}$		26
$a^2bc, a^2b^3, a^{13}b, a^{27}$		27
a^{28}		28
$a^4b^2c, a^4b^4, a^2b^2, a^4b, a^{29}$		29
a^{30}		30
$a^3bcd, a^3b^2c, a^2bc, abcd, a^2b^2, a^{13}b, a^{31}$		31
a^{32}		32
$a^{16}b, a^{33}$		33
$a^{16}b, a^{34}$		34
$a^2b^2cd, a^3b^2c, a^2b^2c^2, a^2bc, a^3b^3, a^4b^2, a^{11}b^2, a^2b, a^{35}$		35
a^{36}		36
$a^{18}b, a^{37}$		37
$a^{18}b, a^{38}$		38
$a^4bcd, a^4b^2c, a^2bc, a^2b^2, a^2b, a^{39}b, a^{39}$		39
a^{40}		40
$a^2b^2c, a^4b^2, a^{13}b^2, a^{20}b, a^{41}$		41
a^{42}		42
$a^2bc, a^2b^3, a^{10}b, a^{43}$		43
$a^4b^2c^2, a^4b^4, a^{14}b^2, a^{44}$		44
a^{45}		45
a^{46}		46
$a^3bcd, a^3bcd, a^4b^2c, a^2bcd, a^3b^2c, a^2b^2c, a^{11}bc, a^2b, a^4b^2, a^{13}b^2, a^{47}$		47
a^{48}		48
$a^4b^2c, a^4b^4, a^{14}b^2, a^{49}$		49
a^{50}		50

X. H

X. How to find out the Smallest number that shall have a given multitude of Aliquot Parts.

First, by the foregoing *Sett.* 9. search out all the Quantities expressible by letters, every one of which may have the number of Aliquot Parts desired; then to the different letters by which every one of those Quantities is exprest, assign the smallest Prime numbers, and find out by continual Multiplication the Products of those Prime numbers correspondent to the said Quantities. Again, let the values of those letters be exprest by the same Prime numbers varied as many ways as is possible, and find out their respective Products, as before. Lastly, all those Products being compared one to another, the least of them shall be the smallest number that hath the prescribed multitude of Aliquot Parts.

Example 1.

Let it be required to find the smallest number that hath 15 Aliquot Parts.

First, all the different Quantities that can be found to have severally 15 Aliquot Parts, (as appears by the precedent *Sett.* 9.) are these, to wit, $abcd, a^3bc, a^2b^2, a^4b, a^5$; then by assigning to a, b, c, d , the smallest Prime numbers, 2, 3, 5, 7, for $abcd$ there will be found 210; (by multiplying 2, 3, 5, 7 one into the other continually,) for a^3bc , 120; for a^2b^2 , 216; for a^4b , 384; and for a^5 , 32768; the least of which Products is 120. But before we can determine whether 120 be the least number or not that hath 15 Aliquot Parts, enquiry must be made by exchanging the values of those letters with the said Prime numbers all manner of ways, *viz.* we may suppose $a=3$; $b=2$; $c=5$; and $d=7$: or, $a=5$; $b=2$; $c=3$; and $d=7$: or again, $a=7$; $b=2$; $c=3$; and $d=5$; and many other ways the values of a, b, c, d may be exprest by the said Prime numbers 2, 3, 5, 7: and consequently from those variations, the quantities $abcd, a^3bc, a^2b^2, a^4b, a^5$ will be expounded by various numbers, which must be compared together, and then the least among them all is the number sought. So after all variations are made, it will appear that a^3bc is that Quantity by which 120, the smallest number having 15 Aliquot Parts and 16 Divisors will be found out.

Example 2.

Again, if the least number that hath 23 Aliquot Parts, or 24 Divisors, be desired:

First, by *Sett.* 9. all the Quantities which have severally 23 Parts will be found these, to wit, $abcd, a^3b^2c, a^2b^2c, a^4b^2, a^5b^2, a^6b^2, a^{11}b, a^{23}$. Then, by assuming for the values of a, b, c, d the least Prime numbers 2, 3, 5, 7; for a^3b^2c there will be found 420; for a^2b^2c , 360; for a^4b^2 , 480; for a^5b^2 , 864; for a^6b^2 , 1152; for $a^{11}b$, 6144; and for a^{23} , 8388608: and after all other possible variations made with the said letters and Prime numbers, by taking sometimes one, sometimes another of the said numbers for the value of a, b , &c. it will at length appear that a^3b^2c finds out 360, the least number that hath the desired multitude of 23 Aliquot Parts, and 24 Divisors.

If there be not occasion to find the least, but any number that hath a given multitude of Aliquot Parts, suppose 15, then you may indifferently use any one of these five quantities, $abcd, a^3bc, a^2b^2, a^4b, a^5$, by assigning to a, b, c, d Prime numbers at pleasure; and taking sometimes one, sometimes another of those numbers; or always new Prime numbers for the values of a, b, c, d ; whence innumerable numbers may be found out, every one of which shall have 15 Aliquot Parts. As, if we suppose $a=2$; $b=3$; and $c=5$; there will be found for a^3bc , 120: but by putting $a=3$; $b=2$; and $c=5$; there will be found for a^3bc , 170. Or also by assuming $a=7$; $b=11$; and $c=13$; there will be produced for a^3bc , 49049: or if we put $a=17$; $b=19$; and $c=23$; then $a^3bc=2146981$. And in like manner you may use every one of the other four quantities $abcd, a^2b^2, a^4b, a^5$. The like also is to be understood of every one of these, $abcd, a^3bc, a^2b^2, a^4b, a^5$; for the finding out innumerable numbers which have severally 23 Aliquot Parts, and 24 Divisors.

Lastly, to find the least number that hath 42 Parts, and 43 Divisors, for as much as a Quantity having this multitude of Parts and Divisors can be designed only in one manner, *viz.* by writing a^{42} ; let the least Prime number 2 be taken for the value of a , and then seek the forty-second Power of the Root 2, by writing down 1 forty-two times separately, and multiplying those numbers one into another, according to the Rule of Continual Multiplication, so the last Product will be 4398046511104; which is the least number that hath the desired multitude of 42 Aliquot Parts. And so of others.

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For further illustration, the Learner may view the following Table, which shews the least number that hath any given multitude of Aliquot Parts under 51. *Note*, That the number of Divisors to any number is always more by one than its number of Aliquot Parts; for albeit a number cannot properly be called a Part of it self, yet 'tis contained in it self once, and therefore may be said to be a Divisor to it self.

Each number in the first of these Columns is the smallest that can be found to have such a multitude of Aliquot Parts as is express'd in the latter Column.

	hath	1	Aliquot Part.
	hath	2	Aliquot Parts.
2		3	
4		4	
6		5	
16		6	
12		7	
64		8	
24		9	
36		10	
48		11	
1024		12	
60		13	
4096		14	
192		15	
144		16	
120		17	
65536		18	
180		19	
262144		20	
240		21	
576		22	
3072		23	
4194304		24	
360		25	
1296		26	
12288		27	
900		28	
960		29	
268435456		30	
720		31	
1073741824		32	
840		33	
9216		34	
196608		35	
5184		36	
1260		37	
68719476736		38	
786432		39	
36864		40	
1680		41	
1099511627776		42	
2880		43	
4398046511104		44	
15360		45	
3600		46	
12582912		47	
70368744177664		48	
2520		49	
46656		50	
6480			
589824			

C H A P. IX.

The Arithmetick both of Surd Numbers, and Surd Quantities express'd by Letters. The Constitution and Invention of six Binomials in numbers; agreeable to those expounded in Prop. 49, 50, 51, 52, 53, 54. Elem. 10. Euclid. with Rules to extract the Square Root out of every one of them; as also, what Root you please out of any Binomial in numbers, having such a Binomial Root as is desired.

SECT. I. Definitions concerning Surd Roots, and their Fundamental Operations.

Every Absolute (or Ordinary) number, whether it be a whole number, or a Fraction, or a whole number with a Fraction annex to it, is called *Rational*: As, 1, 2, 3, 4, &c. also, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, &c. and $1\frac{1}{2}$, (or $\frac{3}{2}$), $5\frac{1}{3}$, (or $\frac{16}{3}$), $29\frac{1}{4}$, &c. are called Rational numbers; so also a , ab , $\frac{bc}{a}$, $a + \frac{bc}{a}$, &c. represent Rational Quantities.

But when the Square Root, Cubick Root, or any other Root cannot be perfectly extracted out of a Rational number, that Root is called *Irrational* or *Surd*; and because it cannot be exactly express'd by any Rational number, it is usual to set some Character (which is called the Radical Sign) before the Rational number, out of which the Root ought to be extracted, to design or signify the same Root: As, $\sqrt{\quad}$ or $\sqrt{(2)}$ prefix'd before any Rational number, signifies the Square Root of that number; $\sqrt{(3)}$ the Cubick Root; $\sqrt{(4)}$ the Biquadratick Root; $\sqrt{(5)}$ the Root of the fifth Power, &c.

Hence $\sqrt{12}$, or $\sqrt{(2)12}$ denotes or represents the Square Root of 12, which Root is called *Irrational* or *Surd*, because it cannot be perfectly express'd by any Rational number; for 3 multiplied by it self produceth 9, which is less than 12; add 4 multiplied by it self produceth 16, which is greater than 12; and although there be innumerable mixt numbers consisting of 3 and some Fraction, which fall between 9 and 16, yet none of them multiplied into it self quadratically can produce the whole number 12.

In like manner, $\sqrt{(3)5}$, which represents the cubick Root of 5, is called an *Irrational* or *Surd* number, because no number can be found, which being multiplied into it self cubically will produce 5 exactly: so also $\sqrt[4]{a}$, $\sqrt[4]{bc}$, $\sqrt[4]{(3)bb}$, &c. represent Surd quantities.

There are two sorts of *Irrational* or *Surd* numbers, Simple and Compound: a Simple Surd number is express'd by one single term; such are $\sqrt{5}$, $\sqrt{10}$, $\sqrt{(3)16}$, $\sqrt{(4)8}$, &c. but a Compound Surd number consists of many simple or single terms, and is formed by the Addition or Subtraction of Simple terms; such are $\sqrt{5} + \sqrt{2}$, $\sqrt{5} - \sqrt{2}$, $\sqrt{8} + \sqrt{6} - \sqrt{2}$; $\sqrt{(3)17} + \sqrt{2}$: which last is called an *Universal Root*, and signifies the cubick Root of the sum of 7, and the square Root of 2. (See Sect. 28. Chap. 1. Book 1. concerning the designing of Surd numbers.)

The Arithmetick of Surd Numbers, and Surd Quantities design'd by Letters, depends chiefly upon these six Primary or Fundamental Operations in simple Surds, *viz.*

1. The Reduction of Rational numbers and Rational quantities express'd by letters, to the form of surd Roots; which shall have a given radical Sign.
2. The Reduction of simple surd Roots having different radical Signs, to other Surds which shall have one common radical Sign, and be equal in value to the given Surds.
3. Multiplication in simple Surds.
4. Division in simple Surds.
5. The Reduction of a given Surd number or quantity to another more simple, when it may be done.
6. How to discover whether two simple Surd numbers or quantities be *Commensurable*, or not; *viz.* whether their *Raſon* or *Proportion* can be express'd by Rational numbers or quantities, or not. These six Operations I shall handle in order.

Sect. II.

SECT. II. How to Reduce Rational numbers and quantities designed by Letters, to the form of Surd Roots, which shall have the same Radical Sign with any Surd Root prescribed.

Multiply the given Rational number or quantity into it self, so often as is requisite to produce a Power of the same Degree with that Power which is denoted by the radical Sign of the prescribed Surd, and then set the said radical Sign before the Power produced by the said multiplication.

As, to reduce 6 to the form of a surd Root which shall have the same radical Sign with $\sqrt[4]{12}$ (or $\sqrt[4]{(2)12}$), I multiply 6 into it self quadratically, and it makes 36; then $\sqrt[4]{36}$ (that is 6,) and $\sqrt[4]{12}$ have the same radical Sign, to wit, $\sqrt[4]{}$ or $\sqrt[4]{(2)}$.

Again, to reduce 5 to the same radical Sign with $\sqrt[4]{(3)12}$, I multiply 5 into it self cubically, (*viz.* 5 into 5, and the Product into 5,) and it produceth 125; then $\sqrt[4]{(3)125}$ (that is, 5,) and $\sqrt[4]{(3)12}$ have the same radical Sign, to wit, $\sqrt[4]{(3)}$.

Likewise, to reduce 3 to the same radical Sign with $\sqrt[4]{(4)12}$, I seek the fourth Power of 3, which (by multiplying the Square of 3 into it self) will be found 81; then $\sqrt[4]{(4)81}$ and $\sqrt[4]{(4)12}$ are of the same kind. And so of others.

By the help of this Rule, when the radical Sign of a simple Surd Fraction hath reference only to one of its Terms, we may reduce the Fraction to another whose radical Sign shall refer both to the Numerator and Denominator: As if $\frac{\sqrt{2}}{5}$ be proposed, which signifies

that $\sqrt{2}$ is divided or to be divided by 5, we may take $\sqrt{25}$ instead of 5, and then the Fraction will be reduced to this $\frac{\sqrt{2}}{\sqrt{25}}$, whose radical Sign refers as well to the Denominator as the Numerator; *viz.* $\sqrt{\frac{2}{25}}$ signifies that $\sqrt{2}$ is divided by $\sqrt{25}$.

Likewise $\frac{5}{\sqrt[4]{34}}$ may be reduced to $\sqrt[4]{(3)125}$, by setting 125 the Cube of 5, for a Numerator instead of 5, and the radical Sign $\sqrt[4]{(3)}$ against the middle of the Fraction; so that $\sqrt[4]{(3)125}$ (which signifies that $\sqrt[4]{(3)125}$ is divided by $\sqrt[4]{(3)4}$) imports as much as $\frac{5}{\sqrt[4]{34}}$, (that is, 5 divided by $\sqrt[4]{(3)4}$).

Nor will the Operation be otherwise in reducing Rational quantities designed by letters to the form of Surd quantities: (respect being had to the Rules of Algebraical Multiplication before delivered.) As to reduce the quantity a , so as it may have the same radical Sign with $\sqrt[4]{b}$, I multiply a into it self quadratically, and it makes aa ; then $\sqrt[4]{aa}$ (that is, a) and $\sqrt[4]{b}$ have the same radical Sign.

Again, to reduce $a + b$ to the same radical Sign with $\sqrt[4]{bc}$, I square $a + b$ and it makes $aa + 2ab + bb$; then $\sqrt[4]{aa + 2ab + bb}$: (that is, $a + b$) and $\sqrt[4]{bc}$ have the same radical Sign.

Likewise, to reduce b to the same radical Sign with $\sqrt[4]{(3)ab}$, I multiply b into it self cubically, and it makes bbb ; then $\sqrt[4]{(3)bbb}$ (that is, b) and $\sqrt[4]{(3)ab}$ have the same radical Sign, to wit, $\sqrt[4]{(3)}$.

Hence also $\frac{a}{\sqrt[4]{b}}$ may be reduced to $\sqrt[4]{\frac{aa}{b}}$, and $\frac{\sqrt[4]{(3)ab}}{3c}$ to $\sqrt[4]{(3)\frac{ab}{27ccc}}$.

SECT. III. How to reduce two simple Surd numbers or quantities having different radical Signs, to two others that may have a common radical Sign.

This Reduction is like that of reducing Vulgar Fractions to a common Denominator; but how 'tis wrought, I shall shew by Examples, first in Surd numbers, and then in Surd quantities express'd by letters.

Example 1.

Let it be required to reduce $\sqrt[4]{(4)10}$ and $\sqrt[4]{(6)7}$ into two other Roots that may have a common radical Sign, and be equal in value to those given.

First divide the given Indices (4) and (6) by their greatest common Divisor (2), and set the Quotients (2) and (3) under their respective Dividends as here you see; then multiply cross-wise, *viz.* the first Dividend or Index (4), by the second Quotient (3); (or the second Dividend (6) by the first Quotient (2));

$$(2) \quad \begin{array}{r} \sqrt[4]{(4)10} \\ (2) \end{array} \times \begin{array}{r} \sqrt[4]{(6)7} \\ (3) \end{array}$$

$$\sqrt[4]{(12)1000} \quad \sqrt[4]{(12)49}$$

the first Quotient (2), and the Product is (12), before which setting $\sqrt[4]{}$ it gives $\sqrt[4]{(12)}$, which is to be reserved for the common radical Sign sought. Then multiply the Powers of the given Roots according to the altern Quotients, *viz.* multiply the first Power 10 cubically, because the second Quotient is (3); and latter Power 7 quadratically, because the first Quotient is (2): so the Products will be 1000 and 49, before each of which prefixing $\sqrt[4]{(12)}$ the common radical Sign before found; there arise $\sqrt[4]{(12)1000}$ and $\sqrt[4]{(12)49}$ the two surd Roots sought, which are equal in value to the given Surds respectively; *viz.* $\sqrt[4]{(12)1000}$ is equal to $\sqrt[4]{(4)10}$, and $\sqrt[4]{(12)49}$ is equal to $\sqrt[4]{(6)7}$; and the Surds found out have a common Radical Sign, as was required.

Example 2.

In like manner, $\sqrt[4]{(2)5}$ and $\sqrt[4]{(3)6}$ will be reduced to $\sqrt[4]{(6)125}$ and $\sqrt[4]{(6)36}$; and the work will stand as here you see underneath.

$$(1) \quad \begin{array}{r} \sqrt[4]{(2)5} \\ (2) \end{array} \times \begin{array}{r} \sqrt[4]{(3)6} \\ (3) \end{array}$$

$$\sqrt[4]{(6)125} \quad \sqrt[4]{(6)36}$$

Example 3.

Again, if $\frac{\sqrt[4]{7}}{3}$ and $\frac{5}{\sqrt[4]{(3)4}}$ be proposed to be reduced to a common Radical Sign, first by the Rule in the preceding Sect. 2. I reduce them to $\sqrt[4]{\frac{7}{27}}$ (or $\sqrt[4]{(2)7}$) and $\sqrt[4]{(3)\frac{125}{4}}$, which according to the Rule in the first Example of this Section will be reduced to these, to wit, $\sqrt[4]{(6)\frac{7}{27}}$ and $\sqrt[4]{(6)\frac{125}{4}}$; and the work will stand as here you see.

$$(1) \quad \begin{array}{r} \sqrt[4]{(2)7} \\ (2) \end{array} \times \begin{array}{r} \sqrt[4]{(3)\frac{125}{4}} \\ (3) \end{array}$$

$$\sqrt[4]{(6)\frac{7}{27}} \quad \sqrt[4]{(6)\frac{125}{4}}$$

The like work is to be done in reducing two Surd quantities express'd by letters, which have different radical Signs, to two others which shall have a common radical Sign, as will appear in the following Examples.

Example 4.

Suppose it be desired to reduce $\sqrt[4]{(2)a}$ and $\sqrt[4]{(6)aa}$ to a common radical Sign.

First, I divide the given Indices (2) and (6) severally by their greatest common Divisor (2) and set the Quotients (1) and (3) under their respective Dividends as here you see; then I multiply cross-wise, *viz.* the first Dividend (2) by the second Quotient (3), (or the latter Dividend (6) by the first Quotient (1), and the Product is (6); before which setting $\sqrt[4]{}$ it gives $\sqrt[4]{(6)}$ for the Common Radical sign sought. Then I multiply the Powers of the given Roots according to the alternate Quotients, *viz.* the first Power a cubically, because the latter Quotient is (3), but the second Power aa , because the first Quotient (1) is a lateral Index, is not to be multiplied into it self at all. So the Products are aaa and aa , before each of which prefixing $\sqrt[4]{(6)}$, (the common radical Sign before found,) there arise $\sqrt[4]{(6)aaa}$ and $\sqrt[4]{(6)aa}$ the two surd Roots sought, which are equal in value to the given Surds respectively, *viz.* $\sqrt[4]{(6)aaa}$ is equal to $\sqrt[4]{(2)a}$, and $\sqrt[4]{(6)aa}$ is equal to $\sqrt[4]{(6)aa}$; and the surd Roots found out have a common radical Sign, to wit, $\sqrt[4]{(6)}$. Therefore that is done which was required.

Example 5.

After the same manner $\sqrt[4]{(4)3b}$ and $\sqrt[4]{(10)5ac}$ will be reduced to $\sqrt[4]{(20)24; bbbbbb}$ and $\sqrt[4]{(20)25aacc}$, and the work will stand as here you see.

$$(2) \quad \begin{array}{r} \sqrt[4]{(4)3b} \\ (2) \end{array} \times \begin{array}{r} \sqrt[4]{(10)5ac} \\ (5) \end{array}$$

$$\sqrt[4]{(20)24; bbbbbb} \quad \sqrt[4]{(20)25aacc}$$

Sect. IV. Multiplication in simple Surd Quantities.

Before Addition and Subtraction can be perform'd in Surd Quantities, the manner of their Multiplication and Division must first be learnt; I shall therefore begin with Multiplication, which requires that the surd Roots proposed to be multiplied be of the same kind; and therefore if they be of different kinds, they must first of all be reduced to the same Radical sign, (by the Rule in the foregoing Sect. 3.) Then,

1. Multiply the numbers or quantities standing next after their common Radical sign one into another, without any regard had to the said Sign; and to the Product of that multiplication prefix the common Radical sign: so this new Root shall be the Product sought.

As, for example, to multiply $\sqrt{5}$ by $\sqrt{3}$, I multiply 5 by 3 and it makes 15, to which I prefix $\sqrt{}$ (the Radical sign of each of the Surds given to be multiplied,) and there ariseth $\sqrt{15}$ for the Product sought.

Likewise if $\sqrt{6}$ be multiplied by $\sqrt{5}$, it produceth $\sqrt{30}$.

Also, $\sqrt{2}$ multiplied by $\sqrt{2}$, makes $\sqrt{4}$.

And $\sqrt{2\frac{1}{2}}$ (or $\sqrt{\frac{5}{2}}$) into $\sqrt{2\frac{1}{2}}$ (or $\sqrt{\frac{5}{2}}$) gives $\sqrt{12\frac{1}{2}}$.

Again, $\sqrt{3\frac{1}{4}}$ multiplied by $\sqrt{(\frac{1}{3})}$, produceth $\sqrt{(\frac{3}{4})20}$.

Likewise $\sqrt{(\frac{1}{4})\frac{1}{2}}$ into $\sqrt{(\frac{1}{4})\frac{1}{2}}$, produceth $\sqrt{(\frac{1}{4})\frac{1}{4}}$.

And if $\sqrt{(\frac{1}{2})5}$ be to be multiplied into $\sqrt{(\frac{1}{3})6}$, the Product will be $\sqrt{(\frac{6}{3})4500}$; for, first, the given Roots being of different kinds are reduced to these, to wit, $\sqrt{(\frac{1}{6})125}$ and $\sqrt{(\frac{1}{6})36}$, which multiplied one into another make $\sqrt{(\frac{6}{6})4500}$.

After the same manner, Multiplication in simple Surd quantities express'd by Letters is performed: as, if \sqrt{a} be to be multiplied by \sqrt{b} , the Product will be \sqrt{ab} . For (according to the Rule of Algebraical Multiplication) the quantity a multiplied by the quantity b , produceth ab ; to which I prefix the given Radical sign $\sqrt{}$, and it gives \sqrt{ab} the Product sought.

Likewise \sqrt{ab} into \sqrt{cd} , produceth \sqrt{abcd} .

And $\sqrt{\frac{2ab}{3c}}$ multiplied by $\sqrt{\frac{9ad}{2b}}$, maketh $\sqrt{\frac{3aad}{c}}$.

Again, to multiply $\sqrt{(\frac{1}{2})d}$ by $\sqrt{(\frac{1}{3})ab}$, first (by the Rule in the foregoing Sect. 3) I reduce them to $\sqrt{(\frac{1}{6})add}$ and $\sqrt{(\frac{1}{6})aabb}$, which multiplied one into another, give $\sqrt{(\frac{1}{6})addaabb}$ for the Product required.

2. When any Surd Root is to be multiplied into it self according to the Index of its own Power, viz. if a surd Square Root be to be squared, or a surd Cubick Root be to be cubed; cast away the Radical sign, and take the number or quantity remaining for the Product sought, which in this case is alwayes Rational: as, to multiply $\sqrt{5}$ into it self, I cast away the Radical sign $\sqrt{}$, and take 5 for the Product, or Square of $\sqrt{5}$; (for $\sqrt{5}$ into $\sqrt{5}$ makes $\sqrt{25}$, that is, 5.) Likewise, the Square of $\sqrt{8}$ is 8, and the Square of $\sqrt{4b}$ is 4b.

In like manner, to multiply $\sqrt[3]{5}$ into it self cubically, I take 5 for the Product; to wit, the Cube of $\sqrt[3]{5}$; (for $\sqrt[3]{5}$ into $\sqrt[3]{5}$ makes $\sqrt[3]{25}$, and this again into $\sqrt[3]{5}$ produceth $\sqrt[3]{125}$, that is, 5.)

Again, $\sqrt{(\frac{1}{4})12}$ multiplied into it self biquadratically, produceth 12; for $\sqrt{(\frac{1}{4})12}$ maketh $\sqrt{(\frac{1}{4})144}$, (which is the Square of $\sqrt{(\frac{1}{4})12}$;) then $\sqrt{(\frac{1}{4})144}$ again into $\sqrt{(\frac{1}{4})12}$ makes $\sqrt{(\frac{1}{4})1728}$, (which is the Cube of $\sqrt{(\frac{1}{4})12}$;) lastly, $\sqrt{(\frac{1}{4})1728}$ again into $\sqrt{(\frac{1}{4})12}$ produceth $\sqrt{(\frac{1}{4})20736}$, that is 12; which is the fourth Power of $\sqrt{(\frac{1}{4})12}$, the Root proposed.

The like is to be done in Surd quantities express'd by Letters; as, if \sqrt{ab} be to be multiplied into it self, or squared, I cast away the Radical sign, and write ab for the Product, or Square of \sqrt{ab} . Likewise, if $\sqrt[3]{bcd}$ be to be multiplied into it self cubically, the Product or Cube thereof will be bcd .

3. When a Surd quantity is given to be multiplied by a Rational quantity, reduce the Rational into the form of a Surd of the same kind with the given Surd, (by the foregoing Rule in Sect. 2.) and then multiply according to the first Rule of this fourth Section; as, to multiply $\sqrt{8}$ by 2, I first reduce 2 to $\sqrt{4}$, then $\sqrt{8}$ into $\sqrt{4}$ gives $\sqrt{32}$; the Product desired: likewise $\sqrt[3]{7}$ multiplied by 5, that is, by $\sqrt[3]{125}$, gives the Product $\sqrt[3]{875}$.

Again, if $\sqrt{(\frac{1}{3})6}$ be to be multiplied by 2, I reduce 2 to $\sqrt{(\frac{1}{3})8}$, (by multiplying 2 into it self cubically;) then $\sqrt{(\frac{1}{3})6}$ multiplied by $\sqrt{(\frac{1}{3})8}$, gives $\sqrt{(\frac{1}{3})48}$ for the Product desired.

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Likewise $\sqrt{(\frac{1}{4})8}$ multiplied by 5, that is, by $\sqrt{(\frac{1}{4})625}$, gives $\sqrt{(\frac{1}{4})5000}$ for the Product sought.

After the same manner, to multiply the Surd quantity \sqrt{a} by the Rational quantity b ; I first reduce b to \sqrt{bb} , then \sqrt{a} into \sqrt{bb} makes \sqrt{abb} the Product sought; likewise $\sqrt{(\frac{1}{3})a}$ into b makes $\sqrt{(\frac{1}{3})abb}$, (b being first reduced to $\sqrt{(\frac{1}{3})bbb}$.)

Again, $\sqrt{3}$ into $4a$ gives the Product $\sqrt{48aa}$.

4. But when a Surd quantity is given to be multiplied by a Rational quantity, it will oftentimes be very convenient to omit their multiplication, and only to connect them so as that the Rational quantity may stand on the left hand of the given Surd, to signify the Product of their multiplication; as, to multiply $\sqrt{8}$ by 2, I write $2\sqrt{8}$ for the Product; which signifies twice the Square Root of 8. Likewise $20\sqrt{3}$ represents the Product of the multiplication of $\sqrt{3}$ by 20, viz. it imports $\sqrt{3}$ to be taken 20 times, which amounts to as much as $\sqrt{1200}$, found out by the preceding third Rule of this Section.

Again, $\frac{2}{3}\sqrt{7}$ signifies the Product of $\sqrt{7}$ multiplied by $\frac{2}{3}$, (or $\frac{2}{3}$ by $\sqrt{7}$;) and $\frac{2}{3}\sqrt{7}$ denotes the Product of $\frac{2}{3}$ multiplied into $\sqrt{7}$, (or $\sqrt{7}$ into $\frac{2}{3}$;) also, 4 into $20\sqrt{3}$ makes $80\sqrt{3}$, that is, $20\sqrt{3}$ taken four times. Likewise $2\sqrt{(\frac{1}{3})6}$ signifies twice the Cubick Root of 6, and is of equal value with $\sqrt{(\frac{1}{3})48}$; likewise $\frac{2}{3}\sqrt{(\frac{1}{3})80}$ denotes the Product of the Cubick Root of 80 multiplied by $\frac{2}{3}$, or $\frac{2}{3}$ of $\sqrt{(\frac{1}{3})80}$, which is equivalent to $\sqrt{(\frac{1}{3})\frac{2222}{27}}$; and $3\sqrt{(\frac{1}{3})5}$ multiplied by 6 makes $18\sqrt{(\frac{1}{3})5}$, that is, $\sqrt{(\frac{1}{3})19160}$.

The like may be done in Surd quantities express'd by Letters. As, if \sqrt{a} be to be multiplied by b , I write $b\sqrt{a}$ to signify the Product; also, 5 into $b\sqrt{a}$ makes $5b\sqrt{a}$; and c into $b\sqrt{a}$, gives the Product $cb\sqrt{a}$; likewise, $4a$ into $\sqrt{3}$ makes $4a\sqrt{3}$.

Again, if \sqrt{ab} be to be multiplied by $b-d$, the Product may be express'd thus; $b-d \times \sqrt{ab}$; or thus, $b-d\sqrt{ab}$.

Also, if $\sqrt{(\frac{1}{3})\frac{2ab}{c}}$ be to be multiplied by d , the Product may be express'd thus, $d\sqrt{(\frac{1}{3})\frac{2ab}{c}}$; and $\sqrt{(\frac{1}{3})a}$ into b , makes $b\sqrt{(\frac{1}{3})a}$, which is equivalent to $\sqrt{(\frac{1}{3})abb}$.

5. When two Rational quantities, whether they be equal or unequal, are multiplied severally into one common surd square Root, according to the method in the preceding fourth Rule, and it is desired to multiply those Products one into the other, (which Products are called Commensurable quantities, for the reason hereafter given in Sect. 7.) multiply the Rational by the Rational, and that which is produced multiply by the said common Surd, omitting its Radical sign, so the last Product is that which is sought, and will be entirely Rational.

As, for example, to multiply $3\sqrt{5}$ by $2\sqrt{5}$, I multiply 3 by 2, and the Product 6 by 5, so it makes 30; which is the Product of $3\sqrt{5}$ multiplied by $2\sqrt{5}$, (or of $\sqrt{45}$ into $\sqrt{45}$.)

Likewise, $2\sqrt{3}$ multiplied by $2\sqrt{3}$, (viz. the Square of $2\sqrt{3}$;) makes 12; and $20\sqrt{3}$ into $8\sqrt{3}$ makes 480, (by multiplying 20, 8 and 3 one into another continually;) again, $\frac{2}{3}\sqrt{12}$ into $5\sqrt{12}$, produceth 160.

After the same manner, to multiply $a\sqrt{c}$ by $b\sqrt{c}$, I multiply a by b , and the Product ab by c ; so there ariseth abc for the Product sought. The Reason of this Rule is evident, for \sqrt{ac} , (that is, $a\sqrt{c}$) multiplied into \sqrt{bce} , (that is, $b\sqrt{c}$) makes \sqrt{anbbcc} , that is abc ; as before.

In like manner, $5\sqrt{b}$ into $5\sqrt{b}$ produceth $25b$, to wit, the Square of $5\sqrt{b}$; and $2a\sqrt{b}$ into $5a\sqrt{b}$ gives the Product $10aab$; also, $5a\sqrt{12d}$ multiplied by $3a\sqrt{12d}$, produceth $160aad$.

But here is to be noted, that this fifth Rule of multiplication takes place only when the common surd Root into which Rational numbers are multiplied is a surd square Root; so that if $4\sqrt{(\frac{1}{3})5}$ be to be multiplied by $2\sqrt{(\frac{1}{3})5}$, the said fifth Rule will be ineffective, and the Product is to be found out by the following sixth Rule.

6. When two Rational quantities, whether they be equal or unequal, are multiplied into two unequal surd Roots of the same kind, or into one common Surd above the quadratic kind, according to the method in the foregoing fourth Rule of this Section, and it is desired to multiply those Products one into another; multiply the Rational by the Rational; and the Surd by the Surd, and joyn these Products together, so as the Rational Product may stand on the left hand; then those two Products so connected shall be the Product sought.

As, for example, to multiply $5\sqrt{8}$ by $2\sqrt{3}$, I multiply 5 by 2, and the Product is 10; also, $\sqrt{8}$ into $\sqrt{3}$ make $\sqrt{24}$; then those two Products connected make $10\sqrt{24}$, (that is, $\sqrt{2400}$.)

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$\sqrt[2]{2400}$, the Product sought. In like manner, $2\sqrt[3]{8}$ into $2\sqrt[3]{3}$ makes $4\sqrt[3]{24}$, that is, $\sqrt[3]{864}$.

Again, $20\sqrt[5]{5}$ multiplied by $18\sqrt[5]{3}$ produceth $360\sqrt[5]{15}$; and $8\sqrt[5]{27}$ into $2\sqrt[5]{3}$ makes $16\sqrt[5]{81}$, that is, 144 ; also, $5\sqrt[5]{(3)4}$ into $3\sqrt[5]{(3)5}$ produceth $15\sqrt[5]{(3)20}$, that is, $\sqrt[5]{(3)3375}$; likewise, $4\sqrt[5]{(3)5}$ into $2\sqrt[5]{(3)5}$ maketh $8\sqrt[5]{(3)25}$; and $3\sqrt[5]{(4)5}$ into $2\sqrt[5]{(4)6}$, makes $6\sqrt[5]{(4)30}$.

After the same manner, to multiply $a\sqrt[5]{bc}$ into $g\sqrt[5]{ad}$, first, I multiply a by g , and it makes ag ; then, $\sqrt[5]{bc}$ into $\sqrt[5]{ad}$ produceth $\sqrt[5]{bcad}$; lastly, ag into $\sqrt[5]{bcad}$ gives $ag\sqrt[5]{bcad}$, the Product sought.

Likewise, $2\sqrt[5]{ab}$ multiplied by $3\sqrt[5]{bc}$, produceth $6\sqrt[5]{abbc}$; and $2\sqrt[5]{a}$ into $2\sqrt[5]{b}$ makes $4\sqrt[5]{ab}$.

Also, $\frac{2bc}{a}\sqrt[5]{ddd}$ multiplied by $\frac{aa}{2c}\sqrt[5]{ac}$, gives the Product $ab\sqrt[5]{acddd}$; and $b\sqrt[5]{(3)dd}$ into $c\sqrt[5]{(3)f}$ makes $bc\sqrt[5]{(3)df}$; again, $a\sqrt[5]{(3)c}$ into $b\sqrt[5]{(3)e}$ makes $ab\sqrt[5]{(3)ec}$.

7. When a simple Surd quantity whose Radical sign hath for its Index some even number greater than 2 is to be squared, prefix a Radical sign whose Index is half the given Index, before the Power of the given Surd; so shall this new Surd be the Square of the given. As, if $\sqrt[4]{(4)5}$ be to be squared or multiplied into it self, take $\sqrt[2]{(2)5}$, or $\sqrt[2]{5}$, for the Square or Product sought: likewise, the Square of $\sqrt[6]{(6)10}$ is $\sqrt[3]{(3)10}$; and $\sqrt[8]{(8)10}$ into $\sqrt[8]{(8)10}$ makes $\sqrt[4]{(4)10}$.

After the same manner, to multiply $\sqrt[4]{(4)bc}$ into it self quadratically, I write $\sqrt[2]{(2)bc}$, or $\sqrt[2]{bc}$, for the Product, or Square of $\sqrt[4]{(4)bc}$: likewise, the Square of $\sqrt[8]{(8)10k}$ is $\sqrt[4]{(4)10kc}$; and $\sqrt[10]{(10)a}$ into $\sqrt[10]{(10)a}$, makes $\sqrt[5]{(5)a}$: moreover, $2ab\sqrt[4]{(4)d}$ into $3\sqrt[4]{(4)d}$ makes $6ab\sqrt[4]{d}$; for $2ab$ into 3 makes $6ab$, and $\sqrt[4]{(4)d}$ being squared makes $\sqrt[2]{(2)d}$, or $\sqrt[2]{d}$.

But when a simple Surd quantity whose Radical sign hath for its Index some ternary number greater than 3, as 6, 9, &c. is to be multiplied into it self cubically, prefix a Radical sign with an Index that may be a third part of the given Index before the Power of the given surd Root; so shall this new Surd be the Cube of that given: As, if $\sqrt[6]{(6)64}$ be to be multiplied into it self cubically, then $\sqrt[2]{(2)64}$ or $\sqrt[2]{64}$ shall be the Cube sought; likewise, the Cube of $\sqrt[9]{(9)512}$ is $\sqrt[3]{(3)512}$.

More Examples to exercise the precedent Rules of Multiplication in simple Surd Numbers.

Multiply by $\sqrt[5]{8}$	$\frac{\sqrt[5]{(3)4}}{\sqrt[5]{(3)7}}$	$\frac{\sqrt[5]{(4)8}}{\sqrt[5]{(4)2}}$
Product $\sqrt[5]{40}$	$\frac{\sqrt[5]{(3)28}}{\sqrt[5]{(4)16}}$	that is, 2.
Multiply by $\sqrt[3]{32}$	Multiply these three continually, $\sqrt[3]{(3)50}$	
Product 32	$\sqrt[3]{(3)50}$	
Multiply by $\sqrt[6]{27}$	$\frac{12}{\sqrt[3]{(3)5}}$	
Product $6\sqrt[27]{27}$, or, $\sqrt[9]{72}$	$12\sqrt[3]{(3)5}$, or, $\sqrt[3]{(3)8640}$	
Multiply by $\frac{18\sqrt[5]{5}}{4\sqrt[5]{5}}$	$\frac{24\sqrt[5]{6\frac{2}{5}}}{5\sqrt[5]{6\frac{2}{5}}}$	$\frac{6\sqrt[5]{7}}{5\sqrt[5]{3}}$
Product 360	765	$30\sqrt[5]{21}$
Multiply by $\sqrt[8]{(4)5}$	that is, $\sqrt[4]{(6)512}$	
Product $\sqrt[4]{(6)8192}$	$\sqrt[4]{(6)16}$	$\frac{4\sqrt[5]{5}}{4\sqrt[5]{5}}$
Multiply by $\frac{5\sqrt[5]{8}}{4}$	$\frac{12\sqrt[5]{(3)4}}{2\frac{1}{2}}$	$\frac{\sqrt[5]{(4)12}}{\sqrt[5]{(4)12}}$
Product $20\sqrt[5]{8}$	$30\sqrt[5]{(3)4}$	$\sqrt[5]{12}$

More Examples to exercise the precedent Rules of Multiplication in simple Surd Quantities express'd by Letters.

Multiply by	$\sqrt[5]{12a}$ $\sqrt[5]{3a}$		$\frac{\sqrt[5]{ab}}{\sqrt[5]{2ac}}$
Product	$\sqrt[5]{36aa}$, or, $6a$		$\sqrt[5]{4aabc}$, or, $2a\sqrt[5]{bc}$.
Multiply by	$\frac{\sqrt[5]{a}}{\sqrt[5]{(3)aa}}$	} that is, {	$\frac{\sqrt[5]{(6)aaa}}{\sqrt[5]{(6)aaaa}}$
Product			$\sqrt[5]{(6)a^4}$.
Multiply by	$\sqrt[5]{27aa}$ $\sqrt[5]{27aa}$	} Multiply these three continually, {	$\frac{\sqrt[5]{(3)aa}}{\sqrt[5]{(3)aa}}$ $\sqrt[5]{(3)aa}$
Product	$27aa$		aa
Multiply by	$\frac{\sqrt[5]{3bc}}{2}$		$\frac{5b}{\sqrt[5]{(3)2d}}$
Product	$2\sqrt[5]{3bc}$, or, $\sqrt[5]{12bc}$		$5b\sqrt[5]{(3)2d}$, or, $\sqrt[5]{(3)250abbb}$.
Multiply by	$\frac{3a\sqrt[5]{5}}{2b\sqrt[5]{5}}$	$\frac{7\sqrt[5]{bc}}{4\sqrt[5]{bc}}$	$\frac{\frac{2}{3}a\sqrt[5]{bc}}{\frac{2}{3}b\sqrt[5]{bc}}$
Product	$3cab$	$28bc$	$2abbc$.
Multiply by	$5\sqrt[5]{ab}$	$3a\sqrt[5]{5}$	$\frac{2bc}{a}\sqrt[5]{d}$
		$2b\sqrt[5]{6}$	$\frac{aa}{2c}\sqrt[5]{d}$
Product	$15\sqrt[5]{anbc}$	$6ab\sqrt[5]{30}$	abd .

The certainty of the first Rule of this fourth Section, (upon which all the rest depend) for the multiplication of two simple Surd numbers of the same kind, may be Demonstrated in manner following. First, let there be two square Roots given to be multiplied, suppose $\sqrt[5]{5}$ and $\sqrt[5]{3}$, then (by the said Rule) the Product of their Multiplication is $\sqrt[5]{15}$; now we must prove that $\sqrt[5]{15}$ is the true Product of $\sqrt[5]{5}$ multiplied by $\sqrt[5]{3}$.

Demonstration.

By the Definition of Multiplication, these are Proportionals, viz. $1 \cdot \sqrt[5]{5} :: \sqrt[5]{3} : \sqrt[5]{15}$ Product, Therefore their Squares shall be also Proportionals, (per 22. prop. 6. Elem. Euclid.) viz. $1^2 \cdot 5 :: 3^2 : 15$ Square of the Product. But these are Proportionals, (per 19. prop. 7. Elem. Euclid.) $1 : 5 :: 3 : 15$. Therefore, from the two last Analogies, 15 is equal to the Square of the Product; and consequently $\sqrt[5]{15}$ is the Product of $\sqrt[5]{5}$ into $\sqrt[5]{3}$: which was to be proved.

Likewise in Cubick Roots, if $\sqrt[5]{(3)5}$ be to be multiplied by $\sqrt[5]{(3)4}$, the Product (by the same Rule) is $\sqrt[5]{(3)20}$. For, By the Definition of Multiplication, these are Proportionals, viz. $1 \cdot \sqrt[5]{(3)5} :: \sqrt[5]{(3)4} : \sqrt[5]{(3)20}$ Product, Therefore their Cubes are also Proportionals, (per prop. 37. Elem. 11. Euclid.) viz. $1 : 5 :: 4 : 20$. But, as $1 : 5 :: 4 : 20$. Therefore

Therefore 20 is equal to the Cube of the Product; and consequently the cubick Root of 20; to wit; $\sqrt[3]{20}$ is the Product of $\sqrt[3]{3}$ multiplied by $\sqrt[3]{4}$; which was to be proved.

Moreover, because (by *Sett. 11. Chap. 5.*) if four numbers be Proportionals, their fourth Powers, fifth Powers, &c. are also Proportionals, this Demonstration may be extended to prove the certainty of the said Rule for multiplying any two simple Surd numbers of the same kind.

Sett. V. Division in simple Surd Quantities.

As before in Multiplication, so here in Division, if the given Surd Roots, to wit, the Dividend and Divisor be not of the same kind, they must be reduced to a common Radical sign by the preceding *Sett. 3.* Then,

1. Divide the Number or Quantity following the Radical sign of the Dividend, by the Number or Quantity following the same Radical sign of the Divisor, without any regard to the Sign, and to the Quotient prefix the said common Radical sign; so this new Root shall be the Quotient sought.

As, for example, to divide $\sqrt{15}$ by $\sqrt{3}$, I divide 15 by 3, and there ariseth 5; before which I prefix $\sqrt{}$ (the Radical sign common to the given Surds,) so $\sqrt{5}$ is the Quotient sought.

Likewise, if $\sqrt{30}$ be divided by $\sqrt{5}$, the Quotient is $\sqrt{6}$.

Also, $\sqrt{\frac{1}{2}}$ divided by $\sqrt{\frac{1}{4}}$ gives the Quotient $\sqrt{\frac{1}{2}}$.

And $\sqrt{\frac{1}{2}}$ or $\sqrt{\frac{1}{2}}$ divided by $\frac{1}{2}$, or $\frac{1}{2}$, gives the Quotient $\frac{1}{2}$.

Again, $\sqrt[3]{320}$ divided by $\sqrt[3]{3}$, gives the Quotient $\sqrt[3]{\frac{320}{3}}$; for 26 divided by 5 gives 4, before which setting $\sqrt[3]{}$ the Radical sign belonging to each of the given Surds, there ariseth $\sqrt[3]{4}$ for the Quotient sought.

Likewise $\sqrt[4]{45}$ divided by $\sqrt[4]{\frac{1}{2}}$, gives the Quotient $\sqrt[4]{90}$.

Moreover, if $\sqrt[4]{64500}$ be given to be divided by $\sqrt[4]{25}$, the Quotient will be $\sqrt[4]{2580}$; for first, the given Roots being of different kinds are reduced to these, to wit, $\sqrt[4]{64500}$ and $\sqrt[4]{64500}$, then by dividing $\sqrt[4]{64500}$ by $\sqrt[4]{64500}$ there ariseth $\sqrt[4]{64500}$, whose square Root being extracted, (because 36 is a square number, and the Index 4 an even number,) it gives $\sqrt[4]{36}$ for the Quotient sought.

After the same manner, Division is perform'd in simple Surd Quantities express'd by Letters. As, to divide \sqrt{ab} by \sqrt{a} , I divide ab by a and there ariseth b , then setting $\sqrt{}$ before b , it gives \sqrt{b} for the Quotient sought; to wit, the Quotient that ariseth by dividing \sqrt{ab} by \sqrt{a} .

Also, \sqrt{b} divided by \sqrt{a} , gives the Quotient $\sqrt{\frac{b}{a}}$.

Likewise, \sqrt{abcd} divided by \sqrt{ab} gives the Quotient \sqrt{cd} .

Also, $\sqrt[3]{\frac{2aad}{c}}$ divided by $\frac{2ab}{3c}$ gives the Quotient $\sqrt[3]{\frac{9ad}{3b}}$.

Again, to divide $\sqrt[4]{64ddaaabb}$ by $\sqrt[4]{3}ab$, I first reduce them to $\sqrt[4]{64ddaaabb}$ and $\sqrt[4]{64aabb}$, then I divide $\sqrt[4]{64ddaaabb}$ by $\sqrt[4]{64aabb}$, and there ariseth $\sqrt[4]{64dd}$, that is, $\sqrt[4]{2}$, for the Quotient sought.

2. When a Rational number or quantity is to be divided by its square Root, that Root is the Quotient; as, if 5 be divided by its square Root, to wit by $\sqrt{5}$, the Quotient will be $\sqrt{5}$; also, 8 divided by $\sqrt{8}$ gives $\sqrt{8}$ for the Quotient.

In like manner if the quantity bc be divided by its square Root, to wit, by \sqrt{bc} , the Quotient will be \sqrt{bc} ; and $5a$ divided by $\sqrt{5a}$, gives the Quotient $\sqrt{5a}$.

3. When a Surd number or quantity is to be divided by a Rational number or quantity, or a Rational number or quantity by a Surd, reduce the Rational into the form of a Surd, (by *Sett. 2. of this Chap.*) and then divide according to the first Rule of this *Sett. 5.*

As, to divide $\sqrt{32}$ by 2, I first reduce 2 to $\sqrt{4}$; then by dividing $\sqrt{32}$ by $\sqrt{4}$, there ariseth $\sqrt{8}$ for the Quotient.

Likewise $\sqrt{175}$ divided by 5, that is, $\sqrt{25}$, gives the Quotient $\sqrt{7}$.

Also 12, that is, $\sqrt{144}$, divided by $\sqrt{3}$ gives the Quotient $\sqrt{48}$.

Again, if $\sqrt[3]{48}$ be to be divided by 2, I first reduce 2 to $\sqrt[3]{8}$, then by dividing $\sqrt[3]{48}$ by $\sqrt[3]{8}$, there ariseth $\sqrt[3]{6}$ for the Quotient sought; also, $\sqrt[4]{45000}$ divided by 5, (that is, by $\sqrt[4]{625}$) gives the Quotient $\sqrt[4]{48}$.

After

After the same manner, to divide the quantity \sqrt{abb} by b , I first reduce b to \sqrt{bb} , and then by dividing \sqrt{abb} by \sqrt{bb} , there ariseth \sqrt{a} the Quotient sought. Again, $\sqrt[4]{8aa}$ divided by $\sqrt[4]{4}$, that is by $\sqrt[4]{16aa}$, gives the Quotient $\sqrt[4]{3}$. Also $\sqrt[3]{(3)abbb}$ divided by b , that is by $\sqrt[3]{3bbb}$, gives the Quotient $\sqrt[3]{a}$.

Likewise, to divide the Rational quantity $\frac{bc}{a}$ by $\sqrt[3]{(3)bbcc}$, I first reduce $\frac{bc}{a}$ to $\sqrt[3]{\frac{bbcc}{aaa}}$, then I divide $\sqrt[3]{\frac{bbcc}{aaa}}$ by $\sqrt[3]{(3)bbcc}$, and there ariseth $\sqrt[3]{\frac{bc}{aaa}}$, or $\sqrt[3]{(3)bc}$, the Quotient sought.

4. When the Product of a Rational number or quantity multiplied into a Surd number or quantity is to be divided by the same Surd, the Quotient will be the said multiplying Rational number or quantity. As, $5\sqrt{3}$ divided by $\sqrt{3}$ gives the Quotient 5; also, $20\sqrt[3]{4}$ divided by $\sqrt[3]{4}$ gives the Quotient 20.

In like manner, $5a\sqrt{b}$ divided by \sqrt{b} gives the Quotient $5a$; and $4b\sqrt[3]{(3)12}$ divided by $\sqrt[3]{(3)12}$ gives the Quotient $4b$.

5. When the Dividend and Divisor are the Products of two Rational numbers or quantities multiplied severally into one common Surd, according to the fourth Rule of Multiplication in *Sett. 4.* (which Products are called Commensurable Surd Roots, as hereafter will appear in *Sett. 7.* of this *Chap.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and that which ariseth shall be the Quotient sought. As, for example, to divide $6\sqrt{3}$ by $2\sqrt{3}$, I divide 6 by 2, and there ariseth 3 the Quotient sought; (for $2\sqrt{3}$ multiplied by 3, produceth $6\sqrt{3}$.)

Again, $5\sqrt{6}$ divided by $2\sqrt{6}$ gives the Quotient $\frac{5}{2}$, or $2\frac{1}{2}$.

Also, $2\sqrt{6}$ divided by $5\sqrt{6}$ gives the Quotient $\frac{2}{5}$; and $2\sqrt{5}$ divided by $2\sqrt{5}$, gives the Quotient 1.

So also $8\sqrt[3]{(3)7}$ divided by $4\sqrt[3]{(3)7}$, gives the Quotient 2; and $3\sqrt[4]{(4)5}$ divided by $4\sqrt[4]{(4)5}$, gives $\frac{3}{4}$ for the Quotient.

In like manner, to divide $4a\sqrt{7}$ by $2a\sqrt{7}$, I divide $4a$ by $2a$, and there ariseth 2, the Quotient sought; (for $2a\sqrt{7}$ into 2 produceth $4a\sqrt{7}$;) also, $3\sqrt{b}$ divided by $5\sqrt{b}$ gives the Quotient $\frac{3}{5}$; and $2\sqrt{b}$ divided by $2\sqrt{b}$, gives the Quotient 1.

Again, $5a\sqrt[3]{b}$ divided by $3a\sqrt[3]{b}$ gives the Quotient $\frac{5}{3}$.

And $7ab\sqrt[3]{(3)dd}$ divided by $3b\sqrt[3]{(3)dd}$, gives the Quotient $\frac{7}{3}a$.

6. When the Dividend and Divisor are the Products of two Rational numbers or quantities multiplied into two unequal Surd numbers or quantities, according to the fourth Rule of Multiplication in the preceding *Sett. 4.* (which Products are called Incommensurable Surd Roots, as hereafter will appear,) divide the Rational part of the Dividend by the Rational part of the Divisor, and the Surd part by the Surd part, then connect the Quotients so as the Rational quotient may stand on the left hand, and this new quantity shall be the Quotient sought.

As, for example, if $4\sqrt{15}$ be to be divided by $2\sqrt{5}$, first I divide 4 by 2, and there ariseth 2; also I divide $\sqrt{15}$ by $\sqrt{5}$, and there ariseth $\sqrt{3}$; then those two Quotients joyned together make $2\sqrt{3}$ (or $\sqrt{12}$;) the Quotient sought.

In like manner $4\sqrt[3]{12}$ divided by $3\sqrt[3]{2}$ gives the Quotient $\frac{4}{3}\sqrt[3]{6}$; for 4 divided by 3, (to wit, the Rational by the Rational,) gives $\frac{4}{3}$; and $\sqrt[3]{12}$ divided by $\sqrt[3]{2}$, (to wit, the Surd by the Surd,) gives $\sqrt[3]{6}$; then by joyning together those two Quotients there ariseth $\frac{4}{3}\sqrt[3]{6}$, or $1\frac{1}{3}\sqrt[3]{6}$, (or $\sqrt[3]{\frac{16}{3}}$;) for the Quotient sought.

Again, $2\sqrt[3]{7}$ divided by $3\sqrt[3]{5}$ gives the Quotient $\frac{2}{3}\sqrt[3]{\frac{7}{5}}$; and $2\sqrt[3]{3}$ divided by $2\sqrt[3]{5}$ gives the Quotient $\frac{1}{\sqrt[3]{5}}$, or $\sqrt[3]{\frac{1}{5}}$.

Likewise to divide $4\sqrt[3]{(3)64}$ by $2\sqrt[3]{(3)8}$, I divide 4 by 2, and it gives 2; also, $\sqrt[3]{(3)64}$ divided by $\sqrt[3]{(3)8}$ gives $\sqrt[3]{(3)8}$; then those two Quotients joyned together make $2\sqrt[3]{(3)8}$, that is 4, the Quotient sought. Moreover, $5\sqrt[3]{(3)20}$ divided by $3\sqrt[3]{(3)4}$ gives the Quotient $\frac{5}{3}\sqrt[3]{(3)5}$.

After the same manner, $4a\sqrt{b}$ divided by $2a\sqrt{f}$ gives the Quotient $2\sqrt{\frac{b}{f}}$; for $4a$ divided by $2a$ gives 2; and \sqrt{b} divided by \sqrt{f} gives $\sqrt{\frac{b}{f}}$; then connecting those two Quotients there ariseth $2\sqrt{\frac{b}{f}}$ for the Quotient sought.

So also, $6ab\sqrt[3]{cd}$ divided by $6a\sqrt[3]{df}$ gives the Quotient $b\sqrt[3]{\frac{c}{f}}$.

And

And $a\sqrt{(3)cc}$ divided by $b\sqrt{(3)dd}$, gives the Quotient $\frac{a}{b}\sqrt{(3)\frac{cc}{dd}}$.

The Demonstration of the aforefaid first Rule of Division (which is the Rule of all the rest) may be formed like that of Multiplication in the preceding Sect. 4. if there be laid, as a ground-work, this Analogy; *vis.* As the Divisor is to 1 (or Unity) so is the Dividend to the Quotient. But waving the Demonstration, I shall give more Examples of Division in simple Surds, both in Numbers and quantities exprest by Letters.

More Examples to exercise Division in simple Surd Numbers.

Dividend	$\sqrt{117}$	$\sqrt{(3)16\frac{2}{3}}$, or, $\sqrt{(?)\frac{46}{3}}$	$\sqrt{(4)256}$
Divisor	$\sqrt{6\frac{2}{3}}$	$\sqrt{(3)3\frac{2}{3}}$, or, $\sqrt{(3)}$	$\sqrt{(4)16}$
Quotient	$\sqrt{18}$	$\sqrt{(3)4\frac{2}{3}}$, or, $\sqrt{(3)\frac{14}{3}}$	2
Dividend	$\sqrt{(12)6125}$	that is, $\sqrt{(6)125}$	
Divisor	$\sqrt{(4)5}$	$\sqrt{(12)125}$	
Quotient	$\sqrt{(12)49}$, or, $\sqrt{(6)7}$	
Dividend	12	$5\sqrt{8}$	$16\sqrt{(3)25}$
Divisor	$\sqrt{12}$	$\sqrt{8}$	$\sqrt{(3)25}$
Quotient	$\sqrt{12}$	5	16
Dividend	$\sqrt{245}$	$\sqrt{(3)686}$	$\sqrt{(5)23328}$
Divisor	$3\frac{1}{2}$	$3\frac{1}{2}$	6
Quotient	$\sqrt{20}$	$\sqrt{(3)16}$	$\sqrt{(5)3}$
Dividend	$20\sqrt{14}$	$\frac{2}{3}\sqrt{20}$	$5\sqrt{(3)3}$
Divisor	$2\sqrt{14}$	$\frac{2}{3}\sqrt{20}$	$2\sqrt{(3)3}$
Quotient	10	5	$\frac{5}{2}, 08, 2\frac{1}{2}$
Dividend	$15\sqrt{18}$	$3\sqrt{8}$	$6\sqrt{(3)24}$
Divisor	$3\sqrt{6}$	$3\sqrt{3}$	$9\sqrt{(3)4}$
Quotient	$5\sqrt{3}$	$\sqrt{\frac{2}{3}}$	$\frac{2}{3}\sqrt{(3)5}$

More Examples to exercise Division in simple Surd quantities exprest by Letters.

Dividend	$\sqrt{15bc}$	$\sqrt{(3)bbdd}$	$\sqrt{(4)32aa}$
Divisor	$\sqrt{3a}$	$\sqrt{(3)4bb}$	$\sqrt{(4)2aa}$
Quotient	$\frac{bc}{a}$	$\sqrt{(3)dd}$, or, d	$\sqrt{(4)16}$, or, 2
Dividend	$\sqrt{(6)675aaaaabbbb}$	that is, $\sqrt{(6)675a^4b^4}$	
Divisor	$\sqrt{(2)3ab}$	$\sqrt{(6)27a^2b^2}$	
Quotient	$\sqrt{(6)25aab}$, or, $\sqrt{(3)5ab}$	
Dividend	$\sqrt{80aaabbb}$	$9bcd$ (or, $\sqrt{81bbccdd^4}$)	
Divisor	$4ab$, (or, $\sqrt{16aabb}$)	$\sqrt{27bcd}$	
Quotient	$\sqrt{5ab}$	$\sqrt{3bcd}$	
Dividend	bc	$b\sqrt{df}$	$2d\sqrt{(3)bb}$
Divisor	\sqrt{bc}	\sqrt{df}	$\sqrt{(3)bb}$
Quotient	\sqrt{bc}	b	2d

Dividend

Dividend	$12\sqrt{dc}$	$\frac{2bc}{a}\sqrt{d}$	$ab\sqrt{(3)f}$
Divisor	$3\sqrt{dc}$	$\frac{2c}{b}\sqrt{d}$	$b\sqrt{(3)f}$
Quotient	4	$\frac{bb}{a}$	a
Dividend	$2bc\sqrt{d}$	$b\sqrt{af}$	$6aa\sqrt{(3)bbdd}$
Divisor	$c\sqrt{a}$	$c\sqrt{f}$	$2a\sqrt{(3)d}$
Quotient	$2b\sqrt{\frac{d}{a}}$	$\frac{b}{c}\sqrt{a}$	3ab

Note. By the help of Division, Surd quantities may oftentimes be reduced into others more simple, which being a very useful work, I shall explain it in the next Section.

Sect. VI. How to reduce a Surd quantity to another more simple, when it may be done.

When the Power of a Surd quantity, the Radical sign being omitted, can be divided just without any Remainder, by a Power which hath a Rational Root of the same kind with that which is denoted by the said Radical sign, then divide the Surd quantity proposed by that Rational Root, and prefix this Root before the Quotient: so you have a new Surd quantity equal to that proposed, and in more simple Terms.

As, if $\sqrt{53}$ be proposed, because 53 may be divided by the square number 9 without any Remainder, I divide $\sqrt{53}$ by $\sqrt{9}$, (that is, by 3) and it gives the Quotient $\sqrt{7}$, before which I set the Rational Divisor 3, and it makes $3\sqrt{7}$, (that is 3 into the square Root of 7, or thrice the square Root of 7,) which is equal to $\sqrt{63}$ first proposed; (for the Quotient $\sqrt{7}$ multiplied by the Divisor 3 makes the Dividend $\sqrt{63}$;) so that instead of $\sqrt{53}$ I write $3\sqrt{7}$.

Likewise, instead of $\sqrt{50}$ we may write $5\sqrt{2}$, (which signifies five times the square Root of 2;) for in regard 50 divided by the Square 25 gives 2, I divide $\sqrt{50}$ by $\sqrt{25}$, that is, by 5, and the Quotient is $\sqrt{2}$; and because every Quotient multiplied by the Divisor produceth the Dividend: Therefore $5\sqrt{2}$ shall be equal to the Dividend $\sqrt{50}$.

After the same manner, instead of $\frac{\sqrt{75}}{2}$, or $\sqrt{\frac{15}{2}}$, we may write $\frac{1}{2}\sqrt{3}$; for $\frac{15}{2}$ divided by the square number $\frac{1}{4}$ gives the Quotient 3; and consequently, $\sqrt{\frac{15}{2}}$ divided by $\sqrt{\frac{1}{4}}$, that is by $\frac{1}{2}$, gives the Quotient $\sqrt{3}$: Therefore $\frac{1}{2}\sqrt{3}$ shall be equal to $\frac{\sqrt{75}}{2}$ or $\sqrt{\frac{15}{2}}$.

Again, instead of $\sqrt{(3)40}$, we may write $2\sqrt{(3)5}$, (which signifies twice the cubick Root of 5;) for 40 divided by the Cube 8 gives the Quotient 5; and consequently, $\sqrt{(3)40}$ divided by $\sqrt{(3)8}$, that is by 2, gives $\sqrt{(3)5}$: Therefore $2\sqrt{(3)5}$ shall be equal to $\sqrt{(3)40}$.

Likewise for $\sqrt{(3)\frac{16}{27}}$, (or $\frac{\sqrt{(3)64}}{27}$;) we may write $\frac{1}{27}\sqrt{(3)2}$; for $\frac{16}{27}$ divided by the Cube $\frac{1}{27}$ gives 2; and consequently $\sqrt{(3)\frac{16}{27}}$ divided by $\sqrt{(3)\frac{1}{27}}$, that is by $\frac{1}{3}$, will give $\sqrt{(3)2}$: Wherefore $\frac{1}{27}\sqrt{(3)2}$ shall be equal to $\sqrt{(3)\frac{16}{27}}$.

The like Operation is to be done, in reducing Surd quantities exprest by Letters to others more simple: as, if $\sqrt{75aa}$ be proposed; For as much as 75aa divided by the Square 25aa gives the Quotient 3, and consequently $\sqrt{75aa}$ divided by $\sqrt{25aa}$, that is, by 5a will give $\sqrt{3}$; Therefore the Divisor 5a multiplied into the Quotient $\sqrt{3}$, produceth $5a\sqrt{3}$, equal to the Dividend $\sqrt{75aa}$; and therefore instead of $\sqrt{75aa}$, we may write $5a\sqrt{3}$.

After the same manner $\sqrt{10aabb}$ may be reduced to $ab\sqrt{10}$; also $\sqrt{5aa}$ to $a\sqrt{5}$; and $\sqrt{(3)add}$ to $d\sqrt{(3)a}$.

Again, for as much as $aaab + aabb$ may be divided by the Square aa and there ariseth $ab + bb$, and consequently $\sqrt{aaab + aabb}$ divided by \sqrt{aa} , that is by a, gives the Quotient $\sqrt{ab + bb}$: therefore a into $\sqrt{ab + bb}$ shall be equal to $\sqrt{aaab + aabb}$: So that instead of $\sqrt{aaab + aabb}$ we may write a into $\sqrt{ab + bb}$: or $a\sqrt{ab + bb}$.

Likewise;

Likewise, for $\sqrt{aabc + 2afbc + fbbc}$: we may write $a + f$ into \sqrt{bbc} , or $a + f/\sqrt{bbc}$; for $aabc + 2afbc + fbbc$ divided by the Square $aa + 2af + ff$ gives bbc , and consequently $\sqrt{aabc + 2afbc + fbbc}$ divided by $\sqrt{aa + 2af + ff}$ that is by $a + f$, gives the Quotient \sqrt{bbc} : Therefore $a + f/\sqrt{bbc}$ imports as much as $\sqrt{aabc + 2afbc + fbbc}$:

After the same manner, instead of $\sqrt[3]{\frac{27aaaabbb}{8b-8a}}$ we may write $\frac{3ab}{2}$ into $\sqrt[3]{\frac{a}{b-a}}$ or $\frac{3ab}{2}\sqrt[3]{\frac{a}{b-a}}$; for since the Power of the Surd proposed is produced by the multiplication of $\frac{a}{b-a}$ into the Cube $\frac{27aaaabbb}{8}$ whose cubick Root is $\frac{3ab}{2}$, and consequently $\sqrt[3]{\frac{27aaaabbb}{8b-8a}}$ divided by $\sqrt[3]{\frac{27aaaabbb}{8}}$, that is by $\frac{3ab}{2}$ gives the Quotient $\sqrt[3]{\frac{a}{b-a}}$. Therefore $\frac{3ab}{2}\sqrt[3]{\frac{a}{b-a}}$ shall be equal to $\sqrt[3]{\frac{27aaaabbb}{8b-8a}}$.

So also, for $\sqrt{\frac{aaomm + 4aamm}{ppcz}}$: we may write $\frac{am}{pc}$ into $\sqrt{oo + 4mp}$: for, if the Power of the Surd proposed be divided by the Square $\frac{aamm}{ppcz}$ the Quotient will be $oo + 4mp$, and consequently, if the Surd proposed be divided by $\sqrt{\frac{aamm}{ppcz}}$: that is, by $\frac{am}{pc}$, the Quotient will be $\sqrt{oo + 4mp}$: Therefore the Divisor $\frac{am}{pc}$ multiplied into the Quotient $\sqrt{oo + 4mp}$: (viz. $\frac{am}{pc}\sqrt{oo + 4mp}$) denotes as much as $\sqrt{\frac{aaomm + 4aamm}{ppcz}}$: the Surd proposed.

Likewise, for $\sqrt{\frac{oozz + 4mpcz}{aa}}$: we may write $\frac{z}{a}\sqrt{oo + 4mp}$:

But when a Square, or Cube, &c. by which the Division necessary to such Contraction is to be performed, cannot be readily discerned, first, (by the Rules of the preceding eighth Chapter) search out all the Divisors of the Power of the Surd quantity proposed, and then see whether any of them be a Square or Cube, &c. to wit, such a Power as the Radical sign denotes, which if you find, you may use in the aforesaid manner to free the Surd quantity, in part, from the Radical sign.

As, if $\sqrt{288}$ be proposed, because among the Divisors of 288 there are found the Square numbers 4, 9, 16, 36 and 144, which dividing 288 will give the Quotients 72, 32, 18, 8 and 2; instead of $\sqrt{288}$ we may write $2\sqrt{72}$, or $3\sqrt{32}$, or $4\sqrt{18}$, or $6\sqrt{8}$, or lastly, $12\sqrt{2}$.

In like manner, if $\sqrt{aaab + aabb}$ be proposed, because among the Divisors of the quantity $aaab + aabb$, there is found the Square aa , the said $\sqrt{aaab + aabb}$ may be reduced to $a\sqrt{aa + bb}$ as before.

Again, for as much as $a^3b - aabb + 2abc - abcc - ab^3 + bbcc - 2b^2c + b^4$ is produced by the multiplication of $a + b$ into the Square $aa + 2ac + cc - 2ab - 2bc + bb$, whose Root is $a + b$; we may instead of $\sqrt{a^3b - aabb + 2abc - abcc - ab^3 + bbcc - 2b^2c + b^4}$ write $a + b$ into $\sqrt{ab + bb}$ or $a + b\sqrt{ab + bb}$.

Likewise, because among the Divisors of $1200aabb$ there are found the Squares $4aabb$, $16aabb$, $25aabb$, $100aabb$ and $400aabb$, which dividing the said $1200aabb$, will give the Quotients 300, 75, 48, 12 and 3; we may for $\sqrt{1200aabb}$ write $2ab\sqrt{300}$, or $4ab\sqrt{75}$, or $5ab\sqrt{48}$, or $10ab\sqrt{12}$, or lastly, $20ab\sqrt{3}$.

SECT. VII. Two Surd Roots being given, to find whether they be Commensurable or Incommensurable.

Commensurable Surd Roots are such whose Reason or Proportion to one another may be express'd by Rational Numbers, or Quantities; and those Surd Roots whose Proportion cannot be express'd by Rational Numbers or Quantities are called Incommensurable.

The

The Rule to try whether two Surd Roots of the same kind, (that is, such as have a common Radical sign,) be Commensurable or not, is this that follows, viz.

Divide the given Roots severally by their greatest Common Divisor, then if the Quotients be Rational Numbers or Quantities, the Roots proposed are Commensurable; but if the Quotients be Irrational or Surd, the given Roots are Incommensurable.

As, for example, to try whether $\sqrt{12}$ and $\sqrt{3}$ be Commensurable or not, I divide them severally by their greatest common Divisor $\sqrt{3}$; and find the Quotients $\sqrt{4}$ and $\sqrt{1}$, that is, 2 and 1 to be Rational numbers, whence I conclude that $\sqrt{12}$ is $2\sqrt{3}$, hath such Proportion to $\sqrt{3}$, that is $1\sqrt{3}$, as 2 to 1, viz. as a Rational number to a Rational number; and consequently $\sqrt{12}$ and $\sqrt{3}$ (according to the Definition above given) are Commensurable. But that $\sqrt{12}$ is to $\sqrt{3}$ as 2 to 1, may be demonstrated thus, viz. It is evident (by reason of the common Factor $\sqrt{3}$) that $2\sqrt{3} : 1\sqrt{3} :: 2 : 1$; and (by Division as above,) $\sqrt{12} = 2\sqrt{3}$, and $\sqrt{3} = 1\sqrt{3}$; therefore $\sqrt{12} : \sqrt{3} :: 2 : 1$. Otherwise thus,

For as much as 12 and 3 divided severally by their common

Divisor 3 give the Quotients 4 and 1, therefore, As $12 : 3 :: 4 : 1$.

Wherefore the Square Roots of those Proportionals shall be $\sqrt{12} : \sqrt{3} :: 2 : 1$.

Proportionals also, (per 22. Prop. 6. Elem. Euclid.) viz. $\sqrt{12} : \sqrt{3} :: 2 : 1$.

Which was to be demonstrated.

After the same manner, $\sqrt{18}$ and $\sqrt{8}$ will be found Commensurable, for the former is to the latter as 3 to 2, to wit, as a Rational number to a Rational number; for if $\sqrt{18}$ and $\sqrt{8}$ be severally divided by their greatest common Divisor $\sqrt{2}$, the Quotients will be $\sqrt{9}$ and $\sqrt{4}$, that is, 3 and 2. Therefore $\sqrt{18}$ is to $\sqrt{8}$ as 3 to 2, and instead of $\sqrt{18}$ and $\sqrt{8}$ we may write $3\sqrt{2}$ and $2\sqrt{2}$, to wit, the Products of the Rational Quotients 3 and 2 multiplied into the common Divisor $\sqrt{2}$.

Again, $\sqrt{48}$ and $\sqrt{75}$ (that is, $4\sqrt{3}$ and $5\sqrt{3}$) are Commensurable, for the former is to the latter as 4 to 5, (to wit, as a Rational number to a Rational number;) for $\sqrt{48}$ and $\sqrt{75}$ being severally divided by their greatest common Divisor $\sqrt{3}$, give the Quotients $\sqrt{16}$ and $\sqrt{25}$, to wit, 4 and 5. Therefore $\sqrt{48} : \sqrt{75} :: 4 : 5$.

Moreover, $\sqrt{320}$ and $\sqrt{315}$ (that is, $4\sqrt{20}$ and $3\sqrt{35}$) having such proportion one to the other as 4 to 3 are Commensurable, for $\sqrt{320}$ and $\sqrt{315}$ being severally divided by their greatest common Divisor $\sqrt{35}$, will give the Quotients $\sqrt{64}$ and $\sqrt{32}$, to wit, 8 and 4. Therefore, $\sqrt{320} : \sqrt{315} :: 8 : 4$.

So also $\sqrt{43888}$ and $\sqrt{4243}$ (that is, $2\sqrt{4243}$ and $1\sqrt{4243}$) are Commensurable, the former having such proportion to the latter as 2 to 1; for if they be severally divided by their greatest common Divisor $\sqrt{4243}$, the Quotients will be $\sqrt{4}$ and $\sqrt{1}$, to wit, 2 and 1. Therefore, $\sqrt{43888} : \sqrt{4243} :: 2 : 1$.

If two Surd Fractions, or mixt numbers standing fraction-wise, be proposed, and have not a common Denominator, reduce them to their smallest common Denominator, and then try (in like manner as before) whether the new Surd Numerators be Commensurable or not, for if these be Commensurable, the Surd Fractions first proposed shall be also Commensurable. As, if $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{4}{27}}$ be proposed, I reduce them to $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{4}{27}}$, then I divide the new Numerators only, to wit, $\sqrt{2}$ and $\sqrt{4}$, by their greatest common Divisor $\sqrt{2}$, and the Quotients $\sqrt{1}$ and $\sqrt{2}$, that is, 1 and $\sqrt{2}$, are Rational numbers. Therefore $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{4}{27}}$ first proposed are Commensurable, and the former hath such proportion to the latter as 1 to $\sqrt{2}$. For,

As $\frac{2}{3} : \frac{4}{27} :: 50 : 72 :: 25 : 36$,
Therefore, $\sqrt{\frac{2}{3}} : \sqrt{\frac{4}{27}} :: \sqrt{50} : \sqrt{72} :: 5 : 6$.
And because $\sqrt{\frac{2}{3}} = \sqrt{\frac{20}{30}}$ and $\sqrt{\frac{4}{27}} = \sqrt{\frac{16}{81}}$,
Therefore, $\sqrt{\frac{2}{3}} : \sqrt{\frac{4}{27}} :: 5 : 6$.

But if either the Numerators or Denominators of two Surd Fractions, or mixt numbers standing fraction-wise, (the Radical sign being neglected,) be Squares or Cubes, &c. viz. Powers of that kind which is denoted by the Radical sign, then you need not reduce the surd Fractions to a common Denominator, but try whether their Numerators or Denominators be Commensurable or not; for if these be Commensurable, the surd Fractions proposed shall

E e

shall

shall be also Commensurable. As, if $\sqrt[3]{\frac{12}{5}}$ and $\sqrt[3]{\frac{1}{5}}$ be proposed; because the Denominators (the Radical sign being neglected) are Squares, (to wit, Powers of that kind which the Radical sign denotes,) and the Numerators $\sqrt[3]{50}$ and $\sqrt[3]{72}$ are Commensurable; (for if these be divided by their common Divisor $\sqrt[3]{2}$, the Quotients are Rational, to wit, 5 and 6) Therefore the said Fractions proposed are also Commensurable, and have such proportion as $\frac{4}{5}$ to $\frac{3}{5}$. (whose Denominators 4 and 5, to wit, $\sqrt[3]{16}$ and $\sqrt[3]{25}$ are the given Denominators,) or as 25 to 24; and, (according to the preceding Sect. 6.) the said Surd Fractions proposed may be expressed thus, $\frac{2\sqrt[3]{2}}{5}$ and $\frac{\sqrt[3]{2}}{5}$.

When two Surd Roots proposed be of different kinds, they must first of all be reduced to a common Radical sign, (by the preceding Sect. 3. of this Chapt.) before the Rules above said be used, to try whether they be Commensurable or not. As, if $\sqrt[3]{(6)64}$ and $\sqrt[3]{(3)17}$ be given; they may be reduced to $\sqrt[3]{(6)64}$ and $\sqrt[3]{(6)729}$, which divided by their greatest common Divisor $\sqrt[3]{(6)1}$, the Quotients will be the same with the Dividends. Now if $\sqrt[3]{(6)64}$ and $\sqrt[3]{(6)729}$ be Rational, then the Surds first given are Commensurable, but $\sqrt[3]{(6)64}$ is, and $\sqrt[3]{(6)729}$ is 3. Therefore the said Surd Roots proposed are Commensurable, and have such Proportion as 2 to 3.

But if the Quotients arising by the division of two surd Roots by their greatest common Divisor as aforesaid, happen to be Irrational or Surd, then the Roots proposed are Incommensurable, such as $\sqrt[3]{48}$ and $\sqrt[3]{8}$, for if they be divided severally by their greatest common Divisor $\sqrt[3]{8}$, the Quotients are $\sqrt[3]{6}$ and 1, but $\sqrt[3]{6}$ is Irrational, therefore the Proportion which $\sqrt[3]{48}$ hath to $\sqrt[3]{8}$ is not as a Rational number to a Rational number, and consequently $\sqrt[3]{48}$ and $\sqrt[3]{8}$ are Incommensurable, and so are all other Surd Roots whose Proportion cannot be expressed by Rational numbers.

I shall now shew how by the help of the preceding Rules we may discover whether two Surd quantities expressed by letters be Commensurable or not. As, if $\sqrt[3]{27aa}$ and $\sqrt[3]{12aa}$ be proposed, they will be found Commensurable; for if they be severally divided by their greatest common Divisor $\sqrt[3]{3aa}$, the Quotients $\sqrt[3]{9}$ and $\sqrt[3]{4}$, that is, 3 and 2, are Rational numbers, and shew that $\sqrt[3]{27aa}$ is to $\sqrt[3]{12aa}$ as 3 to 2, to wit, as a Rational number to a Rational number; wherefore $\sqrt[3]{27aa}$ and $\sqrt[3]{12aa}$ are Commensurable, and may be expressed thus, $3\sqrt[3]{4aa}$ and $2\sqrt[3]{4aa}$.

Note. If two Surd quantities be divided by some common Divisor, though it be not the greatest, yet if there come forth Rational Quotients, we may thence conclude those Surd quantities to be Commensurable, and oftentimes express them various ways. As, if $\sqrt[3]{27aa}$ and $\sqrt[3]{12aa}$ be again proposed; by dividing them severally by their common Divisor $\sqrt[3]{3}$, there will come forth the Quotients $\sqrt[3]{9aa}$ and $\sqrt[3]{4aa}$, that is, $3a$ and $2a$; whence it is evident that $\sqrt[3]{27aa}$ is to $\sqrt[3]{12aa}$ as $3a$ to $2a$, to wit, as a Rational quantity to a Rational quantity, and consequently $\sqrt[3]{27aa}$ and $\sqrt[3]{12aa}$ are Commensurable. Moreover, according to the latter Division, we may write $3a\sqrt[3]{3}$ for $\sqrt[3]{27aa}$, and $2a\sqrt[3]{3}$ for $\sqrt[3]{12aa}$.

Again, $\sqrt[3]{aaaa+abb}$ and $\sqrt[3]{aabb+bbb}$ are Commensurable; for each of them being divided by $\sqrt[3]{aa+bb}$ there arise $\sqrt[3]{aa}$ and $\sqrt[3]{bb}$, that is, a and b , which are Rational quantities, each of which being multiplied into the common Divisor $\sqrt[3]{aa+bb}$ will give instead of the Surds proposed, $a\sqrt[3]{aa+bb}$ and $b\sqrt[3]{aa+bb}$, which have the same proportion to one another as there is between a and b .

Likewise, $\sqrt[3]{ppzz+4mpz}$ and $\sqrt[3]{aaamm+4aammpp}$ are Commensurable, for each

of them being divided by their common Divisor $\sqrt[3]{aa+4mp}$ there will arise $\sqrt[3]{\frac{aa}{aa+4mp}}$, that is, $\frac{z}{a}$ and $\frac{am}{pz}$ (to wit, Rational quantities,) each of which multiplied into the common Divisor $\sqrt[3]{aa+4mp}$ will produce $\frac{z}{a}\sqrt[3]{aa+4mp}$ and $\frac{am}{pz}\sqrt[3]{aa+4mp}$, which are equal to, but more simply expressed than the Surd Quantities proposed, and have that Proportion to one another as is between $\frac{z}{a}$ and $\frac{am}{pz}$.

So also $\sqrt[3]{aaaa+6aaa+21aa+7a+108}$ & $\sqrt[3]{aaaa-10aaa+37aa-12aa+108}$ are Commensurable, for if they be severally divided by their common Divisor $\sqrt[3]{aa+1}$, there will arise $\sqrt[3]{aa-6a+9}$ and $\sqrt[3]{aa-10a+25}$; that is, $a+3$ and $a+5$, each

which multiplied into the common Divisor $\sqrt[3]{aa+1}$ will produce $a+3$ and $a+5$ and $\sqrt[3]{aa+1}$ which have the same Proportion between themselves as that of $a+3$ to $a+5$, and are of the same value with the Surd Quantities first proposed.

Again, $\sqrt[3]{(3)81abbb}$ and $\sqrt[3]{(3)24abbb}$ are Commensurable, for if each of them be divided by their common Divisor $\sqrt[3]{(3)3a}$ there will arise $\sqrt[3]{(3)27bbb}$ and $\sqrt[3]{(3)8bbb}$, that is, $3b$ and $2b$, therefore the Surds proposed may be reduced to $3b\sqrt[3]{(3)a}$ and $2b\sqrt[3]{(3)a}$, the former of which is to the latter as $3b$ to $2b$; and so of others.

SECT. VIII. Addition and Subtraction in simple Surd quantities.

When two or more equal Surd Roots are to be added together; multiply one of them by the number which expresseth the multitude of the Roots proposed; and the Product shall be their Summ: as, the sum of $\sqrt[3]{6}$ and $\sqrt[3]{6}$ is $\sqrt[3]{12}$; for $\sqrt[3]{6}$ multiplied by 2, that is, by 4, produceth $\sqrt[3]{24}$: also $\sqrt[3]{(3)6}$ and $\sqrt[3]{(3)6}$ added into one, make $\sqrt[3]{(3)12}$; for $\sqrt[3]{(3)6}$ multiplied by 3, that is, by $\sqrt[3]{(3)27}$, makes $\sqrt[3]{(3)162}$.

But when two unequal Surd Roots of the same kind, that is, such as have the same Radical sign prefix before each of them, be to be added together, also when the lesser is to be subtracted from the greater, observe this Rule: First (by the preceding Sect. 7. of this Chapt.) you must try whether they be Commensurable or not, then if they be Commensurable, that is, if after they have been severally divided by their greatest common Divisor the Quotients be Rational quantities, multiply the sum of those Rational quantities, by the said common Divisor, and the Product shall be the Summ of the said Surd Roots proposed; but if the Difference of those Rational Quotients be multiplied by the said common Divisor, the Product shall be the Difference of the Roots proposed.

As; for example, if the Summ and Difference of $\sqrt[3]{50}$ and $\sqrt[3]{8}$ be desired; first, I divide each of them by their greatest common Divisor $\sqrt[3]{2}$, and the Quotients are $\sqrt[3]{25}$ and $\sqrt[3]{4}$, that is, 5 and 2, (which are Rational numbers, expressing the Proportion of the given Roots one to the other;) whose sum 7 multiplied by the common Divisor $\sqrt[3]{2}$, produceth $7\sqrt[3]{2}$, or if you please, $\sqrt[3]{98}$, (for 7, to wit, $\sqrt[3]{49}$ into $\sqrt[3]{2}$ makes $\sqrt[3]{98}$;) which is the desired Summ of the given Roots $\sqrt[3]{50}$ and $\sqrt[3]{8}$. And if $5-2$, that is 3; (the Difference of the Rational Quotients before found) be multiplied by the said common Divisor $\sqrt[3]{2}$, the Product will be $3\sqrt[3]{2}$, that is, $\sqrt[3]{18}$; which is the desired Difference of $\sqrt[3]{50}$ and $\sqrt[3]{8}$, the Roots first proposed.

Likewise, the Summ of $\sqrt[3]{(3)500}$ and $\sqrt[3]{(3)108}$ will be found $8\sqrt[3]{(3)4}$, that is, $\sqrt[3]{(3)1048}$; and their Difference $2\sqrt[3]{(3)4}$, that is, $\sqrt[3]{(3)8}$, as will appear by the following Work: viz. First, I divide each of the given Roots $\sqrt[3]{(3)500}$ and $\sqrt[3]{(3)108}$ by their greatest common Divisor $\sqrt[3]{(3)4}$, and the Quotients are $\sqrt[3]{(3)125}$ and $\sqrt[3]{(3)27}$, that is, 5 and 3; then by multiplying 8 (to wit, $5+3$ the sum of the Rational Quotients,) by the common Divisor $\sqrt[3]{(3)4}$, the Product is $8\sqrt[3]{(3)4}$, that is, $\sqrt[3]{(3)2048}$; (for 8, to wit, $\sqrt[3]{(3)512}$ into $\sqrt[3]{(3)4}$, makes $\sqrt[3]{(3)2048}$;) which is the Summ of $\sqrt[3]{(3)500}$ and $\sqrt[3]{(3)108}$, the Roots proposed.

And by multiplying 2, (that is, $5-3$, the Difference of the Rational Quotients) by the said common Divisor $\sqrt[3]{(3)4}$, the Product is $2\sqrt[3]{(3)4}$, that is, $\sqrt[3]{(3)8}$; (for 2, to wit, $\sqrt[3]{(3)8}$ into $\sqrt[3]{(3)4}$, makes $\sqrt[3]{(3)32}$;) which is the Difference of $\sqrt[3]{(3)500}$ and $\sqrt[3]{(3)108}$, the Roots proposed.

Here follow Contractions of the Work in the two last preceding Examples; with others of like nature, to illustrate the Rule before given for the Addition and Subtraction of such simple Surd Roots as are Commensurable.

Example I.

What is the Summ and Difference of $\sqrt[3]{50}$ and $\sqrt[3]{8}$?

The Operation.

$$\begin{array}{lcl} \sqrt[3]{2} \sqrt[3]{50} \sqrt[3]{25}, \text{ that is, } 5; & \text{Therefore} & 5\sqrt[3]{2} = \sqrt[3]{50}. \\ \sqrt[3]{2} \sqrt[3]{8} \sqrt[3]{4}, \text{ that is, } 2; & \text{Therefore} & 2\sqrt[3]{2} = \sqrt[3]{8}. \\ \hline \text{The Summ,} & & 7\sqrt[3]{2} = \sqrt[3]{50} + \sqrt[3]{8}. \\ \text{Or,} & & \sqrt[3]{98} = \sqrt[3]{50} + \sqrt[3]{8}. \\ \hline \text{The Difference,} & & 3\sqrt[3]{2} = \sqrt[3]{50} - \sqrt[3]{8}. \\ \text{Or,} & & \sqrt[3]{18} = \sqrt[3]{50} - \sqrt[3]{8}. \end{array}$$

E 2

Example 2.

Example 2.

What is the Summ and Difference of . . . $\sqrt{(3)500}$ and $\sqrt{(3)108}$?

The Operation.

$$\begin{array}{l} \text{I. } \sqrt{(3)4} \sqrt{(3)500} \text{ (} \sqrt{(3)125} \text{, that is, 5.} \\ \text{II. } \sqrt{(3)4} \sqrt{(3)108} \text{ (} \sqrt{(3)27} \text{, that is, 3.} \\ \text{From Division I. } 5\sqrt{(3)4} = \sqrt{(3)500.} \\ \text{From Division II. } 3\sqrt{(3)4} = \sqrt{(3)108.} \\ \text{The Summ, } 8\sqrt{(3)4} = \sqrt{(3)500} + \sqrt{(3)108}; \\ \text{Or, } \sqrt{(3)2048} = \sqrt{(3)500} + \sqrt{(3)108.} \\ \text{The Difference, } 2\sqrt{(3)4} = \sqrt{(3)500} - \sqrt{(3)108}; \\ \text{Or, } \sqrt{(3)32} = \sqrt{(3)500} - \sqrt{(3)108.} \end{array}$$

Example 3.

What is the Summ and Difference of . . . $\sqrt{147}$ and $\sqrt{12}$?

The Operation.

$$\begin{array}{l} \sqrt{3} \sqrt{147} \text{ (} \sqrt{49} \text{, that is, 7; Therefore } 7\sqrt{3} = \sqrt{147}. \\ \sqrt{3} \sqrt{12} \text{ (} \sqrt{4} \text{, that is, 2; Therefore } 2\sqrt{3} = \sqrt{12}. \\ \text{The Summ, } 9\sqrt{3} = \sqrt{147} + \sqrt{12}; \\ \text{Or, } \sqrt{243} = \sqrt{147} + \sqrt{12}. \\ \text{The Difference, } 5\sqrt{3} = \sqrt{147} - \sqrt{12}; \\ \text{Or, } \sqrt{75} = \sqrt{147} - \sqrt{12}. \end{array}$$

Example 4.

What is the Summ and Difference of . . . $\sqrt{(3)1715}$ and $\sqrt{(3)40}$?

The Operation.

$$\begin{array}{l} \text{I. } \sqrt{(3)5} \sqrt{(3)1715} \text{ (} \sqrt{(3)343} \text{, that is, 7.} \\ \text{II. } \sqrt{(3)5} \sqrt{(3)40} \text{ (} \sqrt{(3)8} \text{, that is, 2.} \\ \text{From Division I. } 7\sqrt{(3)5} = \sqrt{(3)1715}. \\ \text{From Division II. } 2\sqrt{(3)5} = \sqrt{(3)40}. \\ \text{The Summ, } 9\sqrt{(3)5} = \sqrt{(3)1715} + \sqrt{(3)40}; \\ \text{Or, } \sqrt{(3)645} = \sqrt{(3)1715} + \sqrt{(3)40}. \\ \text{The Difference, } 5\sqrt{(3)5} = \sqrt{(3)1715} - \sqrt{(3)40}; \\ \text{Or, } \sqrt{(3)625} = \sqrt{(3)1715} - \sqrt{(3)40}. \end{array}$$

Note. When two Commensurable Surd Roots proposed to be added or subtracted are Fractions, or mixt numbers reduced into the form of Fractions, if they have not a Common Denominator reduce them into others which may have a Common Denominator in the least Terms; then to find out the Rational Quotients, divide only the two new Numerators severally by their greatest Common Divisor, and continue the process as before. The Practice of this Note will be evident in the two following Examples.

Example 5.

What is the Summ and Difference of . . . $\sqrt{\frac{25}{7}}$ and $\sqrt{\frac{3}{7}}$;
Or, $\sqrt{\frac{25}{7}}$ and $\sqrt{\frac{3}{7}}$?

The Operation.

$$\begin{array}{l} \sqrt{\frac{25}{7}} \sqrt{\frac{3}{7}} \text{ (} \sqrt{36} \text{, that is, 6; Therefore } 6\sqrt{\frac{3}{7}} = \sqrt{\frac{25}{7}}. \\ \sqrt{\frac{25}{7}} \sqrt{\frac{3}{7}} \text{ (} \sqrt{25} \text{, that is, 5; Therefore } 5\sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7}}. \\ \text{The Summ, } 11\sqrt{\frac{3}{7}} = \sqrt{\frac{25}{7}} + \sqrt{\frac{3}{7}}; \\ \text{Or, } \sqrt{\frac{289}{7}} = \sqrt{\frac{25}{7}} + \sqrt{\frac{3}{7}}. \\ \text{The Difference, } \sqrt{\frac{3}{7}} = \sqrt{\frac{25}{7}} - \sqrt{\frac{3}{7}}. \end{array}$$

Example 6.

Example 6.

What is the Summ and Difference of . . . $\sqrt[3]{12}$ and $\sqrt[3]{\frac{1}{2}}$;
Or, $\sqrt[3]{\frac{12}{2}}$ and $\sqrt[3]{\frac{1}{2}}$?

The Operation.

$$\begin{array}{l} \sqrt[3]{\frac{1}{2}} \sqrt[3]{\frac{12}{2}} \text{ (} \sqrt[3]{16} \text{, that is, 4; Therefore } 4\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{12}{2}}. \\ \sqrt[3]{\frac{1}{2}} \sqrt[3]{\frac{12}{2}} \text{ (} \sqrt[3]{9} \text{, that is, 3; Therefore } 3\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{1}{2}}. \\ \text{The Summ, } 7\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{12}{2}} + \sqrt[3]{\frac{1}{2}}; \\ \text{Or, } \sqrt[3]{\frac{343}{2}} = \sqrt[3]{\frac{12}{2}} + \sqrt[3]{\frac{1}{2}}. \\ \text{The Difference, } 4\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{12}{2}} - \sqrt[3]{\frac{1}{2}}. \end{array}$$

When two simple Surd Roots given to be added or subtracted be Incommensurable, neither their Summ nor their Difference can be express'd by any simple Root, but they are to be added by +, and to be subtracted by -. As, to add $\sqrt{5}$ and $\sqrt{3}$, I write $\sqrt{5} + \sqrt{3}$ for the Summ, but to subtract $\sqrt{3}$ from $\sqrt{5}$, I write $\sqrt{5} - \sqrt{3}$ for the Remainder: So also, the Summ of $\sqrt{(3)40}$ and $\sqrt{(3)12}$ is $\sqrt{(3)40} + \sqrt{(3)12}$, and their Difference is $\sqrt{(3)40} - \sqrt{(3)12}$.

But Incommensurable square Roots may be added or subtracted by this following Rule; (which is deduced from Prop. 4, & 7. lib. 2. Euclid.)

To the sum of the Squares of the given Surd square Roots, add the double Product of the multiplication of those Roots one into another; so shall the square Root of the sum be the Summ of the Roots proposed to be added: But if the said double Product be subtracted from the said sum of the Squares, the square Root of the Remainder shall be the Difference of the given Surd square Roots. As, if the Summ and Difference of $\sqrt{6}$ and $\sqrt{3}$ be desired, their Summ shall be $\sqrt{6} + \sqrt{3}$; and their Difference $\sqrt{6} - \sqrt{3}$; for the sum of the Squares of the given square Roots $\sqrt{6}$ and $\sqrt{3}$ is 9, and the double Product of their multiplication is $\sqrt{72}$, which I add to and subtract from 9, so the square Root of the sum; to wit, $\sqrt{9 + \sqrt{72}}$ is the Summ desired; and the square Root of the Remainder, to wit, $\sqrt{9 - \sqrt{72}}$ is the Difference.

After the same manner the Addition and Subtraction of simple Surd Quantities express'd by Letters may be performed: As, to add $\sqrt{75aa}$ and $\sqrt{27aa}$, first, (by the preceding Sect. 7.) I find them to be Commensurable; for, if $\sqrt{75aa}$ and $\sqrt{27aa}$ be severally divided by their greatest common Divisor $\sqrt{3aa}$, the Quotients are $\sqrt{25}$ and $\sqrt{9}$, that is, 5 and 3, whose sum 8 multiplied into the common Divisor $\sqrt{3aa}$ makes $8\sqrt{3aa}$; (that is, $\sqrt{192aa}$) for the Summ of $\sqrt{75aa}$ and $\sqrt{27aa}$. But if the Difference of the same Rational Quotients 5 and 3, to wit, 2, be multiplied into the said common Divisor $\sqrt{3aa}$, it makes $2\sqrt{3aa}$ (that is, $\sqrt{12aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$, the Roots first proposed.

Or, we may write $8\sqrt{3}$ (instead of $8\sqrt{3aa}$) for the Summ, and $2\sqrt{3}$ (instead of $2\sqrt{3aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$ before proposed; for these divided severally by their common Divisor $\sqrt{3}$, give Rational Quotients, to wit, $\sqrt{25aa}$ and $\sqrt{9aa}$, that is, 5a and 3a, whose sum 8a multiplied into the common Divisor $\sqrt{3}$ gives $8a\sqrt{3}$ for the Summ of $\sqrt{75aa}$ and $\sqrt{27aa}$; but if the Difference of the said Rational Quotients 5a and 3a, to wit, 2a, be multiplied into the said common Divisor $\sqrt{3}$, the Product $2a\sqrt{3}$ is the Difference of the said $\sqrt{75aa}$ and $\sqrt{27aa}$.

Again, to add $\sqrt{(3)256aaa}$ and $\sqrt{(3)32aaa}$; first, (by Sect. 7.) I find them to be Commensurable, for if each of them be divided by their common Divisor $\sqrt{(3)4}$, the Quotients are Rational, to wit, $\sqrt{(3)64aaa}$ and $\sqrt{(3)8aaa}$, that is, 4a and 2a; these added together make 6a, which multiplied into the common Divisor $\sqrt{(3)4}$ makes $6a\sqrt{(3)4}$ (that is, $\sqrt{(3)864aaa}$) for the desired Summ of $\sqrt{(3)256aaa}$ and $\sqrt{(3)32aaa}$; but if 2a, the Difference of the same Rational Quotients 4a and 2a, be multiplied into the said common Divisor $\sqrt{(3)4}$, the Product $2a\sqrt{(3)4}$ (that is, $\sqrt{(3)32aaa}$) shall be the Difference of $\sqrt{(3)256aaa}$ and $\sqrt{(3)32aaa}$ first proposed.

More

More Examples of the Addition and Subtraction of Commensurable simple surd Quantities express'd by Letters.

Example 1.

What is the Summ and Difference of . . . $\sqrt{28aa}$ and $\sqrt{7aa}$?

The Operation.

$$\begin{array}{l} \text{I. } \sqrt{7}) \sqrt{28aa} \text{ (} \sqrt{4aa}, \text{ that is, } 2a. \\ \text{II. } \sqrt{7}) \sqrt{7aa} \text{ (} \sqrt{aa}, \text{ that is, } a. \\ \quad \text{From Division I. } 2a\sqrt{7} = \sqrt{28aa}. \\ \quad \text{From Division II. } a\sqrt{7} = \sqrt{7aa}. \\ \quad \text{The Summ, } 3a\sqrt{7} = \sqrt{28aa} + \sqrt{7aa}. \\ \quad \text{The Difference, } a\sqrt{7} = \sqrt{28aa} - \sqrt{7aa}. \end{array}$$

Example 2.

What is the Summ and Difference of . . . $\sqrt{45abc}$ and $\sqrt{20abc}$?

The Operation.

$$\begin{array}{l} \text{I. } \sqrt{5bc}) \sqrt{45abc} \text{ (} \sqrt{9aa}, \text{ that is, } 3a. \\ \text{II. } \sqrt{5bc}) \sqrt{20abc} \text{ (} \sqrt{4aa}, \text{ that is, } 2a. \\ \quad \text{From Division I. } 3a\sqrt{5bc} = \sqrt{45abc}. \\ \quad \text{From Division II. } 2a\sqrt{5bc} = \sqrt{20abc}. \\ \quad \text{The Summ, } 5a\sqrt{5bc} = \sqrt{45abc} + \sqrt{20abc}. \\ \quad \text{The Difference, } a\sqrt{5bc} = \sqrt{45abc} - \sqrt{20abc}. \end{array}$$

Example 3.

What is the Summ and Difference of . . . $\sqrt{(3)81abbb}$ and $\sqrt{(3)24abbb}$?

The Operation.

$$\begin{array}{l} \text{I. } \sqrt{(3)3a}) \sqrt{(3)81abbb} \text{ (} \sqrt{(3)7bbb}, \text{ that is, } 3b. \\ \text{II. } \sqrt{(3)3a}) \sqrt{(3)24abbb} \text{ (} \sqrt{(3)8bbb}, \text{ that is, } 2b. \\ \quad \text{From Division I. } 3b\sqrt{(3)3a} = \sqrt{(3)81abbb}. \\ \quad \text{From Division II. } 2b\sqrt{(3)3a} = \sqrt{(3)24abbb}. \\ \quad \text{The Summ, } 5b\sqrt{(3)3a} = \sqrt{(3)81abbb} + \sqrt{(3)24abbb}. \\ \quad \text{The Difference, } b\sqrt{(3)3a} = \sqrt{(3)81abbb} - \sqrt{(3)24abbb}. \end{array}$$

Example 4.

What is the Summ and Difference of . . . $\sqrt{\frac{1}{2}aad}$ and $\sqrt{\frac{1}{2}aad}$,
Or, $\sqrt{\frac{1}{2}aad}$ and $\sqrt{\frac{1}{2}aad}$?

The Operation.

$$\begin{array}{l} \text{I. } \sqrt{\frac{1}{2}d}) \sqrt{\frac{1}{2}aad} \text{ (} \sqrt{36aa}, \text{ that is, } 6a. \\ \text{II. } \sqrt{\frac{1}{2}d}) \sqrt{\frac{1}{2}aad} \text{ (} \sqrt{25aa}, \text{ that is, } 5a. \\ \quad \text{From Division I. } 6a\sqrt{\frac{1}{2}d} = \sqrt{\frac{1}{2}aad}. \\ \quad \text{From Division II. } 5a\sqrt{\frac{1}{2}d} = \sqrt{\frac{1}{2}aad}. \\ \quad \text{The Summ, } 11a\sqrt{\frac{1}{2}d} = \sqrt{\frac{1}{2}aad} + \sqrt{\frac{1}{2}aad}. \\ \quad \text{The Difference, } a\sqrt{\frac{1}{2}d} = \sqrt{\frac{1}{2}aad} - \sqrt{\frac{1}{2}aad}. \end{array}$$

If two Surd Quantities express'd by letters be Incommensurable, their Summ is given by +, and their Difference by —; as, to add $\sqrt{5a}$ and $\sqrt{3a}$, I write $\sqrt{5a} + \sqrt{3a}$ for the Summ: and to subtract $\sqrt{3a}$ from $\sqrt{5a}$, I write $\sqrt{5a} - \sqrt{3a}$ for the Remainder or Difference.

SECT. IX. Addition and Subtraction in Compound Surd Quantities.

The Arithmetick of Compound Surds depends upon the Rules of the Simple, and the Rules of +, and — in Algebraical Addition, Subtraction, Multiplication and Division; but how those Rules are applied to the Arithmetick of Compound Surds, I shall shew in this and the following tenth and eleventh Sections, by Examples both in Surd Numbers and Surd Quantities express'd by Letters.

Example

Examples of Addition and Subtraction in Commensurable simple surd numbers connected to Rational numbers by + or —, as also in compound surd numbers compos'd of Commensurable simple Surds.

To and from	$6 + \sqrt{18} \text{ (} 3\sqrt{2} \text{)}$	$\sqrt{192} \text{ (} 8\sqrt{3} \text{)} + 3$
Add and Subtr.	$4 + \sqrt{8} \text{ (} 2\sqrt{2} \text{)}$	$\sqrt{75} \text{ (} 5\sqrt{3} \text{)} - 3$
Summ,	$10 + \sqrt{50} \text{ (} 5\sqrt{2} \text{)}$	$\sqrt{507} \text{ (} 13\sqrt{3} \text{)} + 0$
Difference,	$2 - \sqrt{2}$	$\sqrt{27} \text{ (} 3\sqrt{3} \text{)} + 6$
To and from	$+ \sqrt{242} \text{ (} 11\sqrt{2} \text{)} - 12$	$15 - 2\sqrt{2} \text{ (} \sqrt{8} \text{)}$
Add and Subtr.	$- \sqrt{50} \text{ (} -5\sqrt{2} \text{)} + 8$	$7 + \sqrt{2}$
Summ,	$+ \sqrt{72} \text{ (} 6\sqrt{2} \text{)} - 4$	$22 - \sqrt{2}$
Difference,	$+ \sqrt{512} \text{ (} 16\sqrt{2} \text{)} - 20$	$8 - 3\sqrt{2} \text{ (} \sqrt{18} \text{)}$
To and from	$\sqrt{242} + \sqrt{192}$	that is, $\sqrt{11\sqrt{2}} + 8\sqrt{3}$
Add and Subtr.	$\sqrt{50} + \sqrt{75}$	$5\sqrt{2} + 5\sqrt{3}$
Summ,	$\sqrt{512} + \sqrt{507}$	that is, $16\sqrt{2} + 13\sqrt{3}$
Difference,	$\sqrt{72} + \sqrt{27}$	$6\sqrt{2} + 3\sqrt{3}$
To and from	$\sqrt{320} - \sqrt{108}$	that is, $8\sqrt{5} - 6\sqrt{3}$
Add and Subtr.	$\sqrt{80} - \sqrt{27}$	$4\sqrt{5} - 3\sqrt{3}$
Summ,	$\sqrt{720} - \sqrt{243}$	that is, $12\sqrt{5} - 9\sqrt{3}$
Difference,	$\sqrt{80} - \sqrt{27}$	$4\sqrt{5} - 3\sqrt{3}$
To and from	$\sqrt{320} + \sqrt{108}$	that is, $8\sqrt{5} + 6\sqrt{3}$
Add and Subtr.	$\sqrt{80} - \sqrt{27}$	$4\sqrt{5} - 3\sqrt{3}$
Summ,	$\sqrt{720} + \sqrt{27}$	that is, $12\sqrt{5} + 3\sqrt{3}$
Difference,	$\sqrt{80} + \sqrt{243}$	$4\sqrt{5} + 9\sqrt{3}$
To and from	$\sqrt{(3)2058} + \sqrt{(3)54}$	that is, $7\sqrt{(3)6} + 3\sqrt{(3)2}$
Add and Subtr.	$\sqrt{(3)162} + \sqrt{(3)16}$	$3\sqrt{(3)6} + 2\sqrt{(3)2}$
Summ,	$\sqrt{(3)6000} + \sqrt{(3)250}$	that is, $10\sqrt{(3)6} + 5\sqrt{(3)2}$
Difference,	$\sqrt{(3)384} - \sqrt{(3)2}$	$4\sqrt{(3)6} - \sqrt{(3)2}$
To and from	$\sqrt{(4)1875} + \sqrt{(3)250}$	that is, $5\sqrt{(4)3} + 5\sqrt{(3)2}$
Add and Subtr.	$\sqrt{(4)48} - \sqrt{(3)16}$	$2\sqrt{(4)3} - 2\sqrt{(3)2}$
Summ,	$\sqrt{(4)7203} + \sqrt{(3)54}$	that is, $7\sqrt{(4)3} + 3\sqrt{(3)2}$
Difference,	$\sqrt{(4)243} + \sqrt{(3)686}$	$3\sqrt{(4)3} + 7\sqrt{(3)2}$

EXPLICATION.

In the first Example, the Rational numbers 6 and 4 added together make 10, and their Difference is 2, then forasmuch as $\sqrt{18}$ and $\sqrt{8}$ (that is, $3\sqrt{2}$ and $2\sqrt{2}$) are Commensurable, (for the former is to the latter as 3 to 2,) their Summ is $\sqrt{50}$, (that is, $5\sqrt{2}$), and their Difference $\sqrt{2}$, (by Sect. 8.) Wherefore $10 + \sqrt{50}$ (that is, $10 + 5\sqrt{2}$) is the Summ, and $2 - \sqrt{2}$ the Difference of the two Binomials $6 + \sqrt{18}$ and $4 + \sqrt{8}$, propos'd in the first Example.

Likewise in the second Example, the two Commensurable surd Roots $\sqrt{192}$ and $\sqrt{75}$, (that is, $8\sqrt{3}$ and $5\sqrt{3}$) added into one simple Surd make $\sqrt{507}$, (that is, $13\sqrt{3}$), but their Difference is $\sqrt{27}$, (that is, $3\sqrt{3}$;) also, $+3$ and -3 added together make 0, but -3 subtracted from $+3$ makes -6 . Wherefore $\sqrt{507}$ (that is, $13\sqrt{3}$) is the Summ, and $\sqrt{27}$ (that is, $3\sqrt{3}$) $+6$ is the Difference of the Binomial $\sqrt{192} + 3$, and the Residual $\sqrt{75} - 3$ propos'd in the second Example.

Again, in the third Example, where $-\sqrt{50} - 8$ is propos'd to be added to $+\sqrt{242} - 12$, and also to be subtracted from the same; first, $-\sqrt{50}$ added to $+\sqrt{242}$, (that is, $-5\sqrt{2}$ to $+11\sqrt{2}$) makes $+6\sqrt{2}$ (that is, $6\sqrt{2}$;) but $-\sqrt{50}$ subtracted from

from $-1\sqrt{2}$ (that is, $-5\sqrt{2}$ from $+11\sqrt{2}$) leaves the Remainder or Difference $+1\sqrt{2}$, (that is, $16\sqrt{2}$;) also, -8 added to -12 makes -20 , but $+8$ subtracted from -12 , leaves the Remainder or Difference -20 . Wherefore $\sqrt{72}$ (that is $6\sqrt{2}$) -4 is the Summ, and $\sqrt{512}$ (that is, $16\sqrt{2}$) -20 is the Difference of the two Residuals proposed in the third Example. The Operation in the rest of the preceding Examples is after the same manner.

Examples of Addition and Subtraction in compound surd numbers, partly Commensurable, and partly Incommensurable.

To and from	$\sqrt{27} (3\sqrt{3}) + \sqrt{8}$	$\sqrt{10} + \sqrt{8} (2\sqrt{2})$
Add and Subtr.	$\sqrt{12} (2\sqrt{3}) + \sqrt{5}$	$\sqrt{3} - \sqrt{2}$
The Summ,	$\sqrt{75} (5\sqrt{3}) + \sqrt{8} + \sqrt{5}$	$\sqrt{10} + \sqrt{3} + \sqrt{2}$
Or,	$\sqrt{75} (5\sqrt{3}) + \sqrt{13} + \sqrt{160}$	$\sqrt{13} + \sqrt{120} + \sqrt{2}$
The Difference,	$\sqrt{3} + \sqrt{8} - \sqrt{5}$	$\sqrt{10} - \sqrt{3} + \sqrt{18} (3\sqrt{2})$
Or,	$\sqrt{3} + \sqrt{13} - \sqrt{160}$	$\sqrt{13} - \sqrt{120} + \sqrt{18} (3\sqrt{2})$
To and from	$\sqrt{(3)56} + \sqrt{(3)16}$	$\sqrt{(4)405} - \sqrt{(3)2}$
Add and Subtr.	$\sqrt{(3)7} - \sqrt{(3)12}$	$\sqrt{(4)80} + \sqrt{(3)5}$
Summ,	$3\sqrt{(3)7} + \sqrt{(3)16} - \sqrt{(3)12}$	$5\sqrt{(4)5} + \sqrt{(3)5} - \sqrt{(3)2}$
Difference,	$\sqrt{(3)7} - \sqrt{(3)16} + \sqrt{(3)12}$	$\sqrt{(4)5} - \sqrt{(3)5} - \sqrt{(3)2}$

EXPLICATION.

In the first of the four last preceding Examples, the Summ of the two Commensurable surd Roots $\sqrt{27}$ and $\sqrt{12}$ (that is, $3\sqrt{3}$ and $2\sqrt{3}$) is $\sqrt{75}$, (that is, $5\sqrt{3}$;) but their Difference is $\sqrt{3}$; and the Summ of the two Incommensurable Roots $\sqrt{8}$ and $\sqrt{5}$ is $\sqrt{8} + \sqrt{5}$, or, $\sqrt{13} + \sqrt{160}$; but their Difference is $\sqrt{8} - \sqrt{5}$, or, $\sqrt{13} - \sqrt{160}$; (according to the Rule before given in Sect. 8. for adding and subtracting two Incommensurable square Roots.) Therefore $\sqrt{75} + \sqrt{8} + \sqrt{5}$, or, $\sqrt{75} + \sqrt{13} + \sqrt{160}$ is the Summ, and $\sqrt{75} + \sqrt{8} - \sqrt{5}$, or, $\sqrt{75} + \sqrt{13} - \sqrt{160}$ is the Difference of the two Binomials $\sqrt{27} + \sqrt{8}$ and $\sqrt{12} + \sqrt{5}$, proposed in the said first Example.

Again, in the third of the said four Examples, where $\sqrt{(3)56} + \sqrt{(3)16}$ and $\sqrt{(3)7} - \sqrt{(3)12}$ are proposed to be added and subtracted, the Summ of the two Commensurable surd Cubick Roots $\sqrt{(3)56}$ and $\sqrt{(3)7}$ is $3\sqrt{(3)7}$, and their Difference is $\sqrt{(3)7}$; also, the Summ of the two Incommensurable cubick Roots $\sqrt{(3)16}$ and $-\sqrt{(3)12}$ is $\sqrt{(3)16} - \sqrt{(3)12}$, but $-\sqrt{(3)12}$ subtracted from $\sqrt{(3)16}$ leaves $\sqrt{(3)16} + \sqrt{(3)12}$. Wherefore $3\sqrt{(3)7} + \sqrt{(3)16} - \sqrt{(3)12}$ is the Summ, and $\sqrt{(3)7} + \sqrt{(3)16} + \sqrt{(3)12}$ is the Difference of the said Binomial and Residual proposed in the third Example.

Examples of Addition and Subtraction in Compound surd quantities express'd by Letters.

Example 1.

To and from	$\sqrt{75aa} + \sqrt{8bb}$	vic. $\{ \begin{array}{l} 5a\sqrt{3} + 2b\sqrt{2} \\ 2a\sqrt{3} + b\sqrt{2} \end{array} \}$
Add and Subtr.	$\sqrt{12aa} + \sqrt{2bb}$	
The Summ is	$\dots\dots\dots 7a\sqrt{3} + 3b\sqrt{2}$	
The Difference is	$\dots\dots\dots 3a\sqrt{3} + b\sqrt{2}$	

EXPLICATION.

First, (by Sect. 7.) I find that $\sqrt{75aa}$ and $\sqrt{12aa}$ are Commensurable, and may be reduced to $5a\sqrt{3}$ and $2a\sqrt{3}$; likewise $\sqrt{8bb}$ and $\sqrt{2bb}$ are Commensurable, and may be reduced to $2b\sqrt{2}$ and $b\sqrt{2}$: then the Summ of $5a\sqrt{3}$ and $2a\sqrt{3}$ is $7a\sqrt{3}$; also, the Summ of $2b\sqrt{2}$ and $b\sqrt{2}$ is $3b\sqrt{2}$: therefore the Summ of the two Binomials proposed in the Example is $7a\sqrt{3} + 3b\sqrt{2}$. But by subtracting $2a\sqrt{3}$ from $5a\sqrt{3}$, the Remainder is $3a\sqrt{3}$; and by subtracting $b\sqrt{2}$ from $2b\sqrt{2}$, the Remainder is $b\sqrt{2}$. Therefore the Difference of the two Binomials proposed is $3a\sqrt{3} + b\sqrt{2}$.

Example 2.

Example 2.

What is the Summ and Difference of this Binomial $\sqrt{(3)1715a^3b^3} + \sqrt{(3)40a^3b^3}$ and Residual, $\dots\dots\dots \sqrt{(3)3}bcd$?

Those reduced give these, to wit, $\dots\dots\dots \sqrt{(3)7ab} \sqrt{(3)5} + \sqrt{(3)3}bcd$
 $\sqrt{(3)2ab} \sqrt{(3)5} - \sqrt{(3)3}bcd$

The Summ, $\dots\dots\dots 9ab\sqrt{(3)5}$
 The Difference, $\dots\dots\dots 5ab\sqrt{(3)5} + 2\sqrt{(3)3}bcd$

Examples of Addition and Subtraction in compound surd numbers altogether Incommensurable.

To and from	$\sqrt{10} + \sqrt{7}$
Add and Subtr.	$\sqrt{3} + \sqrt{2}$
Summ,	$\sqrt{10} + \sqrt{7} + \sqrt{3} + \sqrt{2}$
Or,	$\sqrt{17} + \sqrt{280} + \sqrt{5} + \sqrt{24}$
Difference,	$\sqrt{10} + \sqrt{7} - \sqrt{3} - \sqrt{2}$
Or,	$\sqrt{17} + \sqrt{280} - \sqrt{5} + \sqrt{24}$
To and from	$\sqrt{(3)10} + \sqrt{(3)7}$
Add and Subtr.	$\sqrt{(3)3} - \sqrt{(3)2}$
Summ,	$\sqrt{(3)10} + \sqrt{(3)7} + \sqrt{(3)3} - \sqrt{(3)2}$
Difference,	$\sqrt{(3)10} + \sqrt{(3)7} - \sqrt{(3)3} + \sqrt{(3)2}$

Sect. X. Of Multiplication in Compound Surds.

Example 1.

Multiplicand,	$\sqrt{180} + \sqrt{48}$	that is, $\{ \begin{array}{l} 6\sqrt{5} + 4\sqrt{3} \\ 5\sqrt{5} + 2\sqrt{3} \end{array} \}$
Multiplicator,	$\sqrt{125} + \sqrt{12}$	
Product,	$150 + 20\sqrt{15} + 12\sqrt{15} + 24$	
That is,	$174 + 32\sqrt{15}$	

Example 2.

Multiplicand,	$6 - \sqrt{20}$	that is, $\{ \begin{array}{l} 6 - 2\sqrt{5} \\ 8 - 3\sqrt{5} \end{array} \}$
Multiplicator,	$8 - \sqrt{45}$	
Product,	$48 - 16\sqrt{5} - 18\sqrt{5} + 30$	
That is,	$78 - 34\sqrt{5}$	

Example 3.

Multiplicand,	$\sqrt{18} - 3$	that is, $\{ \begin{array}{l} 3\sqrt{2} - 3 \\ 2\sqrt{2} + 2 \end{array} \}$
Multiplicator,	$\sqrt{8} + 2$	
Product,	$12 - 6\sqrt{2} + 6\sqrt{2} - 6$	
That is,	6	

Example 4.

Multiplicand,	$4\sqrt{5} + 3\sqrt{5}$	that is, $\{ \begin{array}{l} 7\sqrt{5} \\ 7\sqrt{5} \end{array} \}$
Multiplicator,	$4\sqrt{5} + 3\sqrt{5}$	
Product,	245	

EXPLICIT

EXPLICATION.

In the first Example, the two Compound Surd numbers propos'd to be multiplied are $\sqrt{180} + \sqrt{48}$ and $\sqrt{125} + \sqrt{12}$, which are reduced to $6\sqrt{5} + 4\sqrt{3}$ and $5\sqrt{5} + 2\sqrt{3}$; (by Sect. 6. of this Chap.) then $6\sqrt{5}$ multiplied by $5\sqrt{5}$. (according to Rule 5. in Sect. 4. of this Chap.) produceth 150; also, $4\sqrt{3}$ multiplied by $5\sqrt{5}$ (according to Rule 6. in Sect. 4.) produceth $20\sqrt{15}$; again, $6\sqrt{5}$ into $2\sqrt{3}$ makes $12\sqrt{15}$; and $4\sqrt{3}$ into $2\sqrt{3}$, produceth 24; lastly, those Products added together make $174 + 32\sqrt{15}$, the Product sought. The rest of the Examples are wrought in like manner.

When the Multiplicand hath not the same Radical sign with the Multiplier, they must first be reduced to the same Radical sign; (by Sect. 3. of this Chap.) and then the Multiplication is to be made by some of the Rules in Sect. 4. as will be manifest in the following Example.

Multiplicand,	$\sqrt{(5)}6 + \sqrt{(3)}7 + 5$
Multiplier,	$\sqrt{3}$
Product,	$\sqrt{(10)}8748 + \sqrt{(6)}1323 + 5\sqrt{3}$

EXPLICATION.

1. $\sqrt{(5)}6$ and $\sqrt{3}$ are reduced to these having a common Radical sign; to wit, $\sqrt{(10)}6$ and $\sqrt{(10)}143$, which multiplied one into the other, produce $\sqrt{(10)}8748$.
 2. $\sqrt{(3)}7$ and $\sqrt{3}$ are reduced to $\sqrt{(6)}49$ and $\sqrt{(6)}7$, which multiplied one by the other, produce $\sqrt{(6)}1323$.
 3. The Rational number 5 multiplied into $\sqrt{3}$ makes $5\sqrt{3}$, or, $\sqrt{75}$.
- Lastly, those three simple Products added together, give the Product sought, to wit, $\sqrt{(10)}8748 + \sqrt{(6)}1323 + 5\sqrt{3}$ ($\sqrt{75}$).

Three compendious Rules, very useful in the Multiplication of Binomials and Residuals.

1. Because $a + e$ multiplied by $a + e$ produceth $aa + 2ae + ee$; it is evident that the sum of the Squares of the parts (or Names) of any Binomial, together with twice the Product of the parts multiplied one into the other, is equal to the Square of the Sum of the parts. Therefore, to multiply any Binomial by it self, (or to square it,) take the Squares of the parts, and twice the Product of the parts for the Square sought.
2. Because $a - e$ multiplied by $a - e$ produceth $aa - 2ae + ee$; it is manifest that the sum of the Squares of the parts of any Residual, less by the double Product of the parts, is equal to the Square of the difference of the parts. Therefore, to square any Residual, from the sum of the Squares of the parts subtract twice the Product of the parts, and take the Remainder for the Square sought.
3. Because $a + e$ multiplied by $a - e$ produceth $aa - ee$; it is evident that the difference of the Squares of the parts of any Binomial, is equal to the Product made by the multiplication of the sum of the parts into their difference. Therefore, if a Binomial be to be multiplied by its correspondent Residual, that is, by the difference of the parts of the Binomial, take the difference of the Squares of the parts for the Product sought. These three Rules will be exercised by the six Examples next following, and by divers other Examples in this and the following Sections of this Chapter.

Multiplicand,	$3 + \sqrt{5}$	$3 - \sqrt{5}$
Multiplier,	$3 + \sqrt{5}$	$3 - \sqrt{5}$
Product,	$9 + 6\sqrt{5} + 5$	$9 - 6\sqrt{5} + 5$
That is,	$14 + 6\sqrt{5}$	$14 - 6\sqrt{5}$
<hr/>		
Multiplicand,	$3 + \sqrt{5}$	$\sqrt{(3)} 27 + \sqrt{(3)} 8$
Multiplier,	$3 - \sqrt{5}$	$\sqrt{(3)} 27 - \sqrt{(3)} 8$
Product,	$9 - 5$	$\sqrt{(3)} 729 - \sqrt{(3)} 64$
That is,	4	5
<hr/>		
Multiplicand,	$\sqrt{(6)} 7 - \sqrt{(6)} 5$	$\sqrt{(10)} 7 + \sqrt{(10)} 3$
Multiplier,	$\sqrt{(6)} 7 + \sqrt{(6)} 5$	$\sqrt{(10)} 7 - \sqrt{(10)} 3$
Product,	$\sqrt{(3)} 7 - \sqrt{(3)} 5$	$\sqrt{(5)} 7 - \sqrt{(5)} 3$

EXPLI

EXPLICATION.

In the first of the six last Examples, the Binomial $3 + \sqrt{5}$ multiplied into it self or squared, produceth $14 + 6\sqrt{5}$. For the Squares of the parts 3 and $\sqrt{5}$ are 9 and 5, and twice the Product of 3 into $\sqrt{5}$ makes $6\sqrt{5}$, to wit, $\sqrt{180}$; therefore (by the second first of the three preceding Rules,) $9 + 5 + 6\sqrt{5}$; that is, $14 + 6\sqrt{5}$ is the Square of the given Binomial $3 + \sqrt{5}$.

In the second Example, the Residual $3 - \sqrt{5}$ squared or multiplied by it self produceth $14 - 6\sqrt{5}$, (by the second of the said three Rules.)

In the third Example, the Binomial $3 + \sqrt{5}$ multiplied by its correspondent Residual $3 - \sqrt{5}$, produceth 4; which (by the last of the said three Rules) is equal to the difference of the Squares of the parts 3 and $\sqrt{5}$.

Likewise in the fourth Example, the Binomial $\sqrt{(3)} 27 + \sqrt{(3)} 8$, multiplied by its correspondent Residual $\sqrt{(3)} 27 - \sqrt{(3)} 8$, produceth $\sqrt{(3)} 729 - \sqrt{(3)} 64$; to wit, the difference of the Squares of the parts of the given Binomial or Residual.

And in the fifth Example, the Residual $\sqrt{(6)} 7 - \sqrt{(6)} 5$ multiplied by its correspondent Binomial $\sqrt{(6)} 7 + \sqrt{(6)} 5$, produceth $\sqrt{(3)} 7 - \sqrt{(3)} 5$, which is equal to the difference of the Squares of the parts of the given Residual or Binomial. For (by the seventh Rule in Sect. 4. of this Chap.) the Square of $\sqrt{(6)} 7$ is $\sqrt{(3)} 7$, and the Square of $\sqrt{(6)} 5$ is $\sqrt{(3)} 5$.

Examples of Multiplication in Compound Surd quantities express'd by Letters.

Multiplicand,	$\sqrt{abb} + \sqrt{eff}$	that is, $\sqrt{b^2/a} + \sqrt{f^2/c}$
Multiplier,	$\sqrt{add} + \sqrt{caa}$	$\sqrt{d/a} + \sqrt{a/c}$
Product,	$bda + fd/a + ba/a + fac$	$bda + fd + ba + fac$
<hr/>		
Multiplicand,	$2a + 3a/d$	$\sqrt{bc} + a$
Multiplier,	$3c + 2c/d$	$\sqrt{bc} + a$
Product,	$6ac + 9ac/d - 6acd$	$bc + a/bc - aa$
<hr/>		
Multiplicand,	$a + \sqrt{b}$	$\sqrt{ab} + \sqrt{c}$
Multiplier,	$a + \sqrt{b}$	$\sqrt{ac} + \sqrt{d}$
Product,	$aa + 2a/b + b$	$a/bc + c/a + \sqrt{abd} + \sqrt{cd}$
<hr/>		
Multiplicand,	$3bb/d + d/d$	that is, $\sqrt{3bb/d} + \sqrt{d}$
Multiplier,	$3bb/d + d/d$	$\sqrt{3bb/d} + \sqrt{d}$
Product,	$9bbdd + 6bbdd + dd$	or, $9bbdd + 6bbdd + dd + d$

The Operation in these six last Examples will be familiar to him that understands the Rules and Examples before delivered concerning the Multiplication of Surd numbers, and Surd quantities express'd by Letters.

Sect. XI. Division in Compound Surds.

Examples of Division, where the Dividend is a Compound quantity, and the Divisor a Simple quantity.

Dividend,	$\sqrt{21} + \sqrt{15}$	$\sqrt{(3)} 14 - \sqrt{(3)} 8$
Divisor,	$\sqrt{3}$	$\sqrt{(3)} 7$
Quotient,	$\sqrt{7} + \sqrt{5}$	$\sqrt{(3)} 2 - \sqrt{(3)} 4$

ffz

Dividend,

Dividend,	$12\sqrt{6} + 6\sqrt{18} - 2\sqrt{12}$	$\sqrt{20} - \sqrt{(3)10}$
Divisor,	$3\sqrt{6}$	3
Quotient,	$4 + 2\sqrt{3} - \frac{2}{3}\sqrt{2}$	$\sqrt{\frac{20}{9}} - \sqrt{(3)\frac{10}{9}}$
Dividend,	$\sqrt{(4)8} + \sqrt{(5)3}$	$\sqrt{(4)13328} - \sqrt{(4)10368}$
Divisor,	$\sqrt{2}$	6
Quotient,	$\sqrt{(4)4} + \sqrt{(10)\frac{3}{2}}$	$\sqrt{(4)18} - \sqrt{(4)8}$

EXPLICATION.

The first Example is wrought according to Rule 1. in *Self* 5. of this *Chapt.* For first, $\sqrt{20}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{15}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{5}$. Therefore $\sqrt{15} + \sqrt{15}$ divided by $\sqrt{3}$, gives $\sqrt{7} + \sqrt{5}$, the Quotient sought in the first Example.

The second Example is wrought like the first; for $\sqrt{(3)14}$ divided by $\sqrt{(3)7}$ gives $\sqrt{(3)2}$, and $-\sqrt{(3)28}$ divided by $\sqrt{(3)7}$ gives $-\sqrt{(3)4}$. Therefore, $\sqrt{(3)14} - \sqrt{(3)28}$ divided by $\sqrt{(3)7}$, gives $\sqrt{(3)2} - \sqrt{(3)4}$, the Quotient sought in the second Example.

The third Example is wrought according to the fifth and sixth Rules of *Self* 5. of this *Chapt.* For first, $12\sqrt{6}$ divided by $3\sqrt{6}$ gives the Quotient 4, (by the said fifth Rule,) then $6\sqrt{18}$ divided by $3\sqrt{6}$ gives $2\sqrt{3}$, (by the said sixth Rule,) likewise, $-2\sqrt{12}$ divided by $3\sqrt{6}$ gives $-\frac{2}{3}\sqrt{2}$; (for 2 divided by 3 gives $\frac{2}{3}$, and $\sqrt{12}$ divided by $\sqrt{6}$ gives $\sqrt{2}$.) Therefore, $12\sqrt{6} + 6\sqrt{18} - 2\sqrt{12}$ divided by $3\sqrt{6}$, gives $4 + 2\sqrt{3} - \frac{2}{3}\sqrt{2}$, the Quotient sought in the third Example.

In the fourth Example, $\sqrt{20}$ divided by 3, (that is, by $\sqrt{9}$), gives $\sqrt{\frac{20}{9}}$, or $\sqrt{\frac{20}{9}}$; and $-\sqrt{(3)10}$ divided by 3, (that is, by $\sqrt{(3)27}$), gives $-\sqrt{(3)\frac{10}{27}}$.

In the fifth Example, $\sqrt{(4)8}$ and $\sqrt{2}$ are first reduced to $\sqrt{(4)8}$ and $\sqrt{(4)4}$; then $\sqrt{(4)8}$ divided by $\sqrt{(4)4}$ gives $\sqrt{(4)2}$, likewise, $\sqrt{(5)3}$ and $\sqrt{2}$ are reduced to $\sqrt{(10)9}$ and $\sqrt{(10)32}$, then $\sqrt{(10)9}$ divided by $\sqrt{(10)32}$ gives the Quotient $\sqrt{(10)\frac{9}{32}}$. Therefore, $\sqrt{(4)8} + \sqrt{(5)3}$ divided by $\sqrt{2}$, gives $\sqrt{(4)2} + \sqrt{(10)\frac{9}{32}}$, the Quotient sought in the fifth Example. The sixth Example is wrought in like manner; and the Proof in these or the like Examples of Division may be made by Multiplication.

Propositions concerning Division in Surd Quantities, when the Divisor is a Binomial or Trinomial, &c.

When the Divisor is a Binomial or Residual consisting of two square Roots or biquadratic Roots, or of one square Root or biquadratic Root; and of a Rational number, as also when the Divisor is a Trinomial, or Quadrinomial, and none of its Radical Signs exceeds that of the square Root, the work of Division in those cases is grounded upon the five following Propositions, viz.

1. If a Binomial consisting of two simple square Roots connected by $+$, be multiplied by its correspondent Residual, that is, by the difference of those Roots; or if a Residual consisting of two simple square Roots connected by $-$, be multiplied by its correspondent Binomial, that is, by the sum of the same Roots, the Product will be entirely Rational. So the Binomial, $\sqrt{5} + \sqrt{3}$ multiplied by $\sqrt{5} - \sqrt{3}$, (or, the Residual $\sqrt{5} - \sqrt{3}$ by $\sqrt{5} + \sqrt{3}$), gives the Rational Product 2; (by the last of the three Rules before delivered in *Self* 10. of this *Chapt.*)

Likewise, $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$, gives the Rational Product $a - b$.

2. If a Binomial consisting of two Biquadratic simple Roots connected by $+$, be multiplied by its correspondent Residual, to wit, by the difference of those Roots, the Product will be also a Residual consisting of two square Roots connected by $-$, and if this Residual be multiplied by the sum of its Names, (or Parts,) it will give a Product entirely Rational.

As, for example, the Binomial $\sqrt{(4)5} + \sqrt{(4)3}$ multiplied by $\sqrt{(4)5} - \sqrt{(4)3}$ makes $\sqrt{5} - \sqrt{3}$, which multiplied by $\sqrt{5} + \sqrt{3}$ gives the Rational Product 2.

Likewise $\sqrt{(4)81} - 2$, or $\sqrt{(4)81} - \sqrt{(4)16}$, multiplied by $\sqrt{(4)81} + \sqrt{(4)16}$ makes $\sqrt{81} - \sqrt{16}$, which multiplied by $\sqrt{81} + \sqrt{16}$ gives the Rational Product 65; if

3. If a Trinomial consisting of three simple square Roots connected by $+$, or by $+$ and $-$, be multiplied by the same Trinomial, after any one Sign $+$ is changed into $-$, or any one Sign $-$ into $+$, the Product will consist of two Names, (or Parts;) and then if this Product be multiplied by its correspondent Binomial or Residual, (according to the preceding *Prop.* 1.) the last Product will be entirely Rational.

As, for example, the Trinomial $\sqrt{5} + \sqrt{3} + \sqrt{2}$ multiplied by $\sqrt{5} + \sqrt{3} - \sqrt{2}$ gives $2\sqrt{15} + 6$, and this multiplied by $2\sqrt{15} - 6$ gives the Rational Product 24.

Likewise, $\sqrt{30} - \sqrt{5} - \sqrt{3}$ multiplied by $\sqrt{30} + \sqrt{5} - \sqrt{3}$ produceth $28 - 2\sqrt{90}$, and this multiplied by $28 + 2\sqrt{90}$ gives the Rational Product 424.

After the same manner, $\sqrt{a} + \sqrt{b} - \sqrt{c}$ multiplied by $\sqrt{a} + \sqrt{b} + \sqrt{c}$ gives the Product $2\sqrt{ab} + a + b - c$, whose Rational Part $a + b - c$ we may suppose to be equal to some single Quantity d , and then the said Product will be a Binomial $2\sqrt{ab} + d$; which multiplied by its correspondent Residual $2\sqrt{ab} - d$ gives a Product entirely Rational, to wit, $4ab - d^2$. And so of other Trinomials that are qualified as before is supposed.

4. If a Quadrinomial consisting of four simple square Roots connected by $+$, or by $+$ and $-$, be multiplied by the same Quadrinomial after two Signs $+$ are changed into $-$, or two Signs $-$ into $+$, the Product will consist of three Names, (or Parts;) then if this Product be multiplied by its correspondent Trinomial (according to *Prop.* 3.) there will come forth a Binomial or Residual; and lastly, this Binomial or Residual multiplied by its correspondent Residual or Binomial will give a Rational Product.

As, for example, the Quadrinomial $\sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}$ multiplied by $\sqrt{6} + \sqrt{5} - \sqrt{3} - \sqrt{2}$ produceth the Trinomial $6 + 2\sqrt{30} - 2\sqrt{6}$; which multiplied by its correspondent Trinomial $6 + 2\sqrt{30} + 2\sqrt{6}$, (according to the precedent *Prop.* 3.) gives the Binomial $132 + 24\sqrt{30}$; and this multiplied by its correspondent Residual $132 - 24\sqrt{30}$, gives the Rational Product 144.

After the same manner, the Quadrinomial $\sqrt{a} + \sqrt{b} + \sqrt{c} - \sqrt{d}$ multiplied by $\sqrt{a} - \sqrt{b} - \sqrt{c} - \sqrt{d}$ gives the Product $a + d - b - c - 2\sqrt{ad} - 2\sqrt{bc}$, whose Rational part $a + d - b - c$ we may suppose to be equal to some single Quantity f , and then the said Product will be a Trinomial, to wit, $f - 2\sqrt{ad} - 2\sqrt{bc}$; this multiplied by its self after one of its signs $-$ is changed into $+$, (according to *Prop.* 3.) will produce a Residual of two Names (or Parts,) and this Residual multiplied by its correspondent Binomial will give a Rational Product.

5. If two numbers be given for a Dividend and Divisor, and each be multiplied by some number, the first Product divided by the latter will give the same Quotient that ariseth by dividing the given Dividend by the given Divisor. As, if 6 be to be divided by 2, if you multiply each by 4, and divide the first Product 24 by the latter 8, the Quotient 3 is the same that ariseth by dividing 6 by 2. For (by 17 *Prop.* 7. *Elem.* Euclid.) if a number a multiplying two numbers b, c , produce two other numbers ab and ac , the numbers produced shall be, in the same Proportion that the numbers multiplied are, viz. as, $b : c :: ab : ac$, and therefore $\frac{ab}{ac} = \frac{b}{c}$; also, $\frac{ac}{ab} = \frac{c}{b}$. From the foregoing five Propositions the following Rule is deduced, viz.

6. A Rule for Division in Surd Quantities when the Divisor is a Binomial, Trinomial or Quadrinomial of such kind as before is declared.

Reduce the given Divisor to a new Divisor that may be a simple Rational quantity; reduce also the given Dividend to a new Dividend, by multiplying the former by the same quantity or quantities that were Multipliers in reducing the given Divisor to a Rational quantity; then divide the new Dividend by the new Divisor, (according to the method in the Examples at the beginning of this *Self* 11.) so the Quotient shall be the same with that which would arise by dividing the given Dividend by the given Divisor.

As, for example, to divide $\sqrt{8} + \sqrt{6}$ by $\sqrt{4} + \sqrt{2}$, I first multiply the Divisor $\sqrt{4} + \sqrt{2}$ by its correspondent Residual $\sqrt{4} - \sqrt{2}$, and it produceth 2 for a new Divisor; also I multiply the Dividend $\sqrt{8} + \sqrt{6}$ by the said $\sqrt{4} - \sqrt{2}$; and it gives the Product $\sqrt{32} + \sqrt{24} - \sqrt{16} - \sqrt{12}$ for a new Dividend, this divided by 2 (the Divisor before found,) gives $\sqrt{8} + \sqrt{6} - 2 - \sqrt{3}$, the Quotient sought, being equal to that which would arise by dividing $\sqrt{8} + \sqrt{6}$ by $\sqrt{4} + \sqrt{2}$; as will be evident by the Proof; for

for if the said Quotient $\sqrt{8} + \sqrt{6} - 2 - \sqrt{3}$ be multiplied by the given Divisor $\sqrt{4} - \sqrt{2}$, it will produce the given Dividend $\sqrt{8} + \sqrt{6}$.

Likewise, to divide $ab + b\sqrt{bc}$ by $a + \sqrt{bc}$, I multiply each by $a - \sqrt{bc}$, (the Residual correspondent to the Divisor,) and it produceth $aa - bc$ for a new Divisor; and $ab - b\sqrt{bc}$ for a new Dividend, this divided by that gives b for the Quotient sought; for b multiplied into the given Divisor $a + \sqrt{bc}$ makes the given Dividend $ab + b\sqrt{bc}$. Another way of finding out the Quotient in this last Example, is shewn in the first of the last Examples at the latter end of this *Self*. 11.

Again, to divide 10 by $\sqrt{(4)5} - \sqrt{(4)3}$, I multiply each by $\sqrt{(4)5} + \sqrt{(4)3}$; and there comes forth a new Dividend $\sqrt{(4)50000} - \sqrt{(4)30000}$, and a new Divisor $\sqrt{5} - \sqrt{3}$; but this Divisor not being a Rational number, I multiply again both the said new Dividend and Divisor by $\sqrt{5} + \sqrt{3}$, and it produceth another new Dividend $\sqrt{(4)1250000} - \sqrt{(4)750000} + \sqrt{(4)450000} - \sqrt{(4)270000}$, and another new Divisor 2; by this I divide the last Dividend and there ariseth $\sqrt{(4)78125} - \sqrt{(4)46875} + \sqrt{(4)28125} - \sqrt{(4)16875}$ the Quotient sought; for if it be multiplied by the proposed Divisor $\sqrt{(4)5} - \sqrt{(4)3}$ it will produce the given Dividend 10.

Again, to divide $\sqrt{8}$ by $\sqrt{3} + \sqrt{2} + 1$, I first multiply the Divisor by $\sqrt{3} + \sqrt{2} - 1$ and it makes $\sqrt{24} - 4$, this multiplied by its correspondent Residual $\sqrt{24} - 4$ gives the Product 8 for a new Divisor: Now because the given Divisor was first multiplied by $\sqrt{3} + \sqrt{2} - 1$ and the Product by $\sqrt{24} - 4$, the given Dividend must likewise be multiplied first by $\sqrt{3} + \sqrt{2} - 1$, and the Product $\sqrt{24} - 4 - \sqrt{8}$ by $\sqrt{24} - 4$, and there will be produced $8 + \sqrt{128} - \sqrt{192}$ for a new Dividend; so instead of the given Dividend and Divisor we have other numbers in the same Proportion, viz. $8 + \sqrt{128} - \sqrt{192}$ and 8. Therefore (by *Prop. 5*.) the former divided by the latter will give the Quotient sought, to wit, $1 + \sqrt{2} - \sqrt{3}$; but that this is the true Quotient will appear by Multiplication, for if $1 + \sqrt{2} - \sqrt{3}$ be multiplied by the proposed Divisor $\sqrt{3} + \sqrt{2} + 1$, it will produce the given Dividend $\sqrt{8}$.

Note. Although the new Divisor and Dividend found out as aforesaid, may sometimes happen to be Negative quantities, (that is, such whose values are less than nothing,) yet Division being made by them with respect to the Rules of $+$ and $-$, they will give the true Quotient sought. As, for example, suppose 30 to be divided by $2 + \sqrt{9}$, (that is, 30 by 5;) first, the Divisor $2 + \sqrt{9}$ being multiplied by $2 - \sqrt{9}$ gives 4-9, that is, -5 for a new Divisor, and the Dividend 30 multiplied by the said $2 - \sqrt{9}$ gives 60 - $\sqrt{810}$ for a new Dividend, which divided by -5 gives $12 + \sqrt{162}$, which is the same with the Quotient that ariseth by dividing 30 by $2 + \sqrt{9}$, that is, by 5.

Again, let $4 + \sqrt{25}$ be to be divided by $1 - \sqrt{9}$, (that is, 9 by 4, where the Quotient is manifestly $2\frac{1}{2}$;) first, the Divisor $1 - \sqrt{9}$ multiplied by $1 + \sqrt{9}$ produceth $1 - 9$, that is, -8 for a new Divisor; and the Dividend $4 + \sqrt{25}$ multiplied by the said $1 + \sqrt{9}$ makes $4 + \sqrt{25} - 4\sqrt{9} - \sqrt{225}$ for a new Dividend, which divided by -8, (according to the Examples at the beginning of this *Self*. 11.) gives $-\frac{1}{2} - \sqrt{\frac{25}{8}} + \frac{1}{2}\sqrt{9} + \sqrt{\frac{225}{8}}$ the Quotient sought, which after due contraction makes $2\frac{1}{2}$. For $\frac{1}{2}\sqrt{9}$, that is, $\frac{3}{2}$, is equal to $\frac{1}{2}$, and $-\sqrt{\frac{25}{8}}$ is $-\frac{5}{2}$, which added to the said $\frac{1}{2}$ makes $-\frac{4}{2}$; also $-\sqrt{\frac{225}{8}}$ is $-\frac{15}{2}$, which added to $-\frac{4}{2}$, (or $-\frac{19}{2}$;) makes $-\frac{19}{2}$, this added to $\frac{1}{2}$ gives $-\frac{18}{2}$ (or $2\frac{1}{2}$;) the Quotient before found.

7. When the Divisor is a Binomial, or a Residual consisting of two simple Cubic or Biquadratic, &c. Roots, it may be reduced to a Rational Divisor by this following Proposition, viz.

If in the Proportion of the Names (or Parts) of a Binomial or Residual, there be found for many continual Proportionals in multitude as there be Units in the Index of the Radical sign, and that the Radical signs of the Parts of the Binomial or Residual, and also of the Proportionals be the same, but connected in the Binomial by $+$, and in the Proportionals by $-$ and $-$ alternately; or contrarily, in the Proportionals by $+$, and in the Residual by $+$ and $-$, the Product made by the multiplication of the Proportionals by the Binomial or Residual shall be Rational.

As, for example, if there be proposed the Binomial $\sqrt{(3)7} + \sqrt{(3)5}$, find three continual Proportionals, that the first may be to the second, and the second to the third, as $\sqrt{(3)7}$ to $\sqrt{(3)5}$, which may be done by the help of *Self*. 8. *Chap. 5*. of this Book; where it hath been shewn, that aa , ae and ee are continual Proportionals in the Reason of a to e . Therefore if we suppose $\sqrt{(3)7}$ to be a , and $\sqrt{(3)5}$ to be e , then the Square

of $\sqrt{(3)7}$,

of $\sqrt{(3)7}$, to wit, $\sqrt{(3)49}$, shall be the first Proportional (aa); the Product of $\sqrt{(3)7}$ into $\sqrt{(3)5}$, to wit, $\sqrt{(3)35}$, shall be the second Proportional (ae); and the Square of $\sqrt{(3)5}$, to wit, $\sqrt{(3)25}$, shall be the third Proportional (ee); so that these three Cubick Roots, to wit, $\sqrt{(3)49}$, $\sqrt{(3)35}$ and $\sqrt{(3)25}$ are continual Proportionals in the Reason of $\sqrt{(3)7}$ and $\sqrt{(3)5}$. Now I say (according to the Proposition,) If $\sqrt{(3)49} - \sqrt{(3)35} + \sqrt{(3)25}$ be multiplied by $\sqrt{(3)7} + \sqrt{(3)5}$, the Product shall be Rational; also if $\sqrt{(3)49} + \sqrt{(3)35} - \sqrt{(3)25}$ be multiplied by $\sqrt{(3)7} - \sqrt{(3)5}$ the Product shall be Rational, as will appear by the following Operation.

$$\begin{array}{r} \text{Multiplicand,} \quad \sqrt{(3)49} - \sqrt{(3)35} + \sqrt{(3)25} \\ \text{Multiplier,} \quad \sqrt{(3)7} + \sqrt{(3)5} \\ \hline 7 - \sqrt{(3)245} + \sqrt{(3)175} \\ + \sqrt{(3)245} - \sqrt{(3)175} + 5 \\ \hline \text{The Product} \quad 12 \text{ is Rational.} \end{array}$$

$$\begin{array}{r} \text{Multiplicand,} \quad \sqrt{(3)49} + \sqrt{(3)35} - \sqrt{(3)25} \\ \text{Multiplier,} \quad \sqrt{(3)7} - \sqrt{(3)5} \\ \hline 7 + \sqrt{(3)245} - \sqrt{(3)175} \\ - \sqrt{(3)245} + \sqrt{(3)175} - 5 \\ \hline \text{The Product} \quad 2 \text{ is Rational.} \end{array}$$

But for the greater evidence of the certainty of this Proposition in a Binomial and Residual consisting of any two simple Cubick Roots whatever, let there be proposed this Binomial $\sqrt{(3)b} + \sqrt{(3)d}$, and suppose b greater than d ; then three continual Proportionals in the proportion of $\sqrt{(3)b}$ to $\sqrt{(3)d}$ will be found $\sqrt{(3)bb}$, $\sqrt{(3)bd}$ and $\sqrt{(3)dd}$; then multiply as before, viz.

$$\begin{array}{r} \text{Multiplicand,} \quad \sqrt{(3)bb} + \sqrt{(3)bd} + \sqrt{(3)dd} \\ \text{Multiplier,} \quad \sqrt{(3)b} + \sqrt{(3)d} \\ \hline b + \sqrt{(3)b^2d} + \sqrt{(3)b^2d} \\ + \sqrt{(3)b^2d} + \sqrt{(3)b^2d} + d \\ \hline \text{The Product} \quad b + d \text{ is Rational.} \end{array}$$

Again,

$$\begin{array}{r} \text{Multiplicand,} \quad \sqrt{(3)bb} - \sqrt{(3)bd} + \sqrt{(3)dd} \\ \text{Multiplier,} \quad \sqrt{(3)b} - \sqrt{(3)d} \\ \hline b - \sqrt{(3)b^2d} + \sqrt{(3)b^2d} \\ - \sqrt{(3)b^2d} + \sqrt{(3)b^2d} - d \\ \hline \text{The Product} \quad b - d \text{ is Rational.} \end{array}$$

Whence you may observe, that the first Rational Product is the sum of the Names (or Parts,) omitting the Radical signs, of the Cubick Binomial proposed, and the latter Rational Product is the difference of the Parts, omitting the Radical signs, of the Cubick Residual proposed: so that the Rational Product made by the multiplication of the said Proportionals and Binomial or Residual may be discovered without any multiplication.

8. Now, that the Use of the last preceding Proposition may appear, let it be required to divide 10 by $\sqrt{(3)7} - \sqrt{(3)5}$; First, because the Index of the Radical sign is 3, I seek three continual Proportionals in the proportion of $\sqrt{(3)7}$ to $\sqrt{(3)5}$, which Proportionals (as before hath been shewn) are $\sqrt{(3)49}$, $\sqrt{(3)35}$ and $\sqrt{(3)25}$; these I connect by $+$, because the Parts of the given Divisor are connected by $-$, and there ariseth $\sqrt{(3)49} + \sqrt{(3)35} + \sqrt{(3)25}$; then by this common Multiplier I multiply as well the Dividend 10, as the Divisor $\sqrt{(3)7} - \sqrt{(3)5}$; and it produceth $\sqrt{(3)490000} - \sqrt{(3)350000} + \sqrt{(3)250000}$ for a new Dividend; and 2 for a new Divisor: lastly, by dividing the said new Dividend by the new Divisor, there ariseth $\sqrt{(3)6125} + \sqrt{(3)4375} - \sqrt{(3)3125}$ the Quotient sought; for if it be multiplied by the given Divisor $\sqrt{(3)7} - \sqrt{(3)5}$, it will produce the given Dividend 10.

In like manner, to divide 10 by this Binomial $\sqrt[4]{3}5 + \sqrt[4]{3}3$, first I seek three continual Proportionals in the Reason of $\sqrt[4]{3}5$ to $\sqrt[4]{3}3$, which Proportionals will be found $\sqrt[4]{3}25$, $\sqrt[4]{3}15$ and $\sqrt[4]{3}9$; these I connect by $+$ and $-$ alternately, because the Parts of the given Divisor are connected by $+$, viz. to the first Proportional I prefix $+$, to the second $-$, and to the third $+$, so they make $\sqrt[4]{3}25 - \sqrt[4]{3}15 + \sqrt[4]{3}9$, by this, as a common Multiplier, I multiply as well the Dividend 10 as the Divisor $\sqrt[4]{3}5 + \sqrt[4]{3}3$, and there ariseth a new Dividend $\sqrt[4]{3}25000 - \sqrt[4]{3}15000 + \sqrt[4]{3}9000$, and a new Divisor 8, by which I divide the said new Dividend, and there comes forth $\sqrt[4]{3}2125 - \sqrt[4]{3}1125 + \sqrt[4]{3}225$, the Quotient sought.

The same method is to be observed when the Divisor is a Binomial or a Residual consisting of two simple Biquadratic Roots.

As, for example, to divide 10 by $\sqrt[4]{4}5 + \sqrt[4]{4}3$, (which hath already been done after another manner in the third Example of the Rule in the sixth step of this Section;) First, because the Index of the Radical sign is 4, I search out four continual Proportionals in the Reason of $\sqrt[4]{4}5$ to $\sqrt[4]{4}3$ in this manner, viz. For as much as (by Sect. 8. Chap. 5. of this Book,) these are continual Proportionals, to wit, aaa , aa , a and 1 , I suppose $\sqrt[4]{4}5$ to be a , and $\sqrt[4]{4}3$ to be e ; then I multiply $\sqrt[4]{4}5$ into it self cubically, and it gives the first Proportional $\sqrt[4]{4}125$, (to wit, aaa ,) also I multiply the Square of $\sqrt[4]{4}5$ into $\sqrt[4]{4}3$, and it gives the second Proportional $\sqrt[4]{4}75$, (to wit, aa ,) again, I multiply $\sqrt[4]{4}5$ into the Square of $\sqrt[4]{4}3$, and it gives the third Proportional $\sqrt[4]{4}45$, (to wit, a ,) lastly, I multiply $\sqrt[4]{4}3$ into it self cubically, and it gives the fourth Proportional $\sqrt[4]{4}27$, (to wit, eee ,) Then because the two Parts of the given Divisor are connected by $+$, I connect those four Proportionals by $+$ and $-$ alternately; so there ariseth this Compound number $\sqrt[4]{4}125 - \sqrt[4]{4}75 + \sqrt[4]{4}45 - \sqrt[4]{4}27$, by which, as a common Multiplier, I multiply as well the given Dividend 10, as the given Divisor $\sqrt[4]{4}5 + \sqrt[4]{4}3$, and there ariseth a new Dividend 10, as the $\sqrt[4]{4}750000 - \sqrt[4]{4}450000 + \sqrt[4]{4}270000$, and a new Divisor 2, which are the same in every respect with those found in the place before cited.

After the same manner, when the Divisor is a Binomial or a Residual having 5 or 6, &c. for the Index of the common Radical sign of the Roots; it may be reduced to a new Divisor that shall be Rational. But it must be remembered, that when the Roots are of different kinds they must first be reduced to a common Radical sign.

But when the Divisor cannot be reduced to a simple Rational number by any of the foregoing Rules, (which are all that I have met with in Algebraical Authors,) the Dividend may be set as a Numerator over the Divisor as a Denominator, and the Fraction so constituted shall be equal to the Quotient. As, for example, if $\sqrt[4]{48} + \sqrt[4]{3}$ be to be divided by $\sqrt[4]{15} + \sqrt[4]{3}6 - \sqrt[4]{3}$, the Quotient may be represented by this Fraction; to wit,

$$\frac{\sqrt[4]{48} + \sqrt[4]{3}}{\sqrt[4]{15} + \sqrt[4]{3}6 - \sqrt[4]{3}}$$

Examples of Division in Compound Surd quantities express'd by Letters.

Division in Compound Surd quantities express'd by Letters depends upon the Rules of Simple Surds before delivered; as also upon the General method of Division in Sect. 9. Chap. 5. Book 1. as will appear by the following Examples, some of which I shall afterwards explain.

$$\begin{array}{r} \text{Divisor.} \quad \text{Dividend.} \\ a + \sqrt{bc} \quad) \quad ab + b\sqrt{bc} \quad (\quad b, \quad \text{Quotient.} \\ \underline{ab + b\sqrt{bc}} \\ 0 \end{array}$$

$$\begin{array}{r} a + \sqrt{bc} \quad) \quad aa - bc \quad (\quad a - \sqrt{bc} \\ \underline{aa + a\sqrt{bc}} \\ -bc - a\sqrt{bc} \\ \underline{-bc - a\sqrt{bc}} \\ 0 \end{array}$$

\sqrt{ab}

$$\begin{array}{r} \sqrt{ab} - \sqrt{cd} \quad) \quad ab - cd \quad (\quad \sqrt{ab} + \sqrt{cd} \\ \underline{ab - \sqrt{abcd}} \\ -cd + \sqrt{abcd} \\ \underline{-cd + \sqrt{abcd}} \\ 0 \end{array}$$

$$\begin{array}{r} a + \sqrt{bc} \quad) \quad aaa + bc\sqrt{bc} \quad (\quad aa + bc - a\sqrt{bc} \\ \underline{aaa + aa\sqrt{bc}} \\ bc\sqrt{bc} - aa\sqrt{bc} \\ \underline{bc\sqrt{bc} + abc} \\ -aa\sqrt{bc} - abc \\ \underline{-aa\sqrt{bc} - abc} \\ 0 \end{array}$$

$$\begin{array}{r} aa + a\sqrt{bc} \quad) \quad aaab - abbc \quad (\quad ab - b\sqrt{bc} \\ \underline{aaab + aab\sqrt{bc}} \\ -abbc - aab\sqrt{bc} \\ \underline{-abbc - aab\sqrt{bc}} \\ 0 \end{array}$$

$$\begin{array}{r} a - \sqrt{bc} \quad) \quad aab - bbc - ab\sqrt{bc} + \frac{bbc}{a}\sqrt{bc} \quad (\quad ab - \frac{bbc}{a} \\ \underline{aab - ab\sqrt{bc}} \\ -bbc + \frac{bbc}{a}\sqrt{bc} \\ \underline{-bbc + \frac{bbc}{a}\sqrt{bc}} \\ 0 \end{array}$$

EXPLICATION.

In the first Example; first, ab divided by a gives the Quotient b , by which I multiply the whole Divisor $a + \sqrt{bc}$, and it makes $ab + b\sqrt{bc}$, this subtracted from the given Dividend $ab + b\sqrt{bc}$, there remains 0; so the Quotient sought is b .

In the third Example; first, ab divided by \sqrt{ab} gives the Quotient \sqrt{ab} , by which I multiply the whole Divisor $\sqrt{ab} - \sqrt{cd}$ and the Product is $ab - \sqrt{abcd}$, this subtracted from the given Dividend $ab - cd$, there remains to be yet divided, $-cd + \sqrt{abcd}$, then I divide $-cd$ by $-\sqrt{cd}$ and it gives the Quotient $+\sqrt{cd}$, by which I multiply the whole Divisor $\sqrt{ab} - \sqrt{cd}$ and it makes $-cd + \sqrt{abcd}$, this subtracted from the remaining Dividend $-cd + \sqrt{abcd}$ leaves 0; so the Division is finish'd, and the Quotient sought is $\sqrt{ab} + \sqrt{cd}$.

In the sixth and last Example; first, aab divided by a gives the Quotient ab , this multiplying the whole Divisor $a - \sqrt{bc}$ produceth $aab - ab\sqrt{bc}$, which subtracted from the given Dividend leaves to be yet divided $-bbc + \frac{bbc}{a}\sqrt{bc}$, then I divide $+\frac{bbc}{a}\sqrt{bc}$

by $-\sqrt{bc}$ and it gives the Quotient $-\frac{bbc}{a}$, by which I multiply the whole Divisor $a - \sqrt{bc}$ and it produceth $-bbc + \frac{bbc}{a}\sqrt{bc}$, which subtracted from the remaining Dividend $-bbc + \frac{bbc}{a}\sqrt{bc}$ leaves nothing; so the Quotient sought is $ab - \frac{bbc}{a}$.

G g

The

The Arithmetick of Universal Surd Roots, both in Numbers and Quantities express'd by Letters.

SECT. XII. *Multiplication in Universal Surds.*

Universal Roots are the Roots of Compound Numbers or Quantities; how to express Universal Roots, and to find out their values, hath already been shewn in *SECT. 28. Chap. 1. Book I.* I shall therefore proceed to their Multiplication.

1. If the Square Root of any Compound number be to be squared, or multiplied into it self, cast away the universal Radical sign $\sqrt{\quad}$ or $\sqrt[4]{\quad}$, as also the Line that is drawn over the Compound number, and the Compound number it self shall be the Square of the Universal Root proposed. Also, the Cube of the cubick Root of any Compound number is the Compound number it self, the Line drawn over it and the universal Radical sign $\sqrt[3]{\quad}$ being cast away, and so of others.

As, for example, the Square of this universal square Root, $\sqrt{12} + \sqrt{3}$ is $12 + \sqrt{3}$; likewise, the Square of $\sqrt{12} - \sqrt{3}$ is $12 - \sqrt{3}$; also, the Square of $\sqrt{15} + \sqrt{3} + \sqrt{2}$ is $15 + \sqrt{3} + \sqrt{2}$; and the Square of $\sqrt{15} - \sqrt{3} - \sqrt{2}$ is $15 - \sqrt{3} - \sqrt{2}$.

After the same manner, the Cube of this universal cubick Root, $\sqrt[3]{3} + \sqrt[3]{2} + \sqrt[3]{1}$ is $\sqrt[3]{3} + \sqrt[3]{2} + \sqrt[3]{1}$, that is, 8.

Likewise, the Square of $\sqrt{aa + bb}$ is $aa + bb$; and the Cube of $\sqrt[3]{bbb + ccc}$ is $bbb + ccc$; also, the Square of $\sqrt{\frac{1}{2}c + \frac{1}{4}cc - n}$ is $\frac{1}{2}c + \frac{1}{4}cc - n$; and so of others.

2. When an universal Root is to be multiplied by a rational Quantity, or by a simple or compound Surd, or by an universal Root, multiply the Square of the Multiplicand by the Square of the Multiplier, when the universal Radical sign is Quadratick; or, the Cube of the one by the Cube of the other, when the universal Radical sign is Cubick, &c. then before that Product prefix the given universal Radical sign; so shall this new universal Root be the Product sought.

As, for example, if it be desired to double or multiply by 2, this universal square Root, $\sqrt{10} + \sqrt{40}$: I take the Square of 2, which is 4, and the Square of $\sqrt{10} + \sqrt{40}$ which (by the foregoing first Rule of this *SECT.*) is $10 + \sqrt{40}$, then I multiply $10 + \sqrt{40}$ by 4, and it makes $40 + 4\sqrt{40}$, or, $40 + \sqrt{640}$; whose universal square Root, to wit, $\sqrt{40} + \sqrt{40}$, or, $\sqrt{40} + \sqrt{640}$ is the Product of $\sqrt{10} + \sqrt{40}$: multiplied by 2; or the said Product may be express'd thus, $2\sqrt{10} + \sqrt{40}$.

Likewise, if $\sqrt[3]{(3)64} + \sqrt[3]{(3)27}$ be to be doubled, or multiplied by 2, I first multiply each of those numbers cubically, because the Radical sign of the given universal Root is $\sqrt[3]{\quad}$, and their Cubes will be $\sqrt[3]{(3)64} + \sqrt[3]{(3)27}$ and 8; which multiplied one into the other make $8\sqrt[3]{(3)64} + 8\sqrt[3]{(3)27}$, to which Product I prefix the universal Radical sign $\sqrt[3]{\quad}$, and it gives $\sqrt[3]{(3)8\sqrt[3]{(3)64} + 8\sqrt[3]{(3)27}}$; that is, $\sqrt[3]{(3)32} + 24$; or $\sqrt[3]{(3)56}$, which is the Product sought, to wit, the double of $\sqrt[3]{(3)64} + \sqrt[3]{(3)27}$.

After the same manner, if $\sqrt[3]{(3)64} + \sqrt[3]{(2)36} - 3$ be to be multiplied by 5, or $\sqrt[3]{(3)125}$, the Product will be $\sqrt[3]{(3)125\sqrt[3]{(3)64} + 125\sqrt[3]{(2)36} + 375}$; that is, $\sqrt[3]{(3)1625}$.

Again, to multiply $\sqrt{\frac{1}{2} + \sqrt{3}}$ by $\sqrt{5}$, their Squares are $\frac{1}{2} + \sqrt{3}$ and 5, which multiplied one into another make $\frac{5}{2} + \sqrt{3}$; (that is, $\sqrt{20} + \sqrt{75}$) whose universal square Root, to wit, $\sqrt{\frac{5}{2} + \sqrt{3}}$ (or, $\sqrt{250} + \sqrt{75}$) is the Product of $\sqrt{\frac{1}{2} + \sqrt{3}}$: multiplied by $\sqrt{5}$.

Likewise, to multiply $\sqrt{13} + \sqrt{9}$ by $\sqrt{5} + \sqrt{10}$ (that is, 4 by 3; where the Product is manifestly 12); the Squares of the universal Roots proposed are $13 + \sqrt{9}$ and $5 + \sqrt{16}$, which multiplied one into another make $65 + 5\sqrt{9} + 13\sqrt{16} + \sqrt{144}$; whose universal square Root, to wit, $\sqrt{65 + 5\sqrt{9} + 13\sqrt{16} + \sqrt{144}}$: that is, $\sqrt{144}$, or 12 is the Product sought.

Again, to multiply $\sqrt{\frac{1}{2} + \sqrt{22}}$ into $\sqrt{\frac{1}{2} + \sqrt{22}}$: I multiply their Squares $\frac{1}{2} + \sqrt{22}$ and $\frac{1}{2} + \sqrt{22}$ one into another according to the last of the three compendious Rules in *SECT. 10.* of this *Chapt.* and there comes forth $\frac{22}{2} - \frac{22}{2}$, that is, 5; (to wit, the difference

difference between the Squares of $\frac{1}{2}$ and $\sqrt{22}$;) lastly; the square Root of the said 5 is $\sqrt{5}$ for the Product sought.

So also, to multiply $\sqrt{5} + \sqrt{2}$ by $\sqrt{5} + \sqrt{2}$; their Squares $5 + \sqrt{2}$ and $7 + 2\sqrt{10}$ multiplied one into another give $35 + 10\sqrt{10} + 7\sqrt{2} + 2\sqrt{20}$; whose universal square Root, to wit, $\sqrt{35 + 10\sqrt{10} + 7\sqrt{2} + 2\sqrt{20}}$ is the Product sought.

Moreover, to multiply $\sqrt{144} + 4$: by $\sqrt{4} + 2$: by $\sqrt{100} - 1$ (that is, 2 by 3, which will produce 6); I first multiply the Square of $\sqrt{144} + 4$ by the Square of $\sqrt{100} - 1$: viz. $\sqrt{144} - 4$ by $\sqrt{100} - 1$ and it makes $\sqrt{14400} + 4\sqrt{100} - \sqrt{144} - 4$, before which I prefix the universal Radical sign $\sqrt{\quad}$, and it gives $\sqrt{14400 + 4\sqrt{100} - \sqrt{144} - 4}$: which is one of the members of the Product sought; then I multiply in like manner $\sqrt{4} + 2$ by $\sqrt{100} - 1$: and it makes $\sqrt{400} + 2\sqrt{100} - \sqrt{4} - 2$: for the latter member of the Product sought; lastly, both those members being joyned together give $\sqrt{14400 + 4\sqrt{100} - \sqrt{144} - 4} - \sqrt{400} + 2\sqrt{100} - \sqrt{4} - 2$: that is, $\sqrt{144} - \sqrt{36}$; that is, $12 - 6$, or 6; for the Product required.

3. Sometimes the fourth, fifth and sixth Rules in *SECT. 4.* of this *Chapt.* will be useful in the Multiplication of universal Surds: As, if it be desired to multiply $3\sqrt{2} + \sqrt{5}$ by $4\sqrt{2} + \sqrt{5}$: (which are commensurable Roots, for they are in proportion one to the other as 3 to 4.) I multiply 3 by 4, and the Product 12 into $2 + \sqrt{5}$; so there is produced $24 + 12\sqrt{5}$ (that is, $24 + \sqrt{720}$) for the Product sought.

Likewise, $5\sqrt{6} + \sqrt{9}$: multiplied by $2\sqrt{6} + \sqrt{9}$: (that is, 15 by 6,) produceth $60 + 10\sqrt{9}$, (that is, 90.)

Moreover, if $5\sqrt{6} + \sqrt{9}$ be to be multiplied by $3\sqrt{19} - \sqrt{9}$: (that is, 15 by 12,) I first multiply 5 by 3 and it makes 15, then I multiply $\sqrt{6} + \sqrt{9}$ by $\sqrt{19} - \sqrt{9}$: and it produceth $\sqrt{105} - \sqrt{13} - \sqrt{9}$: which latter Product multiplied into the former Product 15 makes $15\sqrt{105} - \sqrt{13} - \sqrt{9}$: (that is, 180,) the Product sought.

4. Sometimes also the three Rules before delivered in *SECT. 10.* of this *Chapt.* concerning the multiplying of Binomials and Residuals will be useful in the Multiplication of Universal surd Roots: As, if this Binomial Root $\sqrt{12} + \sqrt{6}$ be to be squared, or multiplied into it self, the Squares of the Parts are $12 + \sqrt{6}$ and $12 - \sqrt{6}$, whose sum is 24: then the Product made by the multiplication of the Parts one into the other, viz. $\sqrt{12} + \sqrt{6}$ into $\sqrt{12} - \sqrt{6}$ is $\sqrt{138}$, (for the difference of the Squares of 12 and 6 is 138, whose square Root is $\sqrt{138}$;) and the double of the said Product is $2\sqrt{138}$, which added to 24 (the sum of the Squares of the Parts) makes $24 + 2\sqrt{138}$, which is the Square of $\sqrt{12} + \sqrt{6}$: to wit, $\sqrt{24 + 2\sqrt{138}}$ is the sum of the two Parts $\sqrt{12} + \sqrt{6}$: and $\sqrt{12} - \sqrt{6}$: For when the sum of two numbers is multiplied into it self, the square Root of the Product is equal to the same sum.

Likewise if $\sqrt{10} + \sqrt{36}$: $\sqrt{10} - \sqrt{36}$, that is, 2, be to be squared, or multiplied into it self, the Product will be found $20 - 2\sqrt{64}$, that is, 4; and the square Root of this 4, to wit, 2 is the difference of the two Roots or Parts $\sqrt{10} + \sqrt{36}$: and $\sqrt{10} - \sqrt{36}$. For when the difference of two numbers is multiplied into it self, the square Root of the Product is equal to the said difference.

Again, if $6 + \sqrt{20} - \sqrt{16}$ be to be multiplied into $6 - \sqrt{20} - \sqrt{16}$: the Product will be found 20. For (according to Rule 3. in *SECT. 10.* of this *Chapter*;) if $20 - \sqrt{16}$, which is the Square of $\sqrt{20} - \sqrt{16}$ be subtracted from 36 the Square of 6, there will remain $16 + \sqrt{16}$, that is, 20, the Product sought.

Likewise, if $\sqrt{20} + \sqrt{20} - \sqrt{5}$ be to be multiplied into $\sqrt{20} - \sqrt{20} - \sqrt{5}$: the Product will be $\sqrt{5}$.

So also, if $\sqrt{5} + \sqrt{20} - \sqrt{16}$ be to be multiplied by $\sqrt{5} - \sqrt{20} - \sqrt{16}$: (that is, 3 by 11) first, the Squares of the universal Roots proposed are $5 + \sqrt{20} - \sqrt{16}$: and $5 - \sqrt{20} - \sqrt{16}$: these multiplied one by the other, by taking the difference of the Squares of 5 and $\sqrt{20} - \sqrt{16}$: give the Product $5 + \sqrt{16}$; whose universal square Root is $\sqrt{5}$.

$\sqrt{11} + \sqrt{25}$: give the Quotients $\sqrt{9}$ and $\sqrt{4}$, to wit, 3 and 2, which are Rational numbers expressing the Proportion which the given Roots have one to another. Therefore, $3 + 2$, to wit, 5, multiplied into the common Divisor $\sqrt{11} + \sqrt{25}$ gives $5\sqrt{11} + \sqrt{25}$; that is, $\sqrt{275} + \sqrt{15625}$: (to wit, 20,) which is the Summ of the Roots proposed; and $3 - 2$, that is, 1, multiplied into the said $\sqrt{11} + \sqrt{25}$ gives $\sqrt{11} + \sqrt{25}$; (that is, 4,) for the Difference of the given Roots.

Here follow Contractions of the work of Addition and Subtraction in the two last Examples, with others of like nature in Surd quantities express'd by Letters.

Example 1.

What is the Summ and Difference of $\sqrt{8} + \sqrt{43}$: and $\sqrt{2} + \sqrt{3}$?

The Operation.

- I. $\sqrt{2} + \sqrt{3}$:) $\sqrt{8} + \sqrt{43}$: ($\sqrt{4}$, that is, 2.
 II. $\sqrt{2} + \sqrt{3}$:) $\sqrt{2} + \sqrt{3}$: ($\sqrt{1}$, that is, 1.

Therefore from I. $2\sqrt{2} + \sqrt{3}$: = $\sqrt{8} + \sqrt{43}$:

And from II. $1\sqrt{2} + \sqrt{3}$: = $\sqrt{2} + \sqrt{3}$:

The Summ, $3\sqrt{2} + \sqrt{3}$: = $\sqrt{8} + \sqrt{43}$: + $\sqrt{2} + \sqrt{3}$:

The Difference, $1\sqrt{2} + \sqrt{3}$: = $\sqrt{8} + \sqrt{43}$: - $\sqrt{2} + \sqrt{3}$:

Example 2.

What is the Summ and Difference of $\sqrt{99} + \sqrt{25}$: and $\sqrt{44} + \sqrt{25}$?

The Operation.

- I. $\sqrt{11} + \sqrt{25}$:) $\sqrt{99} + \sqrt{25}$: ($\sqrt{9}$, that is, 3.
 II. $\sqrt{11} + \sqrt{25}$:) $\sqrt{44} + \sqrt{25}$: ($\sqrt{4}$, that is, 2.

Therefore from I. $3\sqrt{11} + \sqrt{25}$: = $\sqrt{99} + \sqrt{25}$:

And from II. $2\sqrt{11} + \sqrt{25}$: = $\sqrt{44} + \sqrt{25}$:

The Summ, $5\sqrt{11} + \sqrt{25}$: = $\sqrt{99} + \sqrt{25}$: + $\sqrt{44} + \sqrt{25}$:

The Difference, $1\sqrt{11} + \sqrt{25}$: = $\sqrt{99} + \sqrt{25}$: - $\sqrt{44} + \sqrt{25}$:

Example 3.

What is the Summ and Difference of $\sqrt{aaaa} + \sqrt{abb}$: and $\sqrt{abb} + \sqrt{bbbb}$:?
 Those reduced (by Sect. 6. of this Chap.) give $a\sqrt{a} + \sqrt{bb}$: and $b\sqrt{a} + \sqrt{bb}$:

Therefore their Summ is $a + b$ into $\sqrt{a} + \sqrt{bb}$:

And their Difference is $a - b$ into $\sqrt{a} + \sqrt{bb}$:

Example 4.

What is the Summ and Difference of $\sqrt{ooz + 4mpz}$: and $\sqrt{aaom + 4aamm}$:?
 By dividing each of them by their common Divisor $\sqrt{oo + 4mp}$: there will arise Rational quotients,

to wit,
 Therefore the Surds proposed are Commensurable, and instead of them we may write

Therefore their Summ shall be

That is,

$$\frac{z}{a} \quad \text{and} \quad \frac{am}{pz}$$

$$\frac{z}{a} \sqrt{oo + 4mp} \quad \text{and} \quad \frac{am}{pz} \sqrt{oo + 4mp}$$

$$\frac{z}{a} + \frac{am}{pz} \quad \text{into} \quad \sqrt{oo + 4mp}$$

$$\frac{pz + aam}{apz} \quad \text{into} \quad \sqrt{oo + 4mp}$$

And

And the Difference of the given Surds shall be $\frac{pzz + aam}{apz}$ into $\sqrt{oo + 4mp}$:

Example 5.

What is the Summ and Difference of these two Universal Roots?

$$\sqrt{aaaa} + \sqrt{6aaa} + \sqrt{21aa} + \sqrt{72a} + \sqrt{108} \quad \text{and} \quad \sqrt{aaaa} + \sqrt{10aaa} + \sqrt{37aa} + \sqrt{120a} + \sqrt{300}$$

The Operation.

The given Roots are Commensurable, (as hath been shewn in the last Example but one in Sect. 7. of this Chap.) and may be exprest thus,

$$a + 3\sqrt{aa} + 12: \quad \text{and} \quad a + 5\sqrt{aa} + 12:$$

Therefore their Summ, supposing a to be greater than 5, shall be

$$2a - 2 \quad \text{into} \quad \sqrt{aa + 12}:$$

And their Difference shall be, $8\sqrt{aa + 12}$:

But if we suppose a to be less than 5, then the Summ of the given Surds will be $8\sqrt{aa + 12}$: and their Difference $2a + 2\sqrt{aa + 12}$: that is, $2a + 2$ into $\sqrt{aa + 12}$:

2. When the Root of a Residual is to be added unto, or subtracted from the Root of its correspondent Binomial, those Roots may be connected together by $+$ or $-$; and then the whole being multiplied into it self, the universal Root of the Product shall be the Summ or Difference of the Roots given to be added or subtracted, as before hath been shewn in Rule 4. Sect. 12. of this Chap.

As, if these two Roots be proposed to be added, to wit, $\sqrt{12} + \sqrt{6}$: and $\sqrt{12} - \sqrt{6}$: we may multiply this composed number $\sqrt{12} + \sqrt{6}$: by $\sqrt{12} - \sqrt{6}$: into it self, and there will be produced $24 - 2\sqrt{138}$, whose universal square Root, to wit, $\sqrt{24 - 2\sqrt{138}}$: shall be the sum of the two Roots proposed to be added.

Likewise, if $\sqrt{12} + \sqrt{6}$: - $\sqrt{12} - \sqrt{6}$: be multiplied into it self, the Product will be $24 - 2\sqrt{138}$, whose universal square Root, to wit, $\sqrt{24 - 2\sqrt{138}}$: is the difference of the two Roots proposed.

After the same manner, the sum of these two Roots, $\sqrt{10} + \sqrt{36}$: and $\sqrt{10} - \sqrt{36}$: will be found $\sqrt{120 - 2\sqrt{64}}$: (that is, $\sqrt{36}$, to wit, 6,) but their difference $\sqrt{120 - 2\sqrt{64}}$: (that is, $\sqrt{4}$, to wit, 2.)

Likewise the sum of these Binomial Roots, $\sqrt{a} + \sqrt{bc}$: and $\sqrt{a} - \sqrt{bc}$: will be found $\sqrt{2a + 2\sqrt{a - bc}}$: and their difference $\sqrt{2a - 2\sqrt{a - bc}}$:

3. But if the universal Roots proposed be not commensurable, nor such Binomials and Residuals as are mentioned in the last preceding Rule; then they are to be added by $+$, and subtracted by $-$.

As, if $\sqrt{5} + \sqrt{2}$: and $\sqrt{5} - \sqrt{2}$: be to be added; I write $\sqrt{5 + \sqrt{2}}$: - $\sqrt{5 - \sqrt{2}}$: for the Summ; and to subtract $\sqrt{5} - \sqrt{2}$: from $\sqrt{5} + \sqrt{2}$: I write $\sqrt{5 + \sqrt{2}}$: - $\sqrt{5 - \sqrt{2}}$: for the Remainder.

Likewise the Summ of $\sqrt{aa + bb}$: and $\sqrt{aa - cc}$: is $\sqrt{aa + bb + \sqrt{aa - cc}}$: and their Difference is $\sqrt{aa + bb} - \sqrt{aa - cc}$:

Sect. XV. Concerning the Constitution and Invention of six Binomials in Numbers, agreeable to those expounded in Prop. 49, 50, 51, 52, 53, 54; Elem. 10. Euclid:

By way of preparation to the Construction of the six Binomials in Numbers, I shall premise this

QUESTION.

To find two Square numbers whose difference may be equal to a given Rational number?

CANON.

Take any two numbers, which multiplied one by the other will produce the given number;

number; then half the sum of those two numbers and half their difference shall be the Sides or Roots of the two Squares sought.

As, if 5 be given for the difference of two Squares sought, I take 5 and 1; for the Product of their multiplication is 5; then the half of their sum is 3, and the half of their difference is 2; lastly, the Squares of the said 3 and 2 are 9 and 4, the Squares sought; for their difference is 5, as was prescribed.

Again, the same number 5 being given for the difference of two Squares, take a number at pleasure, as 2, by this divide the given number 5, the Quotient is $\frac{5}{2}$, therefore the Product of the multiplication of the Divisor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the sum and half the difference of the said 2 and $\frac{5}{2}$, to wit, $\frac{9}{2}$ and $\frac{1}{2}$ shall be the sides of the Squares sought; and consequently the Squares themselves are $\frac{81}{4}$ and $\frac{1}{4}$, whose difference is 5, as was desired.

After the same manner innumerable pairs of Squares may be found out in Rational numbers, and the difference of each pair shall be equal to one and the same given number.

The reason of the Canon may be made manifest by this

Theorem.

The Product made by the multiplication of any two unequal numbers is equal to the difference of two Squares, to wit, of the Square of half the sum, and the Square of half the difference of the same two unequal numbers.

As, if c be the greater, and b the lesser of two numbers, then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is $\frac{1}{4}cc + \frac{1}{4}cb + \frac{1}{4}bb$,

The Square of $\frac{1}{2}c - \frac{1}{2}b$ is $\frac{1}{4}cc - \frac{1}{4}cb + \frac{1}{4}bb$,

The difference of those two Squares is cb .

Which difference is manifestly the Product of the multiplication of the two proposed numbers c and b . Wherefore the Theorem, and consequently the Canon first given is manifest.

The Definition of Binomial I.

When the greater Name (or part) of a Binomial is a Rational number, and the lesser part is a Surd Square Root of some Rational number, and the square Root of the difference of the Squares of the parts is a Rational number, the sum of the two parts is called a First Binomial.

Explication.

Let this Binomial be proposed, $3 + \sqrt{5}$

The Squares of the Names, or parts, are $\begin{cases} 9 \\ 5 \end{cases}$

The difference of those Squares is $\begin{cases} 4 \end{cases}$

The Square-Root of that difference is $\begin{cases} 2 \end{cases}$

Because the greater part 3 is a Rational number, and the lesser part $\sqrt{5}$ is a Surd square Root of a Rational number 5, and the difference of the Squares of the parts, viz. 4, is a Square whose Root 2 is a Rational number; the Binomial proposed, to wit, $3 + \sqrt{5}$ is called a First Binomial.

How to find out two such numbers as may constitute a First Binomial.

1. By the Canon of the preceding Question at the beginning of this 15. $\begin{cases} 9 \\ 5 \end{cases}$
Self, find out two Square numbers whose difference may be some Rational number not a Square; such are these Squares, $\begin{cases} 9 \\ 5 \end{cases}$
2. Their difference is $\begin{cases} 4 \end{cases}$
3. Take some Rational number at pleasure for the greater part of the Binomial sought, as $\begin{cases} 6 \end{cases}$
4. Then say, by the Rule of Three, If 9 the greater of the two Squares found out in the first step, give 5 the difference in the second, what shall 36 the Square of the number taken in the third give? $\begin{cases} 420 \end{cases}$
whence the fourth Proportional will be found 20, the square Root whereof is the lesser part, to wit, $\begin{cases} 4 \end{cases}$
5. I say, the sum of the two numbers found out in the third and fourth steps, is a first Binomial, to wit, $\begin{cases} 6 + \sqrt{20} \end{cases}$

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The Definition of Binomial II.

When the lesser part of a Binomial is a Rational number, and the greater part is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the parts is Commensurable to the greater part; the sum of the two Parts is called a Second Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{18} + 4$

The Squares of the Parts are $\begin{cases} 18 \\ 16 \end{cases}$

The Difference of those Squares is $\begin{cases} 2 \end{cases}$

The square Root of that Difference is $\begin{cases} \sqrt{2} \end{cases}$

Because the lesser Part 4 is a Rational number, and the greater Part $\sqrt{18}$ is the Surd square Root of a Rational number 18, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{2}$, is Commensurable to the greater Part $\sqrt{18}$, (for according to the Definition in Sect. 7. of this Chap. $\sqrt{2} \cdot \sqrt{18} = 1 \cdot 3$, that is, as a Rational number to a Rational number,) the proposed number $\sqrt{18} + 4$ is a Second Binomial.

How to find out two such numbers as may constitute a Second Binomial.

1. By the foregoing Canon find out two square numbers whose Difference may be some Rational number not a Square; such are these Squares, $\begin{cases} 9 \\ 4 \end{cases}$
2. Their Difference is $\begin{cases} 5 \end{cases}$
3. Take some Rational number at pleasure for the lesser Part of the Binomial sought, as $\begin{cases} 10 \end{cases}$
4. Then say, If 5 the Difference in the third step gives 9 the greater of the two Squares in the first, what shall 100 the Square of the number taken in the third give? whence you will find 180, whose square Root shall be the greater Part, viz. $\begin{cases} \sqrt{180} \end{cases}$
5. I say the sum of the two numbers found out in the third and fourth steps is a Second Binomial, viz. $\begin{cases} \sqrt{180} + 10 \end{cases}$

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the Parts is Commensurable to the greater Part, the sum of the two Parts is called a Third Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{50} + \sqrt{32}$

The Squares of the Parts are $\begin{cases} 50 \\ 32 \end{cases}$

The Difference of those Squares is $\begin{cases} 18 \end{cases}$

The square Root of that Difference is $\begin{cases} \sqrt{18} \end{cases}$

Because the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd square Roots of two Rational numbers 50 and 32, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{18}$, is Commensurable to the greater Part $\sqrt{50}$, (for $\sqrt{18} \cdot \sqrt{50} = 3 \cdot 5$, that is, as a Rational number to a Rational number,) the proposed number $\sqrt{50} + \sqrt{32}$ is a Third Binomial.

How to find out two such numbers as may constitute a Third Binomial.

1. Find out two Square numbers whose Difference may be some Rational number not a Square; such are these Squares, $\begin{cases} 9 \\ 4 \end{cases}$
2. Their Difference is $\begin{cases} 5 \end{cases}$
3. Take some Rational number not a Square, which may exceed the said Difference 5 by an Unit or two, viz. by 1, when the said Difference increased with 1 makes not a Square; but by 2, when the Difference increased with 1 makes a Square: so in this Example, I take 6, because $5 + 1$ makes not a Square, $\begin{cases} 12 \end{cases}$
4. Again, take some Rational number at pleasure, as $\begin{cases} 12 \end{cases}$

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5. The Square thereof is 144
 6. Then say, If 6 the number taken in the third step, gives 9 the greater of the two Squares in the first, what shall 144 the square number in the fifth give? whence the fourth Proportional is 216, whose square Root, to wit $\sqrt{216}$ shall be the greater Part;
 7. Say again, If the said Square 9 gives 5 the Difference in the second step, what shall 216 the fourth Proportional found out in the fifth give? whence you will find 120, whose square Root, to wit, $\sqrt{120}$ shall be the lesser Part;
 8. I say, the sum of the two numbers found out in the sixth and seventh steps is a third Binomial, to wit, $\sqrt{216} + \sqrt{120}$

The Definition of Binomial IV.

When the greater Part of a Binomial is a Rational number, and the lesser Part is the square Root of a Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part, the sum of the two Parts is called a Fourth Binomial.

Explication.

Let this Binomial be proposed, $5 + \sqrt{12}$
 The Squares of the Parts are 25 and 12
 The Difference of those Squares is 13
 The square Root of that Difference is $\sqrt{13}$

Because the greater Part 5 is a Rational number, and the lesser Part $\sqrt{12}$ is a Surd square Root of a Rational number 12; and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{13}$, is Incommensurable to the greater Part 5; (for $\sqrt{13}$ hath no such proportion to 5 as a Rational number to a Rational number;) the number $5 + \sqrt{12}$ above proposed is a Fourth Binomial.

How to find out two such numbers as may constitute a Fourth Binomial.

1. Take any square number, as 9
 2. Divide that square number 9 into two numbers not Squares, as into 6 and 3
 3. Take a Rational number at pleasure for the greater Part of the Binomial sought, as 5
 4. Then say, If 9 the square number in the first step, give 6 the greater of the two numbers in the second; what shall 36 the Square of the number taken in the third give? so the fourth Proportional will be found 24, whose square Root, to wit $\sqrt{24}$, shall be the lesser Part;
 5. I say, the sum of the two numbers found out in the third and fourth steps, is a Fourth Binomial, viz. $5 + \sqrt{24}$

The Definition of Binomial V.

When the lesser Part of a Binomial is a Rational number, and the greater Part is a Surd square Root of some Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part, the sum of the two Parts is called a Fifth Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{6} + 2$
 The Squares of the Parts are 6 and 4
 The Difference of those Squares is 2
 The square Root of that Difference is $\sqrt{2}$

Because the lesser Part 2 is a Rational number, and the greater Part $\sqrt{6}$ is a Surd square Root of a Rational number 6, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{2}$, is Incommensurable to the greater Part $\sqrt{6}$; (for $\sqrt{2} : \sqrt{6}$ is as 1 : $\sqrt{3}$, not as a Rational number to a Rational number;) the proposed number $\sqrt{6} + 2$ is a Fifth Binomial.

How to find out two such numbers as may constitute a Fifth Binomial.

1. Take any square number, as 9
 2. Divide that square number 9 into two numbers not Squares, as into 6 and 3
 3. Take a Rational number at pleasure for the lesser Part of the Binomial sought, as 2
 4. Then say, If 6 the greater of the two numbers in the second step, gives 9 the square number in the first, what shall 4 the Square of the Rational number taken in the third give? whence you will find the fourth Proportional 6, whose square Root, to wit, $\sqrt{6}$, shall be the greater Part sought;
 I say, the sum of the two numbers found out in the third and fourth steps is a Fifth Binomial, viz. $2 + \sqrt{6}$

The Definition of Binomial VI.

When each of the two Parts of a Binomial is a Surd square Root of some Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part, the sum of the two Parts is called a Sixth Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{5} + \sqrt{3}$
 The Squares of the Parts are 5 and 3
 The Difference of the Squares of the Parts is 2
 The square Root of that Difference is $\sqrt{2}$

Because the two Parts $\sqrt{5}$ and $\sqrt{3}$ are Surd square Roots of two Rational numbers 5 and 3, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{2}$, is Incommensurable to the greater Part $\sqrt{5}$; (for $\sqrt{2}$ hath not such proportion to $\sqrt{5}$ as a Rational number to a Rational number;) the number $\sqrt{5} + \sqrt{3}$ above proposed is a Sixth Binomial.

How to find out two such numbers as may constitute a Sixth Binomial.

1. Take two such Prime numbers that their sum may not be a Square, as 7 and 11
 2. Their sum is 18
 3. Take also any square number, as 9
 4. Take again some Rational number at pleasure, as 2
 5. The Square thereof is 4
 6. Then say, If 9 the square number taken in the third step, gives 12 the sum of the two Prime numbers in the first; what shall 36 the Square in the fifth step give? whence you will find 48, whose square Root, to wit, $\sqrt{48}$, shall be the greater Part;
 7. Say again, If 12 the sum of the two Prime numbers in the first step, gives 7 the greater of those Prime numbers; what shall 48 the fourth Proportional found out in the fifth step give? whence you will find 28, whose square Root, viz. $\sqrt{28}$ shall be the lesser Part;
 I say, the sum of the two numbers found out in the sixth and seventh steps is a Sixth Binomial, viz. $\sqrt{48} + \sqrt{28}$

If of every one of those six Binomials the lesser Part be subtracted from the greater, by interposing the sign $-$, the six Remainders answer to the six Lines which Euclid in Prop. 86, 87, 88, 89, 90, 91. of his Tenth Elem. calls *Apotomes* or *Residual lines*, as,

I.	$3 + \sqrt{5}$	By changing $+$ into $-$, is made Residual.	I.	$3 - \sqrt{5}$
II.	$\sqrt{18} + 4$		II.	$\sqrt{18} - 4$
III.	$\sqrt{50} + \sqrt{32}$		III.	$\sqrt{50} - \sqrt{32}$
IV.	$5 + \sqrt{12}$		IV.	$5 - \sqrt{12}$
V.	$\sqrt{6} + 2$		V.	$\sqrt{6} - 2$
VI.	$\sqrt{5} + \sqrt{3}$		VI.	$\sqrt{5} - \sqrt{3}$

The precedent Constructions of the said six Binomials are demonstrated in Prop. 49, 50, 51, 52, 53, 54. of 10. Elem. Euclid.

Now if any Binomial or Residual be given, we may easily find out another of the same kind in this manner, *viz.* For the first and fourth Binomials, if it be made as the greater Name or Part to the lesser, so any Rational number assumed for the greater Part of a new first or fourth Binomial, to a fourth Proportional number, this number shall be the lesser Part of the new first or fourth Binomial. But for the second and fifth, if it be made as the lesser Part to the greater, so any Rational number taken for the lesser Part of a new second or fifth Binomial to a fourth Proportional, the number so produced shall be the greater Part of the new second or fifth Binomial. And lastly, for the third and sixth Binomials, if it be made as the greater Part to the lesser, (each of which is a Sord square Root,) so any Sord square Root assumed for the greater Part of a new third or sixth Binomial, to a fourth Proportional, there will come forth the lesser Part of a new third or sixth Binomial. (The reason of this Operation is manifest, *per Prop. 15. Elem. 10. Euclid.*) And, after a new Binomial is found out, its correspondent Residual is also made, by changing the sign $+$ into $-$, as before hath been said.

As, for example, if a first Binomial $3 + \sqrt{5}$ be proposed, to find another like to it; I take a Rational number at pleasure, as 8, for the greater Part of the Binomial sought; then by the Rule of Three, as 3 is to $\sqrt{5}$, so 8 to a fourth Proportional, to wit, $\sqrt{128}$, for the lesser Part sought, therefore $8 + \sqrt{128}$ shall be a new first Binomial, and $8 - \sqrt{128}$ a new first Residual, and so of the rest.

SECT. XVI. Concerning the extraction of the Square Root out of Binomials and Residuals constituted in such manner as hath been shewn in the preceding SECT. 15.

Every one of the Binomials and Residuals whose Construction hath been shewn in the preceding SECT. 15. hath a square Root, that is, such a Binomial or Residual that if it be multiplied into it self will produce the given Binomial or Residual; as may be evidently collected out of *Prop. 55, 56, 57, 58, 59, and 60.* Also out of *Prop. 92, 93, 94, 95, 96, and 97.* of the Tenth Book of *Euclid's Elements.*

As, for example, a Binomial of the first kind, suppose $7 + \sqrt{48}$ hath a square Root, to wit, $2 + \sqrt{3}$; for this being squared (or multiplied into it self) produceth that Binomial $7 + \sqrt{48}$; whose greater Part 7 is composed of 4 and 3 the Squares of the Parts of the Root $2 + \sqrt{3}$; and the lesser Part $\sqrt{48}$ is the double of the Product made by the multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2 + \sqrt{3}$: all which is evident by the multiplication of $2 + \sqrt{3}$ into it self. The like effect will be found in every one of the rest of the Binomials constituted in the preceding SECT. 15. Therefore if a Binomial be proposed, and its square Root desired, there is given the sum of the Squares of the Parts of the Root, (which sum is the greater Part of the Binomial proposed;) and the double of the Product of the Parts of the Root, (which double Product is the lesser Part of the Binomial proposed,) to find out the two Parts of the Root severally. And therefore in order to the Extraction of the square Root of a Binomial, it will be requisite to search out a Canon for the solving of this following

QUEST.

The sum (b) of the Squares of two numbers being given; as also (c) the double Product of the multiplication of the same two numbers; to find the numbers severally.

RESOLUTION.

1. For one of the two numbers sought put a
2. Then for as much as the double of the Product of their multiplication is given c , therefore the Product it self is $\frac{c}{2}$
3. Which Product divided by the first number a gives the other number $\frac{c}{2a}$
4. Therefore the Square of the first number is aa
5. And the Square of the other number is $\frac{cc}{4aa}$
6. Therefore the sum of the Squares of the two numbers is $aa + \frac{cc}{4aa}$

7. Which

7. Which sum must be equal to $\frac{b}{2}$, the given sum of the Squares; hence this Equation; $aa + \frac{cc}{4aa} = \frac{b}{2}$
8. From which Equation, after due Reduction, there will arise, $\frac{b}{2}aa - \frac{cc}{4} = \frac{5}{4}cc$
9. And from the last Equation (*per Canon in SECT. 10. Chap. 15. Book 1.*) there will arise this following Canon, to find out the two numbers sought, *viz.*

CANON 1.

$$\sqrt{\frac{b}{2}b + \sqrt{\frac{bb}{2} - \frac{cc}{2}}} = \text{the greater number;}$$

$$\sqrt{\frac{b}{2}b - \sqrt{\frac{bb}{2} - \frac{cc}{2}}} = \text{the lesser number.}$$

That is, in words,

From a quarter of the Square of the given sum of the Squares, subtract a quarter of the Square of the double Product given; then add and subtract the square Root of that Remainder to and from half the given sum of the Squares; to shall the square Roots of the Summ and Remainder of that Addition and Subtraction be the two numbers sought.

10. Moreover, because $\frac{b}{2} + \sqrt{\frac{bb}{2} - \frac{cc}{2}} = \frac{b}{2} + \sqrt{\frac{bb}{2} - \frac{cc}{2}}$
11. Therefore, $\sqrt{\frac{b}{2} + \sqrt{\frac{bb}{2} - \frac{cc}{2}}} = \sqrt{\frac{b}{2} + \sqrt{\frac{bb}{2} - \frac{cc}{2}}}$
12. Likewise because $\frac{b}{2} - \sqrt{\frac{bb}{2} - \frac{cc}{2}} = \frac{b}{2} - \sqrt{\frac{bb}{2} - \frac{cc}{2}}$
13. Therefore, $\sqrt{\frac{b}{2} - \sqrt{\frac{bb}{2} - \frac{cc}{2}}} = \sqrt{\frac{b}{2} - \sqrt{\frac{bb}{2} - \frac{cc}{2}}}$
14. Therefore from the eleventh and thirteenth steps another Canon ariseth to solve the Question; *viz.*

CANON 2.

$$\sqrt{\frac{b}{2} + \sqrt{\frac{bb}{2} - \frac{cc}{2}}} = \text{the greater number;}$$

$$\sqrt{\frac{b}{2} - \sqrt{\frac{bb}{2} - \frac{cc}{2}}} = \text{the lesser number.}$$

That is, in words,

From the Square of the given sum of the Squares subtract the Square of the double Product given; then add and subtract the square Root of the Remainder to and from the given sum of the Squares: to shall the square Root of half the Summ and Remainder of that Addition and Subtraction be the two numbers sought.

By the help of either of those Canons we may extract the square Root of a Binomial or Residual; but I shall use the latter only, whence ariseth

A General Rule for the Extraction of the Square Root out of Binomials and Residuals.

From the Square of the greater part of a given Binomial or Residual, subtract the Square of the lesser; then add the square Root of the Remainder to the greater part; and subtract it also from the same; lastly, connect the square Roots of the half of that Summ and Remainder by the sign $+$ if a Binomial be proposed; but by $-$ if a Residual; so you have the desired square Root of the given Binomial or Residual.

The practice of this Rule will be shewn at large in the following Examples.

Example 1.

Let it be required to extract the square Root of this first Binomial, $27 + \sqrt{704}$

The Operation.

1. From the Square of the greater part 27, *viz.* from 729
2. Subtract the Square of the lesser part $\sqrt{704}$, to wit, 704
3. The Remainder is 25
4. The square Root of that Remainder is 5

5. To

5. To which Square Root add the greater part 27
6. The Summ is 32
7. The half of that Summ is 16
8. The Square Root of the said half Summ is the greater part of the Root fought, to wit, 4
9. Then from the greater part of the given Binomial, viz. from 27
10. Subtract the Square Root before found in the fourth step, to wit, 5
11. The Remainder is 22
12. The half of which Remainder is 11
13. The Square Root of the said half Remainder is the lesser part of the Root fought, to wit, $\sqrt{11}$
14. I say, the two Names or parts in the eighth and thirteenth steps being connected by $+$ shall be the Square Root fought, to wit, $4 + \sqrt{11}$

But if — instead of $+$ be prefix to the lesser part of the said Root, it will give $4 - \sqrt{11}$, which is the Square Root of the first Residual or Apotome $27 - \sqrt{704}$. The former of those two Roots answers to the Irrational line called (in prop. 37, and 55. lib. 10. Elem. Euclid.) a Binomial line; and the latter answers to the Irrational line called (in prop. 74, and 92.) an Apotome or Residual line.

The Proof of the Root above extracted out of the first Binomial; is made by multiplying the Root into it self; thus,

- The Summ of the Squares of the parts of $4 + \sqrt{11}$, $16 + 11$, that is; 27
 The Product of the same parts multiplied one into the other is $4\sqrt{11}$, that is, $\sqrt{176}$
 The double of the said Product is $8\sqrt{11}$, that is, $\sqrt{704}$
 The Summ of the said Summ of the Squares of the parts and the double Product is $27 + \sqrt{704}$

Whence it is manifest that $27 + \sqrt{704}$ is the Square of $4 + \sqrt{11}$, therefore this is the true Square Root of that first Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said Summ of the Squares of the Parts, the Remainder $27 - \sqrt{704}$ is the Square of $4 - \sqrt{11}$; therefore this is the Square Root of that first Residual.

Example 2.

Let it be required to extract the Square Root out of this second Binomial $\sqrt{12} + 6$.

The Operation.

1. From the Square of the greater part $\sqrt{12}$, viz. from 12
2. Subtract the Square of the lesser part 6, to wit, 36
3. The Remainder is $\frac{1}{4}$
4. The Square Root of that Remainder is $\frac{1}{2}$
5. To which Square Root add the greater part, (by the Rule in Sect. 8. of this Chap.) $\sqrt{12} + \frac{1}{2}$
6. The Summ is $\sqrt{48}$
7. The half of which Summ is $\sqrt{12}$
8. The Square Root of that half Summ is the greater part of the Root fought, to wit, $\sqrt{(4)}12$
9. Again, from the greater part of the given Binomial, viz. from $\sqrt{12} + 6$
10. Subtract the Square Root before found in the fourth step, (by the said Rule in Sect. 8.) viz. $\frac{1}{2}$
11. The Remainder is $\sqrt{27}$
12. The half of which Remainder is $\sqrt{\frac{27}{4}}$
13. The Square Root of the said half Remainder is the lesser part of the Root fought, to wit, $\sqrt{(4)}\frac{3}{2}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by the sign $+$ shall be the Root fought, to wit, $\sqrt{(4)}12 + \sqrt{(4)}\frac{3}{2}$

And if — instead of $+$ be prefix to the lesser part of the said Root, it will give $\sqrt{(4)}12 - \sqrt{(4)}\frac{3}{2}$, which is the Square Root of the second Residual $\sqrt{12} - 6$.

The

The former of those two Roots answers to the Irrational line called (in Prop. 38, & 56. lib. 10. Elem. Euclid.) a first Binomial; and the latter answers to the Irrational line called (in Prop. 75, & 93.) a first Medial Residual.

The Proof of the Root above extracted out of the second Binomial.

- The Squares of the Parts of $\sqrt{(4)}12 + \sqrt{(4)}\frac{3}{2}$ the Root found out, are $\sqrt{12}$ and $\sqrt{\frac{3}{4}}$
 Which Squares added together, (as in Example 6. Sect. 8. of this Chap. is manifest,) makes the Summ $7\sqrt{\frac{3}{4}}$, that is, $\sqrt{12}$
 The Product of the Parts, viz. $\sqrt{(4)}12$ into $\sqrt{(4)}\frac{3}{2}$ is $\sqrt{(4)}81$, that is, 3
 The double of the said Product is 6
 Therefore the Summ of the Summ of the Squares of the Parts and the said double Product is $\sqrt{12} + 6$

Whence it is manifest that $\sqrt{12} + 6$ is the Square of $\sqrt{(4)}12 + \sqrt{(4)}\frac{3}{2}$; therefore this is the true Square Root of that second Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said Summ of the Squares of the Parts, the Remainder $\sqrt{12} - 6$ is the Square of $\sqrt{(4)}12 - \sqrt{(4)}\frac{3}{2}$; therefore this is the Square Root of that second Residual.

Example 3.

Let it be required to extract the Square Root of this third Binomial $\sqrt{12} + \sqrt{80}$.

The Operation.

1. From the Square of the greater part $\sqrt{12}$, viz. from 12
2. Subtract the Square of the lesser part, to wit, 80
3. The Remainder is $\frac{1}{4}$
4. The Square Root of that Remainder is $\frac{1}{2}$
5. To which Square Root add the greater part $\sqrt{12} + \frac{1}{2}$
6. The Summ is $\sqrt{48}$
7. The half of which Summ is $\sqrt{12}$
8. The Square Root of that half Summ is the greater part of the Root fought, to wit, $\sqrt{(4)}12$
9. Again, from the greater part of the given Binomial, viz. from $\sqrt{12} + \sqrt{80}$
10. Subtract the Square Root before found in the fourth step, to wit, $\frac{1}{2}$
11. The Remainder is $\sqrt{60}$
12. The half of which Remainder is $\sqrt{15}$
13. The Square Root of the said half Remainder is the lesser part of the Root fought, to wit, $\sqrt{(4)}15$
14. I say, the two parts in the eighth and thirteenth steps, being connected by $+$, shall be the Square Root fought; to wit, $\sqrt{(4)}12 + \sqrt{(4)}15$

And if — instead of $+$ be prefix to the lesser part of the said Root, it gives $\sqrt{(4)}12 - \sqrt{(4)}15$, which is the Square Root of the third Residual $\sqrt{12} - \sqrt{80}$.

The former of those two Roots answers to the Irrational line called (in Prop. 39, & 57. lib. 10. Elem. Euclid.) a second Binomial; and the latter answers to the Irrational line called (in Prop. 76, & 94.) a second Medial Residual.

The Proof of the Root above extracted out of the third Binomial.

- The Squares of the parts of $\sqrt{(4)}12 + \sqrt{(4)}15$, the Root found out, are $\sqrt{12}$ and $\sqrt{15}$
 Which Squares added together, make $7\sqrt{\frac{3}{4}}$, that is, $\sqrt{12}$
 The Product of the parts, viz. $\sqrt{(4)}12$ into $\sqrt{(4)}15$, is $\sqrt{(4)}400$; that is, $\sqrt{20}$
 The double of the said Product is $\sqrt{80}$
 Therefore the Summ of the Summ of the Squares of the Parts and the said double Product is $\sqrt{12} + \sqrt{80}$

Whence it is manifest, that $\sqrt{12} + \sqrt{80}$ is the Square of $\sqrt{(4)}12 + \sqrt{(4)}15$; therefore this is the Square Root of that third Binomial: which was to be proved. Moreover,

Moreover,

Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $\sqrt{2^2} - \sqrt{80}$ is the Square of $\sqrt{(4)}^{\frac{1}{2}} - \sqrt{(4)}^{\frac{1}{2}}$; therefore this is the square Root of that third Residual.

Example 4.

Let it be required to extract the square Root of this fourth Binomial: $7 + \sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, viz. from . . . 49
2. Subtract the Square of the lesser part $\sqrt{20}$, to wit, . . . 20
3. The Remainder is . . . 29
4. The square Root of that Remainder is . . . $\sqrt{29}$
5. To which square Root add the greater part . . . 7
6. The Summ is . . . $7 + \sqrt{29}$
7. The half of which Summ is . . . $\frac{7}{2} + \sqrt{\frac{29}{4}}$
8. The square Root of that half Summ is the greater part of the Root sought, to wit, . . . $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}}$
9. Again, from the greater part of the given Binomial, viz. from . . . 7
10. Subtract the square Root before found in the fourth step, to wit, . . . $\sqrt{29}$
11. The Remainder is . . . $7 - \sqrt{29}$
12. The half of which Remainder is . . . $\frac{7}{2} - \sqrt{\frac{29}{4}}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, . . . $\sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$
14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual) being connected by +, shall be the square Root sought, to wit, . . . $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} + \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$

Which Root answers to the irrational line called (in Prop. 40, & 58. lib. 10. Elem. Euclid.) a Major line.

And if the lesser Name of the said Root be subtracted from the greater, by interposing the sign -, it gives $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} - \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$: which is the Root of the fourth Residual $7 - \sqrt{20}$, and answers to the irrational line called (in Prop. 77, & 95. lib. 10. Elem. Euclid.) a Minor line.

The Proof of the Root above extracted out of the fourth Binomial.

- The Squares of the parts of the Root found out are . . . $\frac{7}{2} + \sqrt{\frac{29}{4}}$ and $\frac{7}{2} - \sqrt{\frac{29}{4}}$
 Therefore the sum of the Squares of the parts is . . . $\frac{7}{2} + \frac{7}{2}$, that is, 7
 The Product of the parts will be found (by Rule 2. Sect. 12. of this Chapt.) . . . $\sqrt{\frac{29}{4}} - \sqrt{\frac{29}{4}}$: that is, $\sqrt{20}$
 The double of the said Product is . . . $\sqrt{20}$
 Therefore the sum of the said sum of the Squares of the parts and the double Product is . . . $7 + \sqrt{20}$.

Whence it is manifest that $7 + \sqrt{20}$ is the Square of $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} + \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$: therefore this is the square Root of that fourth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $7 - \sqrt{20}$ is the Square of $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} - \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$: therefore this is the square Root of that fourth Residual $7 - \sqrt{20}$.

Example 5.

Let it be required to extract the square Root out of this fifth Binomial, $\sqrt{20} + 4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, viz. from . . . 20
2. Subtract the Square of the lesser part 4, to wit, . . . 16
3. The Remainder is . . . 4
4. The square Root of that Remainder is . . . 2
5. To which square Root add the greater part . . . $\sqrt{20}$

6. The

6. The sum is . . . $\sqrt{20} + 2$
7. The half of that sum is . . . $\sqrt{5} + 1$
8. The square Root of the said half sum is the greater part of the Root sought, to wit, . . . $\sqrt{\sqrt{5} + 1}$
9. Again, from the greater part of the given Binomial, viz. from . . . $\sqrt{20}$
10. Subtract the square Root before found in the fourth step, to wit, . . . 2
11. The Remainder is . . . $\sqrt{20} - 2$
12. The half of which Remainder is . . . $\sqrt{5} - 1$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, . . . $\sqrt{\sqrt{5} - 1}$
14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual,) being connected by +, shall be the square Root sought, to wit, . . . $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$

Which Root answers to the irrational line called (in Prop. 41, & 59. lib. 10. Elem. Eucl.) a line containing in Power a Rational and a Medial Rectangle: And if the lesser Name of the said Root be subtracted from the greater, by the interpolation of the sign -, it gives $\sqrt{\sqrt{5} + 1} - \sqrt{\sqrt{5} - 1}$: which is the square Root of the fifth Residual $\sqrt{20} - 4$, and answers to the irrational line which (in Prop. 78, & 96. lib. 10.) is called a line making with a Rational Space the whole Space Medial.

The Proof of the Root above extracted out of the fifth Binomial.

- The Squares of the parts of $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$ are . . . $\sqrt{5} + 1$ and $\sqrt{5} - 1$
 The Root found out, are . . . $\sqrt{5} + 1$ and $\sqrt{5} - 1$
 Therefore the sum of the said Squares of the parts is . . . $\sqrt{5} + \sqrt{5}$; that is, $\sqrt{20}$
 The Product of the parts multiplied one into the other (according to Rule 2. Sect. 12. of this Chapt.) is . . . $\sqrt{5} - 1$; that is, 2
 The double of the said Product is . . . 4
 Therefore the sum of the said sum of the Squares of the parts and double Product is . . . $\sqrt{20} + 4$
 Whence it is manifest that $\sqrt{20} + 4$ is the Square of $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$: therefore this is the square Root of that fifth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $\sqrt{20} - 4$ is the Square of $\sqrt{\sqrt{5} + 1} - \sqrt{\sqrt{5} - 1}$: Therefore this is the square Root of the said fifth Residual $\sqrt{20} - 4$.

Example 6.

Let it be required to extract the square Root of this sixth Binomial, $\sqrt{20} + \sqrt{8}$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, viz. from . . . 20
2. Subtract the Square of the lesser part $\sqrt{8}$, to wit, . . . 8
3. The Remainder is . . . 12
4. The square Root of that Remainder is . . . $\sqrt{12}$
5. To which square Root add the greater part . . . $\sqrt{20}$
6. The sum is . . . $\sqrt{20} + \sqrt{12}$
7. The half of which sum is . . . $\sqrt{5} + \sqrt{3}$
8. The square Root of the said half sum is the greater part of the Root sought, to wit, . . . $\sqrt{\sqrt{5} + \sqrt{3}}$
9. Again, from the greater part of the given Binomial, viz. from . . . $\sqrt{20}$
10. Subtract the square Root before found in the fourth step, to wit, . . . $\sqrt{12}$
11. The Remainder is . . . $\sqrt{20} - \sqrt{12}$
12. The half of that Remainder is . . . $\sqrt{5} - \sqrt{3}$

13. The

13. The

13. The Square Root of the said half Remainder is the lesser part of the Root sought, to wit, $\sqrt{5} - \sqrt{3}$:
 14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual) being connected by $+$, shall be the Square Root sought, to wit, $\sqrt{5 + \sqrt{3}} : + \sqrt{5 - \sqrt{3}}$

Which Root answers to the irrational line which (in Prop. 42, & 60. lib. 10. Elem. Eucl.) is called, a line containing in Power two Medial Rectangles: And, if the lesser part of the said Root be subtracted from the greater by the interpolation of the sign $-$, it gives $\sqrt{5 + \sqrt{3}} - \sqrt{5 - \sqrt{3}}$: which is the Root of the sixth Residual $\sqrt{20 - \sqrt{8}}$, and answers to the irrational line which (in Prop. 79, & 97. lib. 10. Euclid.) is called a line making with a Medial Rectangle a whole Space Medial.

The Proof of the Root above extracted out of the sixth Binomial.

The Squares of the parts of $\sqrt{5 + \sqrt{3}} + \sqrt{5 - \sqrt{3}}$ are $\sqrt{5 + \sqrt{3}}$ and $\sqrt{5 - \sqrt{3}}$
 Therefore the sum of the said Squares of the parts is $\sqrt{5 + \sqrt{3}} + \sqrt{5 - \sqrt{3}}$, that is, $\sqrt{10}$
 The Product of the parts multiplied one into the other is $\sqrt{5 - \sqrt{3}}$: that is, $\sqrt{2}$
 The double of the said Product is $\sqrt{4}$
 Therefore the sum of the said sum of the Squares of the parts and double Product is $\sqrt{20} + \sqrt{8}$.

Whence it is manifest that $\sqrt{20} + \sqrt{8}$ is the Square of $\sqrt{5 + \sqrt{3}} + \sqrt{5 - \sqrt{3}}$: therefore this is that square Root of the sixth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $\sqrt{20} - \sqrt{8}$ is the Square of $\sqrt{5 + \sqrt{3}} - \sqrt{5 - \sqrt{3}}$: therefore this is the square Root of that sixth Residual.

Note. In every Binomial and Residual constituted according to the preceding Sect. 11, the square Root of the difference of the Squares of the Names or parts is equal to the difference of the Squares of the parts of the Root of the Binomial or Residual.

As in the first Binomial $27 + \sqrt{704}$, whose square Root hath before been found $4 + \sqrt{11}$; the Square of 27, to wit, 729, exceeds 704, the Square of $\sqrt{704}$, by 25, whose square Root 5 is equal to the difference of the Squares of the parts of the Root of the Binomial proposed, to wit, the difference between 16 and 11.

This property may be demonstrated thus, let $b + \sqrt{d}$ represent a Binomial Root whose greater part is b ; then the Square of that Root is $bb + 2b\sqrt{d} + d$, this divided into its Names or parts makes the Binomial $bb + d$ more $2b\sqrt{d}$, then the Squares of the parts of this Binomial are $bbb + 2bb\sqrt{d} + dd$ and abd , and the difference of these Squares is $bbb - bbd + dd$, whose square Root $b - \sqrt{d}$ first proposed: which was to be shewn. The like property may be demonstrated in a Residual.

How to extract the Square Root out of a Binomial design'd by Letters, if it hath a Binomial Root.

By the same general Rule which hath before been exercis'd in extracting the square Root out of Binomials express'd by Numbers, we may extract the square Root out of a Binomial design'd by Letters, when it hath a binomial Root, as will be evident by the following Examples; where for the more apparent distinction of the parts of the given Binomial, instead of $+$ I set the word [more] between the parts, and instead of $-$ I set the word [less] between the parts of a given Residual.

Example 1.

Let it be required to extract the square Root out of $bb + d$ more $2b\sqrt{d}$. this Binomial,

The Operation.

1. From the Square of the greater part, (which suppose to be $bbb + 2bb\sqrt{d}$ + dd $bb + d$), viz. from $bbb + 2bb\sqrt{d} + dd$
 2. Subtract the Square of the lesser part $2b\sqrt{d}$, to wit, $4bb\sqrt{d}$
 3. The

3. The Remainder is $bbb - 2bb\sqrt{d} + dd$
 4. The square Root of that Remainder is $bb - d$
 5. To which square Root add the greater part, to wit, $bb + d$
 6. The Summ is $2bb$
 7. The half of that Summ is bb
 8. The square Root of that half Summ is the greater part of the Root sought, to wit, b
 9. Then from the greater part of the given Binomial, viz. from $bb + d$
 10. Subtract the square Root before found in the fourth step, to wit, $bb - d$
 11. The Remainder is $2d$
 12. The half of which Remainder is d
 13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, \sqrt{d}
 14. I say, the two parts in the eighth and thirteenth steps being connected by the sign $+$ shall be the square Root sought, to wit, $b + \sqrt{d}$

Which Root being squared, or multiplied into it self, will evidently produce the given Binomial $bb + d$ more $2b\sqrt{d}$.

Example 2.

Let it be required to extract the square Root out of $mm + \frac{p \times x}{m}$ more $x\sqrt{4pm}$. this Binomial,

The Operation.

1. From the Square of the greater part $mm + \frac{p \times x}{m}$ $mmmm + 2mp \times x + \frac{p \times p \times x \times x}{m}$
 viz. from $mmmm + 2mp \times x + \frac{p \times p \times x \times x}{m}$
 2. Subtract the Square of the lesser part $x\sqrt{4pm}$, to wit, $4mp \times x$
 3. The Remainder is $mmmm - 2mp \times x + \frac{p \times p \times x \times x}{m}$
 4. The square Root of that Remainder is $mm - \frac{p \times x}{m}$
 5. To which square Root add the greater part, to wit, $mm + \frac{p \times x}{m}$
 6. The Summ is $2mm$
 7. The half of which Summ is mm
 8. The square Root of the said half Summ is the greater part of the Root sought, to wit, m
 9. Again, from the greater part of the given Binomial, viz. from $mm + \frac{p \times x}{m}$
 10. Subtract the square Root before found in the fourth step, to wit, $mm - \frac{p \times x}{m}$
 11. The Remainder is $2p \times x$
 12. The half of which Remainder is $\frac{p \times x}{m}$
 13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, $\sqrt{\frac{p \times x}{m}}$ or $x\sqrt{\frac{p}{m}}$
 14. I say, the two parts in the eighth and thirteenth steps, being connected by $+$, shall be the square Root sought; to wit, $m + x\sqrt{\frac{p}{m}}$

Which Binomial Root being squared or multiplied into it self, will produce the given Binomial.

Example 3.

Let it be required to extract the square Root out of $a + b\sqrt{ab}$ more $2ab$. this Binomial,

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8. And from the seventh and second steps the sum $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{4}} =$ sum of the extremes, of the extremes will be also made known, viz. $\frac{1}{2} - \sqrt{\frac{1}{4}} =$ sum of the extremes.
9. Then, (as is manifest by *Quest. 4. Chap. 16. Book 1.*) the sum of the extremes of three numbers continually proportional being given, as also the mean, the extremes shall be given severally by this following

CANON.

From the Square of half the sum of the extremes subtract the Square of the mean, and extract the square Root of the Remainder; then this square Root being added, and subtracted from the said half sum, will give the extremes severally. Therefore,

10. From the Square of the half of $\frac{1}{2} - \sqrt{\frac{1}{4}}$, that is, from $\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4}}$,
 11. Subtract the Square of $\sqrt{\frac{1}{4}} - \frac{1}{2}$, viz. $\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4}}$,
 12. The Remainder is $\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4}}$,
 13. Then the square Root of that Remainder being extracted, (by the General Rule before delivered in *Self. 16. of this Chap.* for extracting the square Root out of Binomials,) will be found $\sqrt{\frac{1}{4}} - \frac{1}{2}$,
 14. Which square Root added to the half of $\frac{1}{2} - \sqrt{\frac{1}{4}}$, gives the greater extreme sought, to wit, $\frac{1}{2}$,
 15. But the said square Root subtracted from the half of $\frac{1}{2} - \sqrt{\frac{1}{4}}$, leaves the lesser extreme, to wit, $\frac{1}{2} - \sqrt{\frac{1}{4}}$,
 16. Wherefore, (in the seventh, fourteenth and fifteenth steps,) three numbers continually proportional are found out, viz. $\frac{1}{2}$, $\sqrt{\frac{1}{4}} - \frac{1}{2}$, and $\frac{1}{2} - \sqrt{\frac{1}{4}}$, whose sum is 6, and the sum of the Squares of the extremes is equal to the triple of the Square of the mean, as will appear by

The Proof.

First, the Product made by the multiplication of the first and third numbers one into the other, that is, of $\frac{1}{2}$ into $\frac{1}{2} - \sqrt{\frac{1}{4}}$, is $\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4}}$, which is also the Square of the second number $\sqrt{\frac{1}{4}} - \frac{1}{2}$, (as will easily appear by Multiplication;) therefore the said three numbers are Proportionals.

Secondly, the sum of the said three proportional numbers is 6; for the mean $\sqrt{\frac{1}{4}} - \frac{1}{2}$, added to $\frac{1}{2} - \sqrt{\frac{1}{4}}$ the lesser extreme, makes 3, to which adding the greater extreme $\frac{1}{2}$, the sum is 6.

Thirdly, the sum of the Squares of the extremes $\frac{1}{2}$ and $\frac{1}{2} - \sqrt{\frac{1}{4}}$, is equal to the triple of the Square of the mean $\sqrt{\frac{1}{4}} - \frac{1}{2}$; for the said sum, as also the said triple Square will by Multiplication be found $\frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4}}$. Therefore all the conditions in the Question are satisfied.

But that the necessity of the Determination annexed to the Question may be made manifest, it remains to prove, That if three unequal numbers be in continual proportion, the sum of the Squares of the extremes is greater than the double of the Square of the mean: Therefore,

Let three unequal numbers in continual proportion be exposed, a, \sqrt{ae}, e :: suppose these,

Then their Squares shall be also Proportionals, (per 22. Prop. 2. Elem. Euclid.) viz. $aa : ae :: ae : ee$

Therefore (by 25. Prop. 5. Elem. Euclid.) $aa + ee > 2ae$

But $aa + ee$ is the sum of the Squares of the extremes of the three Proportionals exposed, and $2ae$ is equal to the double Square of the mean Proportional; wherefore the Theorem is proved; and consequently the Determination is manifestly necessary to be annexed to the Question proposed, that there may be a possibility of finding out what is thereby desired. The Determination may also be easily infer'd from the Canon in the foregoing ninth step.

QUEST. 3.

What is the Product made by the continual multiplication of these four numbers one into another, which differ by an equal excess, to wit, Unity?

$$\left\{ \begin{array}{l} \sqrt{\frac{1}{2}} + \sqrt{101} : -\frac{1}{2} \\ \sqrt{\frac{1}{2}} + \sqrt{101} : -\frac{1}{2} \\ \sqrt{\frac{1}{2}} + \sqrt{101} : -\frac{1}{2} \\ \sqrt{\frac{1}{2}} + \sqrt{101} : -\frac{1}{2} \end{array} \right.$$

Ans.

Ans. The desired Product is exactly 100
 For, (by the last of the three compendious Rules before delivered in *Self. 10. of this Chap.* for the multiplication of Binomials and Residuals,) the Product of the first and fourth numbers is $\sqrt{101} - \frac{1}{2}$
 Likewise, the Product of the second and third number is $\sqrt{101} + \frac{1}{2}$
 Lastly, the two last preceding Products being multiplied one into another (by the same Rule) make 100

QUEST. 4.

1. If a, b, c , be such Quantities that $aa + ca = b$
 What is the value of a ?
 2. Ans. By the Canon in *Self. 6. Ch. 15. Book 1.* $a = \sqrt{b + \frac{1}{4}cc} - \frac{1}{2}c$
 By which value of a , the Equation propos'd may be expounded, as is manifest by the following

Demonstration.

3. If $a = \sqrt{b + \frac{1}{4}cc} - \frac{1}{2}c$
 4. Then consequently by adding $\frac{1}{2}c$ to each part, $a + \frac{1}{2}c = \sqrt{b + \frac{1}{4}cc}$
 5. And by multiplying each part of the last Equation into it self, $aa + ca + \frac{1}{4}cc = b + \frac{1}{4}cc$
 6. Wherefore, by subtracting $\frac{1}{4}cc$ from each part, there remains $aa + ca = b$
 Which was to be proved.

Note. This Demonstration is formed in the way of Composition by the steps of the Resolution of the same Question in *Self. 5. Chap. 15. Book 1.* but in a retrograde or backward order; for the first step in the Composition, (or Demonstration) is the last in the Resolution; the second step in the Composition is the last but one in the Resolution; and so by returning backwards by the steps of the Resolution; the Demonstration ends in the Equation propos'd to be resolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUEST. 5.

1. If a, b, k , be such Quantities that $aa - ba = k$
 What is the value of a ?
 2. Ans. By the Canon in *Self. 8. Ch. 15. Book 1.* $a = \frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + kb}$
 By which value of a , the Equation propos'd may be expounded; as appears by the following

Demonstration.

3. If $a = \frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + kb}$
 4. Then by subtracting $\frac{1}{2}b$ from each part; $a - \frac{1}{2}b = \sqrt{\frac{1}{4}b^2 + kb}$
 5. And by multiplying each part of the last Equation into it self, $aa - ba + \frac{1}{4}bb = \frac{1}{4}b^2 + kb$
 6. Wherefore by subtracting $\frac{1}{4}bb$ from each part, $aa - ba = k$
 Which was to be proved.

QUEST. 6.

1. If c and n be put for such known Quantities, that n not $= \frac{1}{2}cc$,
 2. And if a be put for a Quantity unknown, and What is the value of a ?
 3. Ans. By the Canon in *Self. 10. Chap. 15. Book 1.* these two values of a will be found $a = \frac{1}{2}c + \sqrt{\frac{1}{4}cc - n}$ or $a = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$
 out, viz.

By each of which values of a , the Equation propos'd in the second step may be expounded, viz. if either $\frac{1}{2}c + \sqrt{\frac{1}{4}cc - n}$; or, $\frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$; be put equal to a , then $ca - aa = n$.

Definit.

Demonstration.

4. First, if $a = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$
 5. Then by subtracting $\frac{1}{2}c$ from each part, $a - \frac{1}{2}c = -\sqrt{\frac{1}{4}cc - n}$
 6. And by multiplying each part of the last Equation into it self, $aa - ca + \frac{1}{4}cc = \frac{1}{4}cc - n$
 7. And by adding ca to each part, $aa + \frac{1}{4}cc = \frac{1}{4}cc + ca - n$
 8. And by subtracting $\frac{1}{4}cc$ from each part, $aa = ca - n$
 9. And by adding n to each part, $aa + n = ca$
 10. Wherefore by subtracting aa from each part, $n = ca - aa$
 11. That is, $ca - aa = n$

Which was to be proved.

- Again, If $a = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$
 12. Then by adding $\sqrt{\frac{1}{4}cc - n}$ to each part, $a + \sqrt{\frac{1}{4}cc - n} = \frac{1}{2}c$
 13. And by subtracting a from each part, $\sqrt{\frac{1}{4}cc - n} = \frac{1}{2}c - a$
 14. And by multiplying each part of the last Equation into it self, $\frac{1}{4}cc - n = \frac{1}{4}cc - ca + aa$
 15. And by adding ca to each part, $ca + \frac{1}{4}cc - n = \frac{1}{4}cc + ca - aa$
 16. And subtracting $\frac{1}{4}cc$ from each part, $ca - n = aa$
 17. And by adding n to each part, $ca = aa + n$
 18. Wherefore by subtracting aa from each part, $ca - aa = n$

Which was to be proved.

QUEST. 7.

1. If b and c be put for such known Quantities, that c is greater than b , but less than $2b$, and if a be put for a Quantity unknown;
 2. And if $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$
 What is the value of a ?

RESOLUTION.

3. Because the Squares of equal Quantities are also equal, by multiplying each part of the Equation in the second step into it self, this is produced, viz.

$$\frac{aa}{2} + \sqrt{a^4 - 9b^2} = \frac{baa}{c}$$

 4. Then to the end the Surd Quantity in the Equation in the third step may solely make one part of an Equation, let $\frac{aa}{2}$ be subtracted from each part of that Equation, and this will remain, viz.

$$\sqrt{a^4 - 9b^2} = \frac{baa}{c} - \frac{aa}{2} = \frac{2baa - caa}{2c}$$

5. And to the end the Radical sign in the first part of the last Equation may vanish, let each part be multiplied by it self, so an Equation in Rational quantities will be produced, viz.

$$a^4 - 9b^2 = \frac{4bb^2a^2 - 4bca^2 + cca^2}{c^2}$$

6. And by reducing the last Equation to a common Denominator $4cc$, and then by multiplying each part by the same $4cc$, this Equation in Integers will be produced, viz.

$$cca^4 - 9b^2cc = 4bb^2a^2 - 4bca^2 + cca^2$$

7. And from the Equation in the last preceding step, after due reduction is made to make those Quantities wherein a^4 is found to possess one part, this following Equation aritheth, viz.

$$4bca^2 - 4bb^2a^2 = 9b^2cc$$

8. Then by dividing each part of the last Equation by $4bc - 4bb$, to the end that a^2 may stand alone, this Equation aritheth, viz.

$$a^2 = \frac{9b^2cc}{4bc - 4bb} = \frac{9b^2cc}{4c - 4b}$$

9. But $\frac{9bbcc}{4}$ into $\frac{b}{c-b} = \frac{9b^2cc}{4c-4b}$

10. That

10. Therefore from the two last preceding Equations, by exchanging equal Quantities, this Equation aritheth, viz.

$$a^2 = \frac{9bbcc}{4} \text{ into } \frac{b}{c-b}$$

11. And by extracting the square Root out of each part of the Equation in the tenth step, this aritheth;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}$$

12. Wherefore by extracting the square Root out of each part of the Equation in the eleventh step, the desired value of a is discovered, viz.

$$a = \sqrt{\frac{3bc}{2}} \text{ into } \sqrt{\frac{b}{c-b}}$$

An Example of Quest. 7. in Numbers.

13. If $b = 16$;
 14. And $c = 25$;
 15. And $a =$ a number unknown;
 16. And if $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$

What is the number a ?

17. Answer. From the thirteenth, fourteenth, and twelfth steps, $a = \sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; Then it will follow, that

$$\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (\approx \sqrt{8/8}, \text{ or } \sqrt{12/12})$$

Note. The numbers to express the values of b and c must not be taken at pleasure but such, that the number c may exceed the number b , and be less than $2b$, as is prescribed in the Question; the former part of which Determination is discovered by the Denominator $c - b$ of the surd Fraction in the twelfth step, and the latter part of the Determination is manifest by the latter part of the Equation in the fourth step; where caa is to be subtracted from $2baa$; which cannot be done so as to leave a Remainder greater than nothing; unless c be less than $2b$.

SECT. XVIII. An Explication of Fran. van Schooten's General Rule, to extract what Root you please out of any Binomial in numbers; having such a Binomial Root as is desired.

Preparation.

First, If the given Binomial hath Fractions in it; it must be freed from them, by multiplying the Binomial by their Denominator. As; for example; to extract $\sqrt[3]{(3)}$, that is; the cubic Root, out of $\sqrt[3]{242 + 12\frac{1}{2}}$, I multiply the Binomial by 2; and it makes $\sqrt[3]{498 + 25}$; for $\sqrt[3]{242}$ multiplied by $\sqrt[3]{4}$, (that is; by 2) produceth $\sqrt[3]{968}$; and $12\frac{1}{2}$ into 2, makes 25. Likewise, if there be propos'd $\sqrt[3]{242 + 12\frac{1}{2}}$, I first multiply it by $\sqrt[3]{5}$; and it makes $\sqrt[3]{1210 + 61\frac{1}{2}}$; then this Binomial multiplied by 2 produceth (as before) $\sqrt[3]{2420 + 125}$; and so of others.

Secondly, If neither of the two parts of the given Binomial be Rational, it must be reduced by Multiplication or Division to another Binomial that shall have one of its parts Rational; which Reduction may always be done by the multiplication of either part; but often times more briefly by the multiplication or division of the lesser number. As; for example, $\sqrt[3]{242 + 12\frac{1}{2}}$ may be multiplied by $\sqrt[3]{4}$, and it makes $242 + \sqrt[3]{8808}$; but more conveniently by $\sqrt[3]{2}$, and there comes forth $22 + \sqrt[3]{486}$. After the same manner; $\sqrt[3]{(3)3993 + \sqrt[3]{(6)17578125}}$ may be first multiplied by $\sqrt[3]{(3)3993}$; and the Product again by $\sqrt[3]{(3)3993}$, so there will be produced another Binomial whose Rational part is the absolute number 3993; but more briefly by $\sqrt[3]{(1)9}$; and there will

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be produced another Binomial whose Rational part is 33; and yet more compendiously, if the Binomial proposed be divided by $\sqrt[3]{(3)}$; there will arise $11 \div \sqrt[3]{125}$.

But here is to be noted, that when one part of a Binomial is Rational, whether it be of a Binomial first given, or of another deduced (as above) from that given, then also the Square of the other part ought to be Rational, otherwise no Root can be extracted out of the Binomial or the other deduced from it.

Thirdly, to extract $\sqrt[3]{(6)}$ out of a given Binomial qualified as above is supposed, we must first extract the Square Root, and then out of this the cubick Root; and to extract $\sqrt[3]{(9)}$, we must first extract $\sqrt[3]{(3)}$, and then out of the cubick Root found out we must again extract $\sqrt[3]{(3)}$; and so of any other Root whose Index is a Composite number. But as to the extraction of the Square Root out of a Binomial, a Rule hath been already given and exemplified in the preceding *Sett.* 16. so that here there is need only that I shew how to extract $\sqrt[3]{(3)}$, $\sqrt[3]{(5)}$, $\sqrt[3]{(7)}$, $\sqrt[3]{(11)}$, and such like, whose Indices are Prime numbers.

Fourthly, to extract $\sqrt[3]{(3)}$, $\sqrt[3]{(5)}$, $\sqrt[3]{(7)}$, or the like Root whose Index is a Prime number, we must first of all try whether out of the given Binomial there can be extracted a binomial Root which hath one part Rational, but that may be discovered by subtracting the Square of the lesser part of the given Binomial from the Square of the greater, and extracting the Root out of the Remainder, to wit, the cubick Root, if $\sqrt[3]{(3)}$ be to be extracted out of the given Binomial; or the Root of the fifth Power, if $\sqrt[3]{(5)}$ be to be extracted, and so of others: For if the Root of the said Remainder be not a Rational number, then the Binomial Root sought will certainly want a Rational part, *viz.* each of its parts will be surd; in which case, in order to extract that Root, the given Binomial must be multiplied by the difference of the Squares of the parts, if the Question be concerning the extraction of the cubick Root; or by the Square of the said difference, if $\sqrt[3]{(5)}$ be sought; or by the Cube of the same difference, if $\sqrt[3]{(7)}$ be required; or by the fifth Power of the said difference, if $\sqrt[3]{(11)}$ be sought, and so of the rest. By which multiplication another Binomial will always be produced, wherein the Root of the difference of the Squares of the parts will be the same with the difference of the Squares of the parts of the former Binomial.

As, to extract the cubick Root out of $25 \div \sqrt[3]{968}$; I first subtract 625, the Square of 25, from 968, the Square of $\sqrt[3]{968}$, and there remains 343, whose cubick Root 7 is a Rational number: which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which hath one of its parts Rational.

Likewise, to extract the cubick Root out of $22 \div \sqrt[3]{486}$, we must subtract 484, the Square of 22, from 486, and extract the cubick Root out of the Remainder 2; but because that cannot be done exactly, it shews that the cubick Root of $22 \div \sqrt[3]{486}$ wants a Rational part; and therefore $22 \div \sqrt[3]{486}$ must be multiplied by the said Remainder 2, that there may be a Binomial $44 \div \sqrt[3]{1944}$, wherein the cubick Root of the difference of the Squares of the parts is 2.

So to extract $\sqrt[3]{(5)}$ out of $11 \div \sqrt[3]{125}$, because 121 the Square of 11 subtracted from 125 leaves 4, which considered as a fifth Power hath not an exact Rational Root, we must multiply $11 \div \sqrt[3]{125}$ by 16 the Square of 4, that there may come forth $176 \div \sqrt[3]{32000}$, where $\sqrt[3]{(5)}$ of the difference of the Squares of the parts is 4.

Again, to extract $\sqrt[3]{(7)}$ out of $338 \div \sqrt[3]{114242}$, wherein the difference of the Squares of the parts is 2, because this 2 is not the seventh Power of any Rational number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it makes $2704 \div \sqrt[3]{7311488}$, wherein the $\sqrt[3]{(7)}$ of the difference of the Squares of the parts is 2.

The RULE.

When a Binomial given, or another deduced from it (if need be) by the precedent Preparation, is such, that one of its parts, and the Square of the other part, as also the Root of the difference of the Squares of the parts, (to wit, the cubick Root when $\sqrt[3]{(3)}$, or $\sqrt[3]{(5)}$ when $\sqrt[3]{(5)}$ is sought) are Rational whole numbers; then out of a Binomial so qualified, $\sqrt[3]{(3)}$, or $\sqrt[3]{(5)}$, or $\sqrt[3]{(7)}$, &c. may be extracted, if it hath such a Root, in manner following, *viz.*

First, extract the Root of the difference of the Squares of the parts of the Binomial qualified as aforesaid, *viz.* the cubick Root, when $\sqrt[3]{(3)}$ is sought; but $\sqrt[3]{(5)}$ when $\sqrt[3]{(5)}$, or $\sqrt[3]{(7)}$ when $\sqrt[3]{(7)}$, &c. which Root so extracted is to be reserved for a Divisor.

Secondly,

Secondly, find out a Rational number a little greater than the Root sought, with this caution, that the Rational number found out may not exceed the said Root above $\frac{1}{2}$, which may easily be done by Vulgar Arithmetick, and take the said Rational number for a Divisor.

Thirdly, divide the said Dividend by the said Divisor, and if the Rational part of the given Binomial be greater than the other part, add the Quotient to the said Rational Divisor, and the half of the greatest whole number contained in the sum shall be the Rational part of the Root sought; then from the Square of that Rational part subtract the Root of the difference of the Squares of the parts, (to wit, the Dividend first found out as above,) so the Remainder shall be the Square of the other part, when such a Root as was required can be extracted out of the given Binomial, which you may easily try, by multiplying this Root found out into it self, according to the degree of the Power represented by the given Binomial: for the Root found out being multiplied into it self cubically, if $\sqrt[3]{(3)}$ was sought, or, five times into it self, if $\sqrt[3]{(5)}$ was sought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be less than the other part, then after you have found out the Quotient as above, subtract it from the Rational Divisor, and the half of the greatest whole number contained in the Remainder shall be the Rational part of the Root sought; to the Square of which part if there be added the Dividend first found out as above, the sum will be the Square of the other part, when the Binomial proposed hath a Root; but by multiplying the Root found out into it self (as before) you may easily try whether it be a true Root or not.

Example 1. To extract the Cubick Root out of $20 \div \sqrt[3]{392}$.

First, the difference of the Squares of the parts of the given Binomial, *viz.* the excess of 400, the Square of 20; above 392, the Square of $\sqrt[3]{392}$ is 8, whose cubick Root 2 I reserve for a Dividend.

Secondly, I seek a Rational number that may be greater than the cubick Root of $20 \div \sqrt[3]{392}$, (the given Binomial,) yet so that the excess may not be greater than $\frac{1}{2}$, to which end I extract the Square Root of 392, and find it to be greater than 19, but less than 20; then to 20 the Rational part of the given Binomial I add 19 and 20 severally, and it makes 39 and 40; which are the nearest Rational whole numbers that can express the true value of the given Binomial; whence the cubick Root thereof will be found greater than 3, but less than $3\frac{1}{2}$; this $3\frac{1}{2}$, which, according to the Caution before given, exceeds the true cubick Root of the given Binomial by an excess not greater than $\frac{1}{4}$, I reserve for a Divisor.

Thirdly, I divide 2, the Dividend before reserved, by the said Divisor $3\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now because 20 the Rational part of the given Binomial is greater than the other part $\sqrt[3]{392}$, I add the said Quotient $\frac{4}{7}$ to the said Divisor $3\frac{1}{2}$, and it makes the sum $4\frac{2}{7}$, wherein the greatest whole number is 4, whose half is 2 the Rational part of the Root sought; by the help of which Rational part, the other part is easily discovered; for if from 4 the Square of the said 2, you subtract 2, the cubick Root of the difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2 \div \sqrt[3]{2}$ is the cubick Root of $20 \div \sqrt[3]{392}$ the Binomial proposed, as will appear by the Proof: For $2 \div \sqrt[3]{2}$ being multiplied into it self cubically produceth $20 \div \sqrt[3]{392}$; and for the same reason, $2 \div \sqrt[3]{2}$ is the cubick Root of $20 \div \sqrt[3]{392}$.

Example 2. To extract the Cubick Root out of $44 \div \sqrt[3]{1944}$.

First, the cubick Root of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the Square Root of 1944 is greater than 44, but less than 45; these added severally to 44 the Rational part of the given Binomial, make 88 and 89, whose cubick Roots being extracted, do shew that the cubick Root of the given Binomial is greater than 4, but less than $4\frac{1}{2}$; this Rational number $4\frac{1}{2}$, which according to the Caution before given exceeds the true Root sought by an excess not greater than $\frac{1}{4}$, I take for a Divisor: Thirdly, I divide the said Dividend 2 by the said Divisor $4\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I subtract from the said $4\frac{1}{2}$; (I subtract, because 44 the Rational part of the given Binomial is less than the other part $\sqrt[3]{1944}$;) and there remains $4\frac{1}{9}$; then the half of $4\frac{1}{9}$, the greatest whole number contained in $4\frac{1}{9}$, is 2, which is the Rational part of the Root sought: Lastly, to 4 the Square of the said 2, I add 2 the cubick Root of the difference of the Squares of the parts, and it makes 6 the Square of the other part. So that $2 \div \sqrt[3]{6}$ is the cubick Root sought, as will appear by the Proof: For if it be multiplied into it self cubically, it

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produceth $44 \div \sqrt{1944}$ the Binomial proposed; and for the same reason, $\sqrt{6} - 2$ is the cubic Root of $\sqrt{1944} - 44$.

Example 3. To extract $\sqrt{(5)}$ out of $176 \div \sqrt{32000}$.

First, the difference of the Squares of the parts will be found 1924 , whose $\sqrt{(5)}$ is 4 for a Dividend: Secondly, the sum of the parts will be found greater than 354 , but less than 355 ; and consequently $\sqrt{(5)}$ of the sum of the parts is greater than 3 , but less than $3\frac{1}{2}$: Thirdly, by the said $3\frac{1}{2}$ I divide the said 4 , and the Quotient is $1\frac{1}{2}$, which I subtract from the said Divisor $3\frac{1}{2}$ (because the Rational part of the given Binomial is less than the other part) and there remains $2\frac{1}{2}$; then the half of 2 (the greatest whole number contained in $2\frac{1}{2}$) is 1 , the Rational part of the Root sought: Lastly, the Square of the said 1 , to wit, 1 , added to 4 (the $\sqrt{(5)}$ of the difference of the Squares of the parts of the given Binomial) makes 5 the Square of the other part. So that $1 \div \sqrt{5}$ is the $\sqrt{(5)}$ of the given Binomial $176 \div \sqrt{32000}$, at least if any $\sqrt{(5)}$ can be extracted out of the same; but $1 \div \sqrt{5}$ multiplied into it self five times makes $176 \div \sqrt{32000}$: therefore $1 \div \sqrt{5}$ is manifestly the desired $\sqrt{(5)}$ of $176 \div \sqrt{32000}$.

Example 4. To extract $\sqrt{(7)}$ out of $2704 \div \sqrt{731488}$.

First, the $\sqrt{(7)}$ of the difference of the Squares of the parts is 2 for a Dividend; Secondly, the value of the given Binomial will be found greater than 5407 , but less than 5408 , whence the $\sqrt{(7)}$ thereof will be discovered to be greater than 3 , but less than $3\frac{1}{2}$: Thirdly, by the said $3\frac{1}{2}$ I divide the Dividend before found 2 , and the Quotient is $\frac{2}{3}$, which I add to the Divisor $3\frac{1}{2}$ (because the Rational part 2704 is greater than the other part) and it makes the sum $4\frac{1}{3}$; and therefore $\frac{2}{3}$, the half of the greatest whole number contained in $4\frac{1}{3}$, is the Rational part of the Root sought: Lastly, from 4 , the Square of the said $\frac{2}{3}$ I subtract 2 , to wit, $\sqrt{(7)}$ of the difference of the Squares of the parts of the given Binomial, and there remains 2 the Square of the other part. So that $2 \div \sqrt{2}$ is the desired $\sqrt{(7)}$ of the given Binomial $2704 \div \sqrt{731488}$; for this is the seventh Power of $2 \div \sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial hath been multiplied or divided by some number, and thereby reduced to another Binomial, and the Root of this latter is found out, we must divide or multiply the Root found out by the Root of the number by which the Binomial was multiplied or divided; so there will come forth the Root of the given Binomial.

As, for example, because to extract the cubic Root out of $\sqrt{242} \div 12\frac{1}{2}$, we first multiplied this Binomial by 2 and found $25 \div \sqrt{968}$, whose cubic Root by the Rule before given will be found $1 \div \sqrt{8}$; this must be divided by $\sqrt{(3)2}$, and the Quotient $\sqrt{(3)2} \div \sqrt{(6)128}$ shall be the cubic Root of $\sqrt{242} \div 12\frac{1}{2}$ the Binomial proposed.

But that the reason of the said Division by $\sqrt{(3)2}$ may the more clearly appear, let there be put $d = 1 \div \sqrt{8}$, then it follows that $ddd = 25 \div \sqrt{968}$, and $\frac{ddd}{2} = \sqrt{242} \div 12\frac{1}{2}$ (the Binomial proposed.) Therefore by extracting the cubic Root out of each part of the last Equation, there ariseth $\sqrt{(3)2} \frac{ddd}{2}$, that is, $\frac{d}{\sqrt{(3)2}} = \sqrt{(3)2} \div \sqrt{242} \div 12\frac{1}{2}$: But by supposition $d = 1 \div \sqrt{8}$, therefore $1 \div \sqrt{8}$ divided by $\sqrt{(3)2}$, that is to say, $\sqrt{(3)2} \div \sqrt{(6)128}$ shall be the cubic Root of $\sqrt{242} \div 12\frac{1}{2}$: which was to be shewn.

Example 2. To extract $\sqrt{(3)}$ out of $\sqrt{242} \div \sqrt{242}$.

First, to prepare it for extraction, we multiplied by $\sqrt{5}$, and found $\sqrt{242} \div 12\frac{1}{2}$, whose $\sqrt{(3)}$ (as appears in the last preceding Example) is $\sqrt{(3)2} \div \sqrt{(6)128}$, which divided by $\sqrt{(6)5}$ gives the Quotient $\sqrt{(6)32} \div \sqrt{(6)128}$ for the desired cubic Root of $\sqrt{242} \div \sqrt{242}$. The reason of which division by $\sqrt{(6)5}$ may be thus manifested, let there be put $d = \sqrt{(3)2} \div \sqrt{(6)128}$; then it follows that $ddd = \sqrt{242} \div 12\frac{1}{2} = \sqrt{242} \div \sqrt{242}$ into $\sqrt{5}$, whence $\frac{ddd}{\sqrt{5}} = \sqrt{242} \div \sqrt{242}$; therefore the cubic Root

of each part of the last Equation being extracted there ariseth $\sqrt{(3)2} \frac{ddd}{\sqrt{5}}$, that is, $\frac{d}{\sqrt{(6)5}}$ (for $\sqrt{(3)}$ of $\sqrt{5}$ is $\sqrt{(6)5}$) $= \sqrt{(3)2} \div \sqrt{(6)128} \div \sqrt{242}$: But by supposition, $d = \sqrt{(3)2} \div \sqrt{(6)128}$.

$d = \sqrt{(3)2} \div \sqrt{(6)128}$; therefore $\sqrt{(3)2} \div \sqrt{(6)128}$ divided by $\sqrt{(6)5}$ gives the cubic Root of $\sqrt{242} \div \sqrt{242}$: which was to be shewn.

Example 3. To extract $\sqrt{(3)}$ out of $\sqrt{242} \div \sqrt{242}$.

First, (according to the second Rule of the precedent Preparation) I multiply it by $\sqrt{2}$, and there comes forth $22 \div \sqrt{486}$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44 \div \sqrt{1944}$, whose cubic Root (as before hath been shewn) is $2 \div \sqrt{6}$, which must be divided by $\sqrt{2}$ and there will come forth $\sqrt{2} \div \sqrt{3}$ for the cubic Root sought of $\sqrt{242} \div \sqrt{242}$. But to manifest the reason of dividing $2 \div \sqrt{6}$ by $\sqrt{2}$; let there be put $d = 2 \div \sqrt{6}$, then it follows that $ddd = 44 \div \sqrt{1944} = 22 \div \sqrt{486}$ into 2 , whence $\frac{ddd}{2} = 22 \div \sqrt{486}$; and this Equation divided by $\sqrt{2}$

(because in the Preparation we multiplied by $\sqrt{2}$) gives $\frac{ddd}{\sqrt{2}} = \sqrt{242} \div \sqrt{242}$; therefore $\sqrt{(3)}$ being extracted out of each part of the last Equation there ariseth $\sqrt{(3)2} \frac{ddd}{\sqrt{2}}$, that is, $\frac{d}{\sqrt{2}}$, or $\frac{d}{\sqrt{2}} = \sqrt{(3)2} \div \sqrt{242} \div \sqrt{242}$: But by supposition, $d = 2 \div \sqrt{6}$; therefore $2 \div \sqrt{6}$ divided by $\sqrt{2}$, viz. the Quotient $\sqrt{2} \div \sqrt{3}$, shall be the cubic Root of $\sqrt{242} \div \sqrt{242}$: which was to be shewn.

Example 4. To extract $\sqrt{(5)}$ out of $\sqrt{(3)3993} \div \sqrt{(6)17578125}$.

First, (according to the second preparatory Rule) I divide the given Binomial by $\sqrt{(3)3}$, and then (according to the fourth preparatory Rule) I multiply the Quotient $\sqrt{(3)1331} \div \sqrt{(6)1953125}$ by 16 , and there comes forth $176 \div \sqrt{32000}$, whose $\sqrt{(5)}$ (as hath been shewn) is $1 \div \sqrt{5}$. Now this Root $1 \div \sqrt{5}$ divided by $\sqrt{(5)16}$; and the Quotient multiplied by $\sqrt{(5)3}$ will discover the true $\sqrt{(5)}$ of $\sqrt{(3)3993} \div \sqrt{(6)17578125}$, the reason of which Division and Multiplication may be made manifest thus; let there be put $d = 1 \div \sqrt{5}$, then it follows that $ddd = 176 \div \sqrt{32000}$, and by dividing each part of the last Equation by 16 , (because in the preparatory work we multiplied by 16) there ariseth $\frac{ddd}{16} = \sqrt{(3)1331} \div \sqrt{(6)1953125}$; and by

multiplying each part of this Equation by $\sqrt{(3)3}$, there will be produced $\frac{ddd}{16} \times \sqrt{(3)3} = \sqrt{(3)3993} \div \sqrt{(6)17578125}$: Therefore $\sqrt{(5)}$ being extracted out of each part of the last Equation, there will arise $\sqrt{(5)3} \frac{ddd}{16}$, that is, $\frac{d}{\sqrt{(5)16}}$ equal to $\sqrt{(5)}$ of $\sqrt{(3)3993} \div \sqrt{(6)17578125}$: But by supposition, $d = 1 \div \sqrt{5}$; therefore $1 \div \sqrt{5}$ multiplied into $\sqrt{(5)3}$, and the Product divided by $\sqrt{(5)16}$, or $1 \div \sqrt{5}$ divided by $\sqrt{(5)16}$, and the Quotient multiplied by $\sqrt{(5)3}$ produceth the true $\sqrt{(5)}$ of $\sqrt{(3)3993} \div \sqrt{(6)17578125}$: which was to be shewn.

The Demonstration followeth.

The certainty of the preceding Rule will be made manifest by the three following Propositions.

PROPOSITION 1.

If a Binomial whereof one part and the Square of the other are Rational numbers &c. multiplied into it self cubically, there will be produced another Binomial, the Square of whose lesser part being subtracted from the Square of the greater part; leaves a cubic number, to wit, the Cube of the difference of the Squares of the parts of the Root or first Binomial.

To make this manifest, let there be proposed the Binomial $b \div \sqrt{d}$, this multiplied into it self cubically produceth $bbb \div 3bb\sqrt{d} \div 3bd \div d\sqrt{d}$, to wit, the Cube of $b \div \sqrt{d}$. Here you are to note well, that although in that Cube there be four parts or members, yet they are to be esteemed but as two, one of which, to wit, $bbb \div 3bd$ may design a Rational number, and the other, $3bb\sqrt{d} \div d\sqrt{d}$ (or $bb \div d \times \sqrt{d}$) an irrational or surd number whose Square is Rational, whence it is manifest, first, that the Cube of a Binomial is also a Binomial, viz. $b \div \sqrt{d}$ multiplied into it self cubically produceth this Binomial

Binomial $bbb + 3bd$ more $3bb/d + d/d$ (or $3bb + d \times d$); secondly, the Rational part $bbb + 3bd$ is manifestly composed of the Cube of the Rational part of the Root and of the triple Product made by the multiplication of the same Root into the Square of its other part; and lastly, the difference of the Squares of the said parts $bbb + 3bd$ and $3bb/d + d/d$ is equal to the Cube of $bb - d$, or of $d - bb$, viz. to the Cube of the difference of the Squares of the parts of the Root $b + d$: For the Squares of $bbb + 3bd$ and $3bb/d + d/d$ are $bbbbb + 6bbbd + 3bbdd$ and $9bbbd + 3bbdd + ddd$, and if these Squares be subtracted one from the other, the Remainder is either $bbbbb - 3bbbd + 3bbdd - ddd$, which is the Cube of $bb - d$, or else the Remainder is $ddd - 3bbdd + 3bbbd - bbb$, which is the Cube of $d - bb$.

To illustrate this Proposition by Numbers, let there be put $b = 2$, and $d = 6$; hence the Binomial $2 + \sqrt{6}$ multiplied into itself cubically produceth the Binomial $44 + \sqrt{1944}$, wherein the difference of the Squares of the parts (viz. the Remainder when 1936 the Square of 44 is subtracted from 1944 the Square of $\sqrt{1944}$) is 8 , to wit, the Cube of the difference of the Squares of the parts of the binomial Root $2 + \sqrt{6}$.

Likewise this Binomial $2 + \sqrt{2}$ multiplied into itself cubically produceth the Binomial $20 + \sqrt{392}$, wherein the difference of the Squares of the parts, to wit, 8 , is the Cube of the difference of the Squares of the parts of the Root $2 + \sqrt{2}$.

The same properties adhere also to a Residual Root, viz. the Cube of the Residual Root $b + d$ is also a Residual, to wit, $bbb + 3bd$ is $3bb/d + d/d$, (or $3bb + d \times d$), and the difference of the Squares of the parts of the latter Residual is equal to the Cube of the difference of the Squares of the parts of the Root or first Residual.

PROP. 2.

If a Binomial whereof one part and the Square of the other are Rational numbers, be multiplied by the difference of the Squares of the parts, the Product will be another Binomial, wherein the difference of the Squares of the parts is a Cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root multiplied.

To make this manifest, let there be proposed the Binomial $b + \sqrt{d}$, and suppose b greater than \sqrt{d} ; then $b + \sqrt{d}$ multiplied by $bb - d$, the difference of the Squares of the parts, will produce this Binomial, to wit, $bbb - bd$ more $bb/d - d/d$, the Squares of whose parts are $bbbbb - 2bbbd + bbdd + 3bbbd - 3bbdd - ddd$; then this latter Square subtracted from the former leaves $bbbbb - 3bbbd + 3bbdd - ddd$, which is the Cube of $bb - d$ the difference of the Squares of the parts of the first Binomial $b + \sqrt{d}$. The same property would appear if we supposed b less than \sqrt{d} .

To illustrate this Proposition by Numbers, suppose $b = 22$, and $d = 486$; where the Binomial $22 + \sqrt{486}$ multiplied by 2 , the difference of the Squares of the parts, produceth the Binomial $44 + \sqrt{1944}$; wherein the difference of the Squares of the parts is 8 , which is the Cube of 2 , the difference of the Squares of the parts of the former Binomial $22 + \sqrt{486}$.

PROP. 3.

If the difference of the Squares of any two numbers be divided by a number which doth not exceed the sum of those two numbers above $\frac{1}{2}$; then the Quotient added to the said Divisor will give a number greater than the double of the greater of the said two numbers, but the excess will be less than unity: and if the said Quotient be subtracted from the said Divisor, the Remainder shall be greater than the double of the lesser of the two numbers, but this excess also shall be less than unity.

To manifest this, let a represent the greater of two numbers, and e the lesser; also, let b represent some Fraction not greater than $\frac{1}{2}$: then I say, first, $a + e + b + \frac{aa - ee}{a + e + b}$ is greater than $2a$; but the excess is less than 1 , which I prove thus:

It is evident that $aa + ee + bb + 2ae + 2be + 2ba + aa - ee$ is greater than $2aa + 2ae + 2be$; therefore by dividing each of those two Compound quantities by $a + e + b$, it follows that the first Quotient $a + e + b + \frac{aa - ee}{a + e + b}$ shall be greater than the latter

Quotient $2a$; and if this quantity be subtracted from that, the Remainder $\frac{2be + bb}{a + e + b}$ will be less than 1 . For by supposition b is not greater than $\frac{1}{2}$; therefore $2be$ is less than $a + e$.

$a + e$, and bb less than b ; and consequently the Numerator $2be + bb$ is less than the Denominator $a + e + b$: wherefore $\frac{2be + bb}{a + e + b}$ is less than 1 .

After the same manner it may be proved that $a + e + b - \frac{aa - ee}{a + e + b}$ is greater than $2e$; but this excess also shall be less than 1 : which was to be shewn.

Now to apply the preceding three Propositions to the Demonstration of the Rule before given, let it be required to extract the cubick Root out of the Binomial $100 + \sqrt{7803}$, whose Rational part 100 is greater than the other part $\sqrt{7803}$. Here we may suppose $bbb + 3bd$ to be 100 , and $3bb/d + d/d$ (or $3bb + d \times d$) to be $\sqrt{7803}$; so that $bbb + 3bd$ more $3bb/d + d \times d$ may design the given Binomial $100 + \sqrt{7803}$; and its Cubick root $b + \sqrt{d}$ the Root sought, whose greater part may be b , and the lesser \sqrt{d} : Then, according to the Rule

To extract $\sqrt{(3)}$ out of $100 + \sqrt{7803}$:

First, from the Square of 100 , that is, from	10000
Subtract the Square of $\sqrt{7803}$, that is,	7803
The Remainder is	2197
The Cubick root of that Remainder is	13 (= $bb - d$)

Which Root 13 is (by Prop. 1.) equal to the difference of the Squares of the parts of the Binomial root sought.

Secondly, find out a Rational number greater than the sum of the parts of the Cubick root sought, with this Caution, that the excess may not be above $\frac{1}{2}$, viz.

To the greater part of the given Binomial, that is, to	100
Add the nearest value in whole numbers of the other part	88 or 89
$\sqrt{7803}$, that is,	
So the sum shews, that the value in whole numbers of the	188 and 189
given Binomial falls between	

Whence the Cubick root of the given Binomial is greater than $5\frac{1}{2}$, but less than 6 ; so that the excess of 6 above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out (as above) 13 the true difference of the Squares of the parts of the Cubick root sought, and 6 a Rational number which exceeds not the true sum of the same parts above $\frac{1}{2}$; we may by the help of Prop. 3, and 1. find out the parts severally in this manner, viz.

Divide the said	13
By the said	6
And the Quotient is	$2\frac{1}{6}$
Which added to the said Divisor 6 , makes the sum	$8\frac{1}{6}$

Which sum $8\frac{1}{6}$ doth (by Prop. 3.) exceed the double of the greater (to wit, the Rational) part of the Cubick Root sought, but the excess is less than 1 ; therefore $7\frac{1}{2}$ is less than the said double, but $8\frac{1}{6}$ is greater than the same: and consequently, because the said greater part is supposed to be a Rational whole number, the double thereof must necessarily be 8 , (to wit, the greatest whole number between $7\frac{1}{2}$ and $8\frac{1}{6}$) and therefore the said part itself is 4 ; which being found out, it is easy to find the other part. For, (by Prop. 1.) if from 16 the Square of the said greater part 4 , there be subtracted 13 , the Cubick root of the difference of the Squares of the parts of the given Binomial, there will remain 3 , the Square of the other part; so that the Cubick root found out is $4 + \sqrt{3}$, which will appear by the Proof to be the true Root sought; for $4 + \sqrt{3}$ being multiplied into itself cubically produceth the given Binomial $100 + \sqrt{7803}$. And for the same reason $4 - \sqrt{3}$ is the Cubick root of $100 - \sqrt{7803}$.

Or more briefly, the Proof may be made thus.

To the Cube of 4 the Rational part of the Root found out,	64, that is, bbb
viz. to	
Add the Product of thrice that part multiplied into the	36, that is, $3bd$
Square of the Surd part found out, viz. the Product	
And it makes the sum	100, that is, $bbb + 3bd$

Which

Which sum is the same with the Rational part of the given Binomial, and therefore it proves that $4 + \sqrt[3]{3}$ is the Cubick root sought.

In like manner, to extract $\sqrt[3]{(3)}$ out of $44 + \sqrt[3]{1944}$, where the Rational part 44 is less than the other part $\sqrt[3]{1944}$; we may suppose (as before) $bbb + 3bd$ to be 44, and $3bb + d + \sqrt[3]{d}$ (that is, $3bb + d + \sqrt[3]{d}$) to be $\sqrt[3]{1944}$; so that $bbb + 3bd$ more $3bb + d + \sqrt[3]{d}$ may design the given Binomial $44 + \sqrt[3]{1944}$, and its Cubick root $b + \sqrt[3]{d}$ the Root sought, whose lesser part may be b ; and the greater $\sqrt[3]{d}$. Then, according to the Rule

To extract $\sqrt[3]{(3)}$ out of . . . $44 + \sqrt[3]{1944}$.

First, from the Square of $\sqrt[3]{1944}$, viz. from . . . $\sqrt[3]{1944}$
 Subtract the Square of 44, . . . $\sqrt[3]{1936}$
 The Remainder is . . . $\sqrt[3]{8}$
 The Cubick root of that Remainder is . . . $\sqrt[3]{2} (= d - bb)$

Which Root 2 is (by Prop. 1.) equal to the difference of the Squares of the part of the Binomial root sought.

Secondly, find out a Rational number greater than the sum of the parts of the Cubick root sought, with this Caution, that the excess may not be above $\frac{1}{2}$, which may be done thus, viz.

To the lesser part of the given Binomial, viz. to . . . $\sqrt[3]{44}$
 Add the nearest value in whole numbers of the other . . . $\sqrt[3]{44}$ or 45
 part $\sqrt[3]{1944}$, that is, . . . $\sqrt[3]{44}$ or 45
 So the sum shews that the value in whole numbers of the . . . $\sqrt[3]{88}$ and 89.
 given Binomial, falls between . . . $\sqrt[3]{88}$ and 89.

Whence the Cubick root of the given Binomial is greater than 4, but less than $4\frac{1}{2}$, so that the excess of $4\frac{1}{2}$ above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out 2, the true difference of the Squares of the parts of the Cubick root sought, and $4\frac{1}{2}$ a Rational number which doth not exceed the true sum of the same parts above $\frac{1}{2}$; we may by the help of Prop. 3, and 1. find out the parts severally in this manner, viz.

Divide the said . . . $\sqrt[3]{4\frac{1}{2}}$
 By the said . . . $\sqrt[3]{4\frac{1}{2}}$
 And it gives the Quotient . . . $\sqrt[3]{4\frac{1}{2}}$
 Which subtracted from the said Divisor $4\frac{1}{2}$, there remains . . . $\sqrt[3]{4\frac{1}{2}}$

Which Remainder $4\frac{1}{2}$ doth (by Prop. 3.) exceed the double of the lesser part (which in this Example is the Rational part) of the Cubick root sought, but the excess is less than 1; Therefore $3\frac{1}{2}$ is less than the said double; but $4\frac{1}{2}$ is greater than the same; and consequently because the said lesser part is a Rational whole number, the double thereof must necessarily be 4, to wit, the greatest whole number between $3\frac{1}{2}$ and $4\frac{1}{2}$, and therefore the said part it self is 2: which being found, it is easy to find the other part; for if to the Square of the said lesser part 2, there be added 2 the Cubick root of the difference of the Squares of the parts of the given Binomial, the sum 6 shall be the Square of the other part. So that the Cubick root found out is $2 + \sqrt[3]{6}$, which will appear to be the true Cubick root sought; for $2 + \sqrt[3]{6}$ multiplied into it self cubically produceth the given Binomial $44 + \sqrt[3]{1944}$. And for the same reason $\sqrt[3]{6} - 2$ is the Cubick root of $\sqrt[3]{1944} - 44$.

Or more briefly, the Proof may be made thus:

To the Cube of 2, the Rational part of the Root found . . . $\sqrt[3]{8}$, that is, $\sqrt[3]{8}$
 out, viz. to . . . $\sqrt[3]{8}$
 Add the Product of thrice that part multiplied into the . . . $\sqrt[3]{36}$, that is, $\sqrt[3]{36}$
 Square of the Surd part found out, viz. the Product . . . $\sqrt[3]{36}$
 And the sum is . . . $\sqrt[3]{44}$, that is, $\sqrt[3]{44}$

Which sum is the same with the Rational part of the given Binomial; and therefore it proves that $2 + \sqrt[3]{6}$ is the Cubick root sought.

Lastly, what hath here been shewn concerning the Demonstration of the Extraction of the Cubick Root, may easily be applied to the Extraction of the other Roots before mentioned, so that there is no need of farther discourse in this matter.

CHAP.

C H A P. X.

An Explication of Simon Stevin's General Rule, to extract one Root out of any possible Equation in Numbers, either exactly, or very nearly true.

Equations falling under any of the Forms in the fourteenth and fifteenth Chapters of the first Book of these Elements, are capable (as hath there been shewn) of perfect Resolutions in Numbers; viz. the value of the Root or Roots sought in any of those Equations may be found out and exprest exactly, either by some Rational or Irrational number or numbers; but the perfect Resolution of all manner of Compound Equations in numbers, I have not found in any Author: and since an Exposition of the General Method of Vieta, the Rules of Huddenius and others to that purpose, would make a large Treatise, and after all leave the curious Analyst dissatisfied; I shall not clogg these Elements with a tedious discourse upon those difficult Rules, which at the best are exceeding tedious in Operation, and some of them uncertain too, but rather pursue my first Design, which was to explain Fundamentals, and such Rules as are certain and most important in this profound Art. However, I shall lead the industrious Learner a few steps farther in order to his understanding the Resolution of all manner of Compound Equations in numbers; and in this Chapter explain Simon Stevin's General Rule, which with the help of the Rules in the following eleventh Chapter, will discover all the Roots of any possible Equation in numbers, either exactly, if they be Rational, or very nearly true if Irrational.

QUEST. 1.

If . . . $aaa + 26a = 40188$, what is the number a ?

RESOLUTION.

This Equation not falling under any of the three Forms in Sect. 1. Chap. 15. Book 1. cannot be resolved by any of the Canons in that Chapter, and therefore according to Simon Stevin's general Method I search out the number a by trials, thus, viz.

1. I suppose . . . $a = 1$
 Thence it follows that . . . $aaa = 1$
 And . . . $26a = 26$
 Therefore . . . $aaa + 26a = 27$
 Which 27 ought to have been 40188, but it's too little; whereby I find that by supposing a to be 1, I did not hit upon the true number a , and therefore I make another trial, in like manner as before, viz.

2. I suppose . . . $a = 10$
 Thence it follows that . . . $aaa = 1000$
 And . . . $26a = 260$
 Therefore . . . $aaa + 26a = 1260$
 Which 1260 being yet too little, I make a third trial, viz.

3. I suppose . . . $a = 100$
 Thence it follows, that . . . $aaa + 26a = 1002600$

Which 1002600 exceeds the just Result or absolute number 40188 in the latter part of the Equation first propos'd, and therefore the true number a is less than 100; But the second trial shews it to be greater than 10; and therefore the whole number which expresteth the exact, or at least part of the value of a , must necessarily consist of two Characters, and consequently the first (towards the left hand) must be one of these nine, 1; 2; 3; 4; 5; 6; 7; 8; 9; but because by the second Inquiry 10 was found too little, I now make trial with 2 for the first figure of the Root a , viz.

4. I suppose . . . $a = 20$
 Thence . . . $aaa + 26a = 8520$
 Which Result 8520 being yet less than the just Result 40188, I make trial again, viz.
 5. I suppose . . . $a = 30$
 Thence . . . $aaa + 26a = 27780$

L I

Which

Which is yet too little; therefore,

6. I suppose $a = 40$

Thence $aaa - 26a = 65040$
Which 65040 being greater than 40188, it shews me that the true Root or value of a is less than 40; but by the fifth trial is greater than 30, and consequently the first figure of the Root is 3.

Now the second Character of the Root must necessarily be one of these, *viz.* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and because it hath been discovered that the true value of the Root a is greater than 30, the second Character cannot be 0, I therefore make trial with 1, and suppose $a = 31$, which proving too little, I make trial with 32, 33, 34, &c. severally, in like manner as before, and at length I find 34 to be the true number a sought, by which the Equation propos'd may be expounded; for if $a = 34$, then consequently $aaa - 26a = 40188$.

11. But if after trials made (as before) the value of a the Root sought happens to fall between two whole numbers that differ by Unity; then trials are to be made with the lesser whole number increased with $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. until you have found the value of a in some mixt number consisting of a whole number and some certain tenth parts of an Unit; But if the said value of a happens not to be express'd exactly by the said lesser whole number increased with certain tenth parts, then you are to make trials with the said lesser whole number increased with a decimal Fraction having for its Numerator a number greater than 10, but less than 100, and for its Denominator 100, as with $\frac{11}{100}$, $\frac{111}{1000}$, &c. and by proceeding in that manner you may find the exact value of the Root a when its fractional part is exactly equal to some decimal Fraction, or else approach infinitely near to the said exact value when 'tis irrational or surd, as in this following

QUEST. 2.

If $aaaa - 50a = 184638.6801$; (or, $184638\frac{6801}{10000}$;) what is the number a ?

RESOLUTION.

First, I suppose $a = 1$, but this proving too little I put $a = 10$, this also proving too little, I assume $a = 100$, which after trial I find to be greater than the true number a , and consequently the number a falls between 10 and 100; then making trial with 20 I find it too little, but making trial with 30 I find this too great, and therefore the true Root a falls between 20 and 30. Again, making trial with 21 I find it too great, but 20 was before found too little, therefore the true Root a is between 20 and 21; then I make trial with 20.1, (that is, $20\frac{1}{10}$;) 20.2; 20.3, &c. and at length find 20.7 to be the true number a sought; for if $a = 20.7$ (that is, $20\frac{7}{10}$;) it will make $aaaa - 50a = 184638.6801$ the Equation propos'd.

But if 20.7 had proved too little, and 20.8 too great, then trials must have been made with 20.71, (that is, $20\frac{71}{100}$;) 20.72; 20.73, &c. In like manner if 20.7 had been too little, but 20.71, (that is, $20\frac{71}{100}$;) too great, then trials must have been made with 20.701, (that is, $20\frac{701}{1000}$;) 20.702; 20.703, &c. This will be partly exercis'd in resolving the Equation in this following

QUEST. 3.

If $aaa - 20aa = 1954$, what is the number a ?

Ans. $a = 8.308$, &c. found out by trials, as before.

111. When the value of (a) the required Root of an Equation happens to be less than Unity, then trial is to be made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now suppose 1 (that is, $\frac{1}{1}$;) to be too great, but .01 (that is, $\frac{1}{100}$;) too little; then trial must be made with .02 | .03 | .04 | &c. until you have found out the greatest figure that must stand in the second place of the decimal Fraction expressing the Root sought; supposing then such figure to be found 8, *viz.* that .08 (or $\frac{8}{100}$;) is less, but .09 (or $\frac{9}{100}$;) is greater than the Root, trial must be made with .081, (that is, $\frac{81}{1000}$;) .082 | .083 | &c. as in this following

QUEST. 4.

If $aaa - 324aa = 269$, what is the number a ?

Ans. $a = .083$, &c. that is, $\frac{83}{1000}$, &c.

IV. The

IV. The preceding Examples may suffice to shew the use of this General Method when all the Terms of the unknown part of an Equation are Affirmative; (*viz.* when $+$ is prefix'd to each Term,) in which case there is but one Affirmative Root; in the search whereof by trials (as before) if the numbers assumed severally for the value of the Root sought do ascend greater and greater, then the Absolute numbers resulting from those assumed values will likewise ascend; and contrarily, if the assumed Roots do descend from a greater to a less, the Results will likewise grow less and less; whence by comparing an Absolute number resulting from an assumed Root with the just Absolute number of the Equation propos'd, you may certainly know (if the said Result and just Absolute be not equal to one another) whether you are to take a number greater or less than that last before assumed.

But when the unknown part of an Equation consists of affirmative and negative Terms mingled one with another, then the search by trials will be more intricate and doubtful than before; for sometimes it will be hard to discern whether a following assumed Root must be taken greater or less than that which was taken next before. Moreover, a Compound Equation of this latter kind may happen to be such, that it may be expounded by as many several affirmative Roots as there be Unities in the Index of the highest unknown Power, *viz.* a Cubical Equation may be so constituted that it shall have three different affirmative Roots, a Biquadratic Equation four several Roots; and so of higher Equations, as will be shewn in the following Chap. 11. But in what manner soever any possible Equation is continued in Rational numbers, this general Method will always find out one affirmative Root, either exactly true, or at least very near the truth, as will farther appear by the following Questions.

QUEST. 5.

If $aaa - 22aa - 157a = 360$; what is the number a ?

RESOLUTION.

1. I suppose $a = 1$

Thence it follows that $aaa - 22aa - 157a = 136$

Which 136 is less than the just absolute number 360; and therefore I make another trial, *viz.*

2. I suppose $a = 10$

Thence it follows that $aaa - 22aa - 157a = 370$

Which 370 exceeds the just absolute number 360, and therefore I conclude there is one affirmative value of a , (either rational or irrational) between 1 and 10, which value, after trials made with 2, 3, 4, 5; I find to be 5; this will constitute the Equation propos'd; for if $a = 5$, then $aaa - 22aa - 157a$ will exactly make 360.

But there are two other Roots or values of a , to wit, 8 and 9, each of which will likewise constitute the Equation first propos'd; but how they are found out will be shewn in Sect. 9. of the following Chap. 11.

QUEST. 6.

If $3200a - aaa = 46577$ (just,) what is the number a ?

RESOLUTION.

1. I suppose $a = 1$

Thence $3200a - aaa = 3199$ (less than just.)

2. I suppose $a = 10$

Thence $3200a - aaa = 31000$ (less than just.)

3. I suppose $a = 100$

Thence $3200a - aaa = 680000$ (less than just.)

Now because the second Result (or absolute number) -31000 is Affirmative, and the last Result -680000 is Negative, I make trials with numbers between 10 and 100 for the value of a , for if the Equation propos'd be possible, before the affirmative Results fall off to negatives, there will be a Root or value of a producing an affirmative Result either exactly equal, or very near to the just Result 46577; therefore,

4. I suppose $a = 20$

Thence $3200a - aaa = 56000$ (greater than just.)

Let 2

Now

Now because by taking 20 for the value of a , the Result 56000 exceeds the just Result 46577; but by taking 10 for a , the Result 31000 happened to be less than the said 46577; it shews there is one affirmative Root or value of a between 10 and 20, which Root, after tryals made with intermediate numbers (as in former Examples) will be found 15.7, &c. Moreover, because by supposing $a = 20$, the Result 56000 happened to exceed the just Result 46577, but by putting $a = 100$ the Result — 68000 proved to be less than the same 46577, it shews there is an Affirmative value of a between 20 and 100, which value after tryals made will be found 47: so that there are two affirmative Roots or values of a found out, to wit, 15.7, &c. (or 15.73, &c.) and 47; the former of which will nearly, and the latter exactly constitute the Equation proposed.

V. *Florismond de Beaune* in the latter of two small Treatises printed in 1659, concerning the Nature, Constitution and Limits of Equations, shews how to find out Limits within which the Roots of all compound Equations not ascending above the Biquadratic kind are confined; which Limits when they may be discovered without much trouble, and are not very wide asunder, will help to lessen the tryals in the general Method before delivered: As, in the last Example, where

The Equation proposed was } $3200a - aaa = 46577$
 First, because aaa must be subtracted from 3200a } $aaa \approx 3200a$
 and leave a Remainder equal to 46577, it presupposeth }
 Therefore by dividing each part by a , } $aa \approx 3200$
 And by extracting the square Root out of each part, } $a \approx 56.5, \&c.$
 it follows that }
 Again, from the Equation propos'd, by transpo- } $3200a - 46577 = aaa$
 sition 'tis evident that }
 Whence 'tis also manifest that } $3200a \approx 46577$
 And consequently by dividing each part by 3200, } $a \approx 14.5, \&c.$

Thus it is found that the value of a the Root sought is greater than 14.5, &c. but less than 56.5, &c. and therefore tryals according to the general Method aforesaid need not be made with any numbers that are not within those Limits.

From the premises 'tis evident, that this general Method finds not a perfect Root of an Equation, unless such Root be a whole number, or else a Fraction exactly equal to some decimal Fraction; or lastly, a mixt number compos'd of a whole number and a perfect decimal Fraction.

Note. When the Coefficients or known numbers multiplied into any of the unknown Powers under the highest, (which must have no Coefficient but Unity), are vulgar (not decimal) Fractions, or mixt numbers whose fractional parts are vulgar Fractions; likewise, when the Absolute number that solely possesseth the latter part of the Equation propos'd is a vulgar Fraction, or mixt number whose fractional part is a vulgar Fraction; all those vulgar Fractions must be reduced to decimal Fractions, or else the Equation must be reduced to another Equation in Integers (by *Self. 7.* in the following *Chapt. 11.*) before you enter upon the Resolution by tryals as aforesaid.

CHAP. XI.

Extractions out of the Algebraical Treatises of Vieta and Renates des Cartes, concerning the Constitution and Resolution of Compound Equations in Numbers; especially those which have many Roots.

1. **T**HE scope of this Chapter is, first, to shew how to form an Equation that shall have as many different Roots or values of the Quantity sought as shall be desired; then how to free an Equation from Fractions, and to cast away the second Term; and lastly, how to find out the Roots of all manner of Compound Equations in numbers, either exactly, if they be Rational, or very near the truth if irrational.

BUT

But that the Learner may the more easily perceive my meaning, I shall premise a few Definitions in three Sections next following.

11. When the known Absolute number in an Equation solely possesseth one part thereof, let it be transferr'd to the other part by the sign —, and then there will be an Equation which hath 0 or nothing for one part, and the other part is by *Cartesius* called the Summ of the Equation proposed. As, for example, if this Equation be proposed, *viz.* $aaa - 9aa + 26a = 24$, by transposition of 24 it makes $aaa - 9aa + 26a - 24 = 0$, whose first part is called the Summ of the Equation proposed.

111. In the Equations handled in this Chapter, I put a , e or y to signify an unknown Quantity; and by the first Term of an Equation is meant the highest unknown Power, to wit, that which hath most Dimensions or Degrees of a ; by the second Term that which hath fewer Dimensions by one than the first, and so downwards. As in this Equation, $aaa - 9aa + 26a - 24 = 0$, the first Term is aaa , whose Index is 3; the second Term is $-9aa$, where the Index of aa is 2; the third Term is $+26a$, where the Index of a is 1; and the last Term is -24 , the known Absolute number whose Index is 0.

IV. The Roots of an Equation are of three kinds, *viz.* either Affirmative, or Negative, or Impossible: an affirmative Root is a quantity greater than nothing, as $+5$ or $+20$: a negative Root (which *Cartesius* calls a false Root) expresseth a quantity whose Denomination is opposite to an affirmative, as -5 , or -20 : the former of which wants 5, and the latter 20 of being equal to nothing: lastly, impossible Roots are such whose values cannot be conceived or comprehended either Arithmetically or Geometrically: As in this Equation, $a = 2 - \sqrt{-1}$, where $\sqrt{-1}$, that is, the square Root of -1 is no manner of way intelligible, for no number can be imagined, which being multiplied by it self according to any Rule of Multiplication, will produce -1 .

V. These things premised, I shall proceed to the forming of Equations which shall have many Roots.

PROP. I.

To form an Equation which shall have two Affirmative Roots.

1. Suppose } $a = 2$, that is, $a - 2 = 0$
 } $a = 3$, that is, $a - 3 = 0$
2. Then by multiplying the said $a - 2 = 0$ by }
 $a - 3 = 0$, this Equation is produced, *viz.* } $aa - 5a + 6 = 0$
3. That is, by transposition, } $5a - aa = 6$

Which last Equation falls under the last of the three Forms in *Self. 1. Chap. 15. Book 1.* and may be expounded by either of two Roots or values of a , which by the Canon in *Self. 10.* of the same *Chapt.* will be found 2 and 3, to wit, those from which the said Equation was produced by Multiplication, as above.

Again, if this Equation $aa - 5a + 6 = 0$, (that is, $aa + 6a = 55$), which hath one affirmative Root, to wit, 5, be multiplied by $a - 6 = 0$, there will be produced $aaa - 91a + 330 = 0$, (that is, $91a - aaa = 330$), which hath two affirmative Roots or values of a , to wit, 5 and 6, which may be found out by the Rule hereafter delivered in *Self. 9.* of this *Chapt.*

PROP. II.

To form an Equation which shall have one Affirmative, and one Negative Root.

1. Suppose } $a = 3$; that is, $a - 3 = 0$
 } $a = -2$, that is, $a + 2 = 0$
2. Then by multiplying the said $a - 3 = 0$ by }
 $a + 2 = 0$, this Equation is produced, *viz.* } $aa - a - 6 = 0$
3. That is, } $aa - a = 6$

Which last Equation falls under the second of the three Forms in *Self. 1. Chap. 15. Book 1.* and may be expounded by either of two Roots or values of a , whereof one is Affirmative, and the other Negative; which, after the manner of resolving *Quest. 1.* in *Self. 7.* of the same *Chapt.* will be found $+3$ and -2 , to wit, those from which the said Equation was produced by Multiplication, as before.

PROP.

PROP. III.

To form an Equation which shall have three Affirmative Roots.

1. Suppose $\left. \begin{array}{l} a = 2, \text{ that is, } a - 2 = 0 \\ a = 3, \text{ that is, } a - 3 = 0 \\ a = 4, \text{ that is, } a - 4 = 0 \end{array} \right\}$
2. Then by multiplying the three last Equations (in each of which the latter part is 0) one into another, this Equation will be produced, $aaa - 9aa + 12a - 24 = 0$
3. That is, by transposition of -24 , . . . $aaa - 9aa + 12a = 24$

Which Equation may be expounded by every one of these three affirmative Roots, of values of a , to wit, 2, 3 and 4; which may be found out by the Rule in the following Self. 9. of this Chapter.

The same Equation may likewise be formed altogether by Letters, thus, viz. Let the said known Roots 2, 3 and 4 be represented by b, c, d ; and then

4. Suppose $\left. \begin{array}{l} a = b, \text{ that is, } a - b = 0 \\ a = c, \text{ that is, } a - c = 0 \\ a = d, \text{ that is, } a - d = 0 \end{array} \right\}$
5. Then by multiplying those three last Equations, in each of which the latter part is nothing, one into another, this Equation will be produced, viz.

$$\begin{array}{rcl} & \left. \begin{array}{l} -b \\ -c \\ -d \end{array} \right\} aaa & \left. \begin{array}{l} +bc \\ +bd \\ +cd \end{array} \right\} aa \\ \text{That is, . . . } & aaa - 9aa & + 12a - 24 = 0 \end{array}$$

PROP. IV.

To form an Equation which shall have three Affirmative Roots, and one Negative Root.

1. Suppose $\left. \begin{array}{l} a = 2, \text{ that is, } a - 2 = 0 \\ a = 3, \text{ that is, } a - 3 = 0 \\ a = 4, \text{ that is, } a - 4 = 0 \\ a = -5, \text{ that is, } a + 5 = 0 \end{array} \right\}$
2. Then by multiplying the four last Equations (in each of which the latter part is 0,) one into another, this following Equation will be produced, viz.

$$\begin{array}{rcl} & & aaa - 4aaa - 19aa + 106a - 120 = 0 \\ \text{That is, . . . } & & aaa - 4aaa - 19aa + 106a = 120 \end{array}$$

Which last Equation may be expounded by every one of these three affirmative Roots, or values of a , viz. 2, 3 and 4; and by one negative Root -5 ; every one of which may be found out by the Rule in the following Self. 9. of this Chapter.

The same Equation may likewise be formed altogether by Letters, thus, viz. Let the said known Roots, 2, 3, 4 and -5 be represented by b, c, d and $-f$; then

3. Suppose $\left. \begin{array}{l} a = b, \text{ that is, } a - b = 0 \\ a = c, \text{ that is, } a - c = 0 \\ a = d, \text{ that is, } a - d = 0 \\ a = -f, \text{ that is, } a + f = 0 \end{array} \right\}$
4. Then by multiplying the four last Equations, in each of which the latter part is 0, one into another, this following Equation will be produced, viz.

$$\begin{array}{rcl} & \left. \begin{array}{l} -b \\ -c \\ -d \\ +f \end{array} \right\} aaaa & \left. \begin{array}{l} +bc \\ +bd \\ +cd \\ -cf \end{array} \right\} aa \\ \text{That is, . . . } & aaaa - 4aaa & - 19aa + 106a - 120 = 0 \end{array}$$

After the same manner you may form an Equation which shall have as many Roots as you please, either all Affirmative, or some of them Affirmative and some Negative.

V. I. Obf.

V. I. Observations upon the preceding four Propositions.

1. By what hath been said 'tis evident, that sometimes an Equation may have as many Roots as there be unities in the Index of the highest unknown Term; I say sometimes, not always: for although this Equation $aaa - 6aa + 13a - 10 = 0$, as to its number of Terms and Signs be like to the Equation formed in the preceding Prop. 3. so that one may think it hath three Roots, yet it hath only one affirmative Root, to wit, 2, and no other Root either Affirmative or Negative can constitute the said Equation, for 'tis produced by the multiplication of this impossible Equation $aa - 4a + 5 = 0$ by $a - 2 = 0$; but that $aa - 4a + 5 = 0$, that is, $4a - aa = 5$, is an impossible Equation, the Determination in Self. 9. Quest. 1. Chap. 15. Book 1. makes manifest.

In like manner, although this Equation $aaaa - 60aaa + 1650aa - 22500a + 115344 = 0$, as to its number of Terms and Signs be like to an Equation that hath four affirmative Roots, yet that Equation can be expounded only by two affirmative Roots, to wit, 12 and 18, and by no other Root either Affirmative or Negative; for 'tis made by the multiplication of $aa - 30a + 216 = 0$, which hath two affirmative Roots, 12 and 18, into this impossible Equation $aa - 30a + 534 = 0$.

2. For as much as Division resolves or undoes that which is compos'd or done by Multiplication, if the sum of any Equation which is produced by the multiplication of two or more Equations one into another, (according to the Method in the preceding four Propositions) be divided by a Binomial compos'd of the unknown quantity (a) less by the value of any one of the affirmative Roots, or more by the value of one of the negative Roots, the Quotient shall be an Equation in which the first Term hath fewer Dimensions by one than the first Term of the Equation so divided: And if the Quotient be divided in like manner, there will come forth an Equation whose first Term hath fewer Dimensions by one than the former Quotient. As, for example, let there be proposed the Equation in the preceding Prop. 4. to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, which was made by the continual multiplication of $a - 2 = 0$, $a - 3 = 0$, $a - 4 = 0$, $a + 5 = 0$; I say, if the Equation proposed be divided by any one of those Binomials $a - 2$, $a - 3$, $a - 4$, $a + 5$, the Quotient will be an Equation wherein the first Term hath only three Dimensions, which are fewer by one than those in $aaaa$ the first Term of the Equation proposed: So if the said $aaaa - 4aaa - 19aa + 106a - 120 = 0$ be divided by $a - 2 = 0$, there will arise $aaa - 2aa - 23a + 60 = 0$, as you see by the subsequent Division.

$$\begin{array}{r} a - 2 \quad aaaa - 4aaa - 19aa + 106a - 120 \quad (aaa - 2aa - 23a + 60) \\ \underline{aaaa - 2aaa} \\ -2aaa - 19aa \\ \underline{-2aaa + 4aa} \\ -23aa + 106a \\ \underline{-23aa + 46a} \\ +60a - 120 \\ \underline{+60a - 120} \\ 0 \end{array}$$

Likewise if the Quotient, to wit, the Equation $aaa - 2aa - 23a + 60 = 0$, where the first Term aaa hath three Dimensions, be divided by $a - 3 = 0$, there will arise $aa + a - 20 = 0$; whose first Term aa hath but two Dimensions: And lastly, if the said latter Quotient $aa + a - 20 = 0$ be divided by $a - 4 = 0$, there will come forth a simple Equation, to wit, $a + 5 = 0$, that is, the negative Root $a = -5$.

The like Division may be practised with the literal Equations at the latter end of Prop. 3, and 4. in the preceding Self. 5.

3. If a complete Equation, that is, such in which all the Terms are extrant, be produced by the multiplication of possible Equations one into another, you may discover how many affirmative, and how many negative Roots that Equation hath, by this Rule, viz. As often as — follows next after +, or + next after —; so often there is an affirmative Root; and as often as two signs — or two signs + stand next to one another, so often there is a negative Root: As, for example, in this Equation, (before formed in Prop. 4.)

to wit,

to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, because next after the first Term $-4aaa$ there follows $-4aaa$, it shews there is one affirmative Root; and because next after $-4aaa$ there comes $-19aa$, it shews that the Equation hath one negative Root; again, because next after $-19aa$ there follows $+106a$, it shins there is another affirmative Root; and because next after $+106a$ there follows -120 , it shews there is a third affirmative Root; so that the said Rule discovers the Equation propos'd to have three affirmative Roots, and one negative Root.

4. It is also manifest from the manner of forming Equations according to the Propositions in the preceding *Self*. 5. that in every Equation which hath as many affirmative Roots as there be Dimensions in the first Term, the Coefficient or known quantity in the second Term is equal to the sum of all the affirmative Roots; and the known quantity in the third Term is equal to the sum of the Products of every two of the said Roots multiplied one by the other; and the known quantity in the fourth Term is equal to the sum of the Products of every three of the said Roots, and so forward when there be more Terms, but the last Term, to wit, the Absolute quantity given is equal to the Product of all the Roots multiplied one into another: As in the following Equation (before formed in *Prop*. 3.) viz.

$$\begin{array}{rcl} & -b & \} +bc \\ aaa & -c & \} aa \\ & -d & \} +cd \\ \text{That is, } & -gaa & +16a - 24 = 0 \end{array}$$

First, the sum of 2, 3 and 4, (that is, of b, c, d) the three Roots of that Equation is 9, which is the known number of the second Term $-gaa$; secondly, the sum of the Products of every two of the said Roots multiplied one by the other is 24, that is, $+bc + bd + cd$, which is the known Coefficient of the third Term $+16a$, or $+bc + bd + cd$ into a ; and lastly, the Product of all the three Roots multiplied one into another is 24, or $-bcd$, to which prefixing $-$ it makes -24 , or $-bcd$ the last Term of the Equation propos'd.

The like Properties ensue when the sum of the numbers of multitude of affirmative and negative Roots is equal to the number of Dimensions in the first Term of an Equation, saving that here, in summing up all the Roots which compose the known quantity in the second Term, and likewise the Products which compose the known quantities in the following Terms, respect must be had to the Rules of Addition of $+$ and $-$ in such manner as the Equation propos'd if it be formed altogether by letters will direct; as you may easily perceive by the Equations formed in *Prop*. 4. of the preceding *Self*. 5.

VII. How to free an Equation from Fractions, when 'tis incumberd therewith in the second, third or any of the following Terms, which work is by Vieta called Homoceria.

The Rules in *Chap*. 12. *Book* 1. shew how to reduce an Equation, so, as that the first Term may have no Coefficient but unity; but if after any Equation is so reduced there happens to be any Fraction in the second, third, or any of the following Terms, such Equation may be reduced to another whose Terms shall be all Integers, by the Method in the five Examples next following.

Example 1.

1. Let this Equation be propos'd to be reduced to another in Integers, viz.

$$aaa + \frac{1}{2}a = 225$$

Operation.

- Suppose $e = 2a$, (2a because 2 is the Denominator of the Fraction $\frac{1}{2}$.)
- Then divide each part of the last Equation by 2 (the Denominator aforesaid) and there ariseth
- And by multiplying each part of the Equation in the third step cubically, there comes forth
- Again, by multiplying each part of the Equation in the third step by $\frac{3e}{4}$, (the Fraction in the second Term of the Equation first propos'd,) it makes

6. Then

- Then add the two last Equations into one, and the sum is
- But by supposition in the first step
- Therefore from the two last Equations, (by 1. *Axiom*. 1. *Elem*. Euclid.)
- Which last Equation being reduced to Integers, (by *Self*. 2. *Chap*. 12. *Book* 1.) gives

Therefore an Equation is found out which is altogether express'd by Integers, and when the value of e in the last Equation is discovered, the value of a in the Equation propos'd is consequently known; for by the third step $a = \frac{3e}{4}$; therefore if e be 12, then a shall be 6.

Example 2.

Again, if this Equation be propos'd, it may be reduced in like manner as before in Example 1. to this, viz.

And if e be 10, then a shall be 5.

Example 3.

So likewise this Equation May be reduced to this

And if e be 10, then a is 5.

Example 4.

1. Again, let there be propos'd

Operation.

- Suppose $e = 12a$, (12a, because 12 is the Denominator of the Fraction $\frac{1}{12}$ in the second Term.)
- Then divide each part of the last Equation by 12 (the Denominator aforesaid,) and there ariseth
- And by multiplying cubically the last Equation, it produceth
- And by multiplying the Equation in the third step by $\frac{11e}{144}$, it makes
- And by adding the two last Equations into one, the sum makes
- But by the Equation propos'd,
- Therefore from the two last Equations (by 1. *Axiom*. 1. *Elem*. Euclid.)
- Which Equation reduced to Integers gives

Thus an Equation is found out in Integers, and when the value of e is discovered, the value of a in the Equation propos'd is consequently known; for by supposition in the second step, e is to a as 12 to 1: therefore if e be 12, then a shall be 1.

Example 5.

1. Again, let there be propos'd

Operation.

- Suppose $e = 6a$, (6a, because 6 is the Denominator of the Fraction $\frac{1}{6}$.)
- Then by dividing each part of the last Equation by 6, there ariseth
- And by squaring the last Equation it makes
- Likewise by squaring each part of the last Equation, there will be produced,
- And by multiplying the Equation in the fourth step by that in the third, the Product is

M m

7. And

7. And by multiplying the last Equation by 10, it gives?
this, *viz.* $\frac{10eee}{216} = 10aaa$
8. And by multiplying the Equation in the fourth step?
by $45\frac{1}{2}$, it produceth $\frac{275ee}{216} = 45\frac{1}{2}aaa$
9. And by multiplying the Equation in the third step?
by $104\frac{1}{2}$, the Product will be $\frac{625e}{36} = 104\frac{1}{2}aaa$
10. Then by connecting the Quantities which stand in the first parts of the Equations in the fifth, seventh, eighth and ninth steps, together with 89, by the same signs which respectively belong to each Term of the Equation proposed, the sum shall be equal to the sum of the same Equation, and consequently equal to nothing; hence this Equation ariseth, *viz.*

$$\frac{eee}{1296} - \frac{10eee}{216} + \frac{275ee}{216} - \frac{625e}{36} + 89 = 0$$

11. Which Equation being reduced to Integers (by Sect. 7. Chap. 11. Book 1.) gives

$$eee - 60eee + 165cee - 22500e + 115344 = 0.$$

Thus an Equation is found out whose Terms are all Integers; and the value of the Root e in this Equation is to the value of the Root a in the Equation proposed as 6 to 1; (for, by supposition in the second step, $e = 6a$;) and therefore if e be 12, then a shall be 2; or if e be 18, then a shall be 3.

VIII. How to take away the second Term of a Compound Equation.

The Rule is this; Divide the Coefficient, (that is, the known Quantity) multiplied into the second Term of an Equation proposed, by the Index (or number of Dimensions) of the Power which is the first Term: Then if the signs of the first and second Terms be unlike, (*viz.* if one be + and the other —,) subtract the Quotient from the affirmative Root sought; but if the signs be like, (that is, both + or both —,) add the said Quotient to the affirmative Root: Then equate the said Summ or Remainder to some letter to represent an unknown Quantity, and proceed according to the Method in the following Examples; so far length a new Equation will arise, wherein the second Term is wanting.

Example 1.

1. Let there be proposed this Equation $aa - 6a = 72$
2. That is $aa - 6a - 72 = 0$
3. Here the number of Dimensions in the first Term aa is 2, and the known number multiplied into a making the second Term $6a$ is 6; this divided by the said 2 gives 3, which subtracted from the Root a , (because the signs of the first and second Terms are unlike,) leaves $a - 3$ which is equal to some unknown number, let it be e ; then
4. By supposition $a - 3 = e$
5. And consequently, by adding 3 to each part of that Equation, there ariseth $a = e + 3$
6. And by squaring each part of the last Equation, there comes forth $aa = ee + 6e + 9$
7. And by multiplying each part of the Equation in the fifth step by the Coefficient 6 in the proposed Equation, it makes $6a = 6e + 18$
8. Then by subtracting the last Equation from that in the sixth step, there remains $aa - 6a = ee - 9$
9. And lastly, by subtracting 72 (the last Term of the Equation propos'd) from the Equation in the eighth step, there remains $aa - 6a - 72 = ee - 81 = 0$

Thus you see an Equation is found out, to wit, $ee - 81 = 0$, which is equal to the Equation propos'd, and it wants the second Term; (for there is not any number of e in the Equation found out;) now if the value of e be made known, then the value of a is consequently known: But the Equation found out, to wit, $ee - 81 = 0$, that is, $ee = 81$ gives $e = 9$; and by the fifth step $a = e + 3$; therefore $a = 12$.

Example 1.

Example 2.

1. Again, let there be proposed this Equation, *viz.* $aa + 6a = 216$
2. That is, $aa + 6a - 216 = 0$
3. Here (as before) I divide 6, the Coefficient in the second Term $6a$, by 2, which denotes the number of Dimensions in the first Term aa , and the Quotient 3 I add to the Root a , (because the first and second Terms of the Equation have the same sign +) and the sum $a + 3$ is equal to some unknown number, let it be e ; then
4. By supposition $a + 3 = e$
5. Therefore by subtracting 3 from each part of that Equation, there ariseth $a = e - 3$
6. And by squaring the last Equation, there comes forth $aa = ee - 6e + 9$
7. And by multiplying the Equation in the fifth step by 6, it produceth $6a = 6e - 18$
8. Then by adding the two last Equations into one, the sum is $aa + 6a = ee - 9$
9. And by subtracting 216 (the last Term of the Equation propos'd) from each part of the Equation in the eighth step, there remains $aa + 6a - 216 = ee - 225 = 0$

Thus an Equation is found out, to wit, $ee - 225 = 0$, which wants a second Term, (for there is no number of e in that Equation,) and when the value of e is made known the value of a in the Equation propos'd is known also; but the Equation $ee - 225 = 0$, that is, $ee = 225$ gives $e = 15$; and by the fifth step, $a = e - 3$; therefore $a = 12$ that is, $15 - 3$.

Example 3.

1. Again, let this Equation be propos'd, $aaa - 18aa - 7a + 696 = 0$
2. According to the Rule before given, I divide 18 the known number of the second Term $-18aa$, by 3, which denotes the number of Dimensions in the first Term aaa , and the Quotient is 6, this I subtract from the Root a , (because the signs of the first and second Terms are unlike,) and the Remainder is $a - 6$, which is equal to some unknown number, suppose it to be e ; then
3. By supposition $a - 6 = e$
4. Therefore by adding 6 to each part of that Equation, there ariseth $a = e + 6$
5. And by squaring the last Equation it makes $aa = ee + 12e + 36$
6. And by multiplying the two last Equations one by the other, the Product is $aaa = ee + 18ee + 108e + 216$
7. And by multiplying the Equation in the fifth step by 18, (the Coefficient in the second Term of the Equation propos'd,) it makes $18aa = 18ee + 216e + 648$
8. Likewise, the Equation in the fourth step being multiplied by 7, (the Coefficient in the third Term of the Equation propos'd,) produceth $7a = 7e + 42$
9. Then to the Equation in the sixth step adding 696, (to wit, the last Term of the Equation propos'd,) the sum is $aaa + 696 = ee + 18ee + 108e + 912$
10. Likewise by adding the eighth Equation to the seventh, it makes $18aa + 7a = 18ee + 225e + 690$
11. Lastly, by subtracting the Equation in the tenth step from that in the ninth, this following Equation remains; *viz.* $aaa - 18aa - 7a + 696 = ee - 115e - 222 = 0$.

Thus an Equation is found out, to wit, $eee - 115e - 222 = 0$, which wants the second Term, (to wit, the Power ee ;) and when the value of the Root e is made known, the value of the Root a shall be known also: For by the fourth step, $a = e + 6$; therefore if e be 2, then a shall be 8; and if e be equal to $\sqrt{112} - 1$, then a shall be equal to $\sqrt{112} - 5$.

M m 2

Example 4.

Example 4.

1. Again, let there be proposed . . . $aaaa + 6aaa + 11aa + 6a - 100 = 0$
2. According to the Rule before given, I divide 6 the Coefficient in the second Term $+ 6aaa$, by 4, which denotes the number of Dimensions in the first Term $aaaa$, and the Quotient is $\frac{3}{2}$, which I add to the Root a , (because the signs of the first and second Terms are like) and the sum is $a + \frac{3}{2}$, which is equal to some unknown number, let it be e , then
3. By supposition . . . $a + \frac{3}{2} = e$
4. Therefore . . . $a = e - \frac{3}{2}$
5. The Square of the last Equation is . . . $aa = ee - 3e + \frac{9}{4}$
6. And the two last Equations multiplied one by the other, make . . . $aaa = eee - \frac{3}{2}ee + \frac{3}{2}e - \frac{27}{4}$
7. And the Equation in the sixth step being multiplied by that in the fourth step, will produce . . . $aaaa = eeee - 6eee + \frac{27}{2}ee - \frac{27}{2}e + \frac{81}{4}$
8. And the Equation in the sixth step multiplied by 6 produceth . . . $6aaa = 6eee - \frac{27}{2}ee + \frac{27}{2}e - \frac{27}{2}$
9. And the Equation in the fifth step multiplied by 11 produceth . . . $11aa = 11ee - 33e + \frac{27}{2}$
10. And the Equation in the fourth step multiplied by 6 gives . . . $6a = 6e - 9$
11. Now 'tis manifest, that if from the sum of the first parts of the four last Equations there be subtracted 100, the Remainder will be equal to the sum of the Equation first propos'd equal to 0; therefore also if 100 be subtracted from the sum of the latter parts of the said four Equations the Remainder shall be equal to 0, viz. $eeee - \frac{3}{2}ee - 99\frac{1}{2}e = 0$.
12. In which last Equation, the second Term, to wit, the Power eee is wanting, as was desired: And when the value of e is made known, the value of the Root a in the Equation propos'd shall be known also, for by the fourth step $a = e - \frac{3}{2}$, but (by the Canon in *Self. 8. Chap. 15. Book 1.*) the value of e in the Equation in the eleventh step will be found $\sqrt{1\frac{1}{4} + \sqrt{101}}$: and therefore $a = \sqrt{1\frac{1}{4} + \sqrt{101}} - \frac{3}{2}$.

IX. The use of the preceding Rules of this Chapter, in the Resolution of all manner of Compound Equations in Numbers.

After an adjected or Compound Equation different from any of the three Forms in *Self. 1. Chap. 15. Book 1.* is prepared for Resolution by the Rules of *Chap. 12. Book 1.* and reduced (if need be) to Integers, and the sum of all the Terms made equal to 0, (or nothing,) according to *Self. 7.* and 2. of this *Chapt.* search out (by the Rules of *Chap. 8.* of this *Book*) all the just Divisors to the last Term (that is, the known Absolute number of the Equation so reduced. Then try whether any one of those Divisors connected to the unknown Root a by $-$ or $+$ will divide the total sum of the said reduced Equation without leaving a Remainder; for when such Division succeeds, either the known part of the said Binomial Divisor is the desired value of the Root a ; or at least the Quotient gives an Equation whose first Term hath fewer Dimensions by one than the Equation divided; and then the Root of this new Equation, if its first Term be a Square, may be found out by some of the Canons in *Self. 6, 8, 10.* of *Chap. 15. Book 1.* But if the first Term contains three or more Dimensions, let this Equation be examined by Division, (as before) and if none of those Divisions work off just without a Fraction, then by taking away the second Term, (by the Rule in *Self. 8.* of this *Chapt.*) another Equation more simple; and such as may be resolved by some of the Canons in *Self. 6, 8, 10. Chap. 15. Book 1.* will sometimes arise: But if none of those ways prove effectual, you may by the general Method in the foregoing *Chapt. 10.* find out one Affirmative Root very near a true Root, and then joining this Root found out to the unknown Root a by the sign $-$, you may by this Binomial divide the Equation, and proceed to find out the rest of the Roots very near the truth: all which will be made manifest by the following Questions.

QUEST. 11

QUEST. 1.

If . . . $aaa - 9aa + 26a = 24$
That is, if . . . $aaa - 9aa + 26a - 24 = 0$ } What is the number a ?

RESOLUTION.

First, (by the Method in *Self. 5. Chap. 8.* of this *Book*) I search out all the numbers that will severally divide the last Term 24 without a Remainder, and find them to be these, viz. 1, 2, 3, 4, 6, 8, 12, 24. Then, by examining in order whether the total sum of the Equation propos'd may be divided by $a - 1$, or $a + 1$; by $a - 2$, or $a + 2$; &c. I find it may be exactly divided by $a - 2$ without a Remainder, and the Quotient is $aa - 7a + 12$, as you see by this following Division.

$$\begin{array}{r}
 a-2 \quad aaa - 9aa + 26a - 24 \quad (aa - 7a + 12 \\
 \underline{aaa - 2aa} \\
 -7aa + 26a \\
 \underline{-7aa + 14a} \\
 +12a - 24 \\
 \underline{+12a - 24} \\
 0
 \end{array}$$

Therefore 2 the known number in the Divisor $a - 2$ is one Real or Affirmative Root of the Equation propos'd; for as well the Divisor as the Dividend was supposed equal to nothing, viz. $a - 2 = 0$, whence $a = 2$; the Quotient also is consequently equal to 0, viz. $aa - 7a + 12 = 0$; that is, $7a - aa = 12$; hence (by the Canon in *Self. 10. Chap. 15. Book 1.*) two other Affirmative values of the Root a will be discovered, to wit, 4 and 3. So that three Real values of a , to wit, 2, 3 and 4 are found out, by every one of which the Equation propos'd may be expounded, as the Proof will easily shew.

QUEST. 2.

If . . . $aaa - 22aa + 157a = 360$
That is, if . . . $aaa - 22aa + 157a - 360 = 0$ } What is a ?

RESOLUTION.

First, the Divisors of the last Term 360 will be found these, viz. 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180 and 360; then by examining in order whether the sum of the Equation propos'd may be divided by $a - 1$, or $a + 1$; by $a - 2$, or $a + 2$; by $a - 3$, or $a + 3$; &c. I find that $a - 5$ will precisely divide the said sum without a Fraction, and therefore 5 is one Affirmative Root or value of a , then the Quotient $aa - 17a + 72 = 0$, that is, $17a - aa = 72$ affords two other Affirmative values of a , to wit, 8 and 9. Thus you see three Real values of a ; two other Affirmative values of a , by every one of which the Equation propos'd, to wit $aaa - 22aa + 157a - 360 = 0$ may be expounded, as will appear by the Proof.

QUEST. 3.

If . . . $91a - aaa = 330$
That is, if . . . $aaa - 91a + 330 = 0$ } What is a ?

RESOLUTION.

First, the Divisors of the last Term 330 will be found 1, 2, 3, 5, 6, 10, 11, 15, 22, 30, 55, 66, 110, 165 and 330; then by examining in order whether the sum of the Equation propos'd, to wit, $aaa - 91a + 330$ may be divided by $a - 1$, or $a + 1$; by $a - 2$, or $a + 2$; &c. I find it may be divided by $a - 5$ and leave no Remainder; therefore $a - 5 = 0$ gives $a = 5$, which is one Affirmative Root of the Equation propos'd, and the Quotient $aa + 5a - 66 = 0$, that is, $aa + 5a = 66$ affords another Affirmative value of a , to wit, 6. So that two Real values of a are found out, by each of which the Equation propos'd may be expounded; for if $a = 5$; or $a = 6$, from either supposition it follows that $91a - aaa = 330$.

QUEST. 4.

To find two numbers whose sum shall be 5, and that if the sum of their Squares be multiplied by the sum of their Cubes, the Product may be 455.

RESOL.

RESOLUTION.

This Question may be solved by the Canon of *Quest. 13. Chap. 16. Book 1.* but that Canon being raised from Positions that lye out of the common Road, I shall here solve the Question in the ordinary way, and so it will exercise the preceding Rules of this Chapter. First then,

1. For one of the numbers sought put a
2. Therefore the other number is $5 - a$
3. The Square of the first number is aa
4. The Square of the second is $aa - 10a + 25$
5. The sum of those Squares is $2aa - 10a + 25$
6. The Cube of the first number is aaa
7. The Cube of the second is $-aaa + 15aa - 75a + 125$
8. Therefore the sum of those Cubes is $+15aa - 75a + 125$
9. Which sum being multiplied by the sum of the Squares in the fifth step gives this Product, viz. $30aaa - 300aaa + 1375aa - 3125a + 3125$.
10. But according to the Question, the Product in the last step must be equal to the given Product 455, hence this Equation ariseth,

$$30aaa - 300aaa + 1375aa - 3125a + 3125 = 455.$$

11. And by subtracting 455 from each part of the last Equation, this ariseth,

$$30aaa - 300aaa + 1375aa - 3125a + 2670 = 0.$$

12. And by dividing every Term in the last Equation by 30, this ariseth,

$$aaa - 10aaa + 45\frac{1}{2}aa - 104\frac{1}{6}a + 89 = 0.$$

13. Then by supposing $e = 6a$, and proceeding according to the Example 5. in *Self. 7.* of this *Chapt.* to free the Equation in the preceding twelfth step from Fractions, this will be produced, viz.

$$eee - 6eee + 165ce - 250ce + 115344 = 0.$$

14. Now the Divisors of the last Term 115344 will be found 1, 2, 3, 4, 6, 8, 9, 12, 15, 24, 27, &c. and after trials made by Division (like as in the three last preceding Questions,) I find that $e - 12 = 0$ will precisely divide the sum of the Equation in the thirteenth step, and therefore 12 is one true value of e . Again, the Quotient of that Division being $eee - 48ee + 1074e - 9612$, I seek the Divisors of the last Term, 9612, and find them to be 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, &c. Then after trials made (as before) I find that $e - 18$ will exactly divide the said $eee - 48ee + 1074e - 9612$, and therefore 18 is one other Affirmative value of e ; and because the Quotient of the last mentioned Division, to wit, $ee - 3ce + 534 = 0$, that is, $3ce - ee = 534$, is an impossible Equation, (as is evident by the Determination in *Self. 9. Quest. 1. Chap. 15. Book 1.*) I conclude that the Equation in the thirteenth step hath no other Root or value of e besides 12 and 18 before found. But because by supposition in the thirteenth step, $e = 6a$, $\frac{1}{2}$ of 12 and likewise of 18, that is, 2 and 3 shall be the true values of a to solve the Question; for their sum is 5, and if 13 the sum of their Squares be multiplied by 35 the sum of their Cubes, the Product is 455, as was desired.

Sometimes the taking away of the second Term of an Equation (by the Rule in *Self. 8.* of this *Chapt.*) will be an expedient to find out an Equation resolvable by some of the Canons in *Self. 6, 8 and 10. Chap. 15. Book 1.* when trials by Division (as before) will be in vain, as will appear by the following fifth Question, which I find resolved two manner of ways in Page 319. of *Cartesius* his Geometry, set forth with Comments by *Fran. van Schooten*, and printed at *Amsterdam* in 1659.

QUEST. 5.

To find four numbers in Arithmetical Progression continued, such, that their common difference may be unity, and the Product made by their continual multiplication 100.

RESOLUTION.

1. For the first number put a
2. Then the second shall be $a + 1$
3. The third $a + 2$
4. And the fourth $a + 3$
5. Therefore the Product of their continual multiplication is $aaaa + 6aaa + 11aa + 6a$

6. Which

6. Which Product must be equal to 100, $aaaa + 6aaa + 11aa + 6a = 100$ therefore $aaaa + 6aaa + 11aa - 6a - 100 = 0$
7. That is $aaaa + 6aaa + 11aa - 6a - 100 = 0$
8. Of which Equation the last Term 100 may be divided by 1, 2, 4, 5, 10, 20, 25, 50 and 100, but Division being tried by $a -$ or $+1$, by $a -$ or $+2$, by $a -$ or $+4$, &c. it proves ineffectual. Then by taking away the second Term, (as in Example 4. *Self. 8.* of this *Chapt.*) this Equation ariseth, viz. $eeee - 2\frac{1}{2}ee - 99\frac{1}{2}e = 0$, in which the Root e , (by the Canon in *Self. 8. Chap. 15. Book 1.*) will be found equal to $\sqrt{1\frac{1}{4}} + \sqrt{101}$: but in taking away the second Term, a was put equal to $e - \frac{1}{2}$, and therefore $a = \sqrt{1\frac{1}{4}} + \sqrt{101} - \frac{1}{2}$, and consequently from the first, second, third and fourth steps,

The four numbers sought are these,

$$\begin{cases} \sqrt{1\frac{1}{4}} + \sqrt{101} - \frac{1}{2} \\ \sqrt{1\frac{1}{4}} + \sqrt{101} - \frac{1}{2} \\ \sqrt{1\frac{1}{4}} + \sqrt{101} - \frac{1}{2} \\ \sqrt{1\frac{1}{4}} + \sqrt{101} - \frac{1}{2} \end{cases}$$

Which numbers exceed one another by Unity; and the Product of their multiplication is 100, as before hath been proved in *Quest. 3. Self. 17. Chap. 9.* of this *Book.*

Another way of Resolving Quest. 5.

For the first number put $a = 1\frac{1}{2}$, for the second $a = \frac{1}{2}$, for the third $a = -\frac{1}{2}$; and for the fourth $a = 1\frac{1}{2}$; then by multiplying these four numbers one into another, and comparing the Product to 100, this Equation ariseth, viz. $aaaa - 2\frac{1}{2}aa = 99\frac{1}{2}$; whence the four numbers sought will be found the same as before.

QUEST. 6.

1. If $8a^3 + 63aa - a^4 - 341a = 1304$;
 2. That is, If $a^4 - 8a^3 - 63aa + 341a - 1304 = 0$;
- What is the number a ?

RESOLUTION.

3. The Divisors of the last Term 1304 are 1, 2, 4, 8, 163, 326 and 1304; then after trials made by Division (as in the preceding Questions,) I find that $a - 8 = 0$ will exactly divide the sum of the Equation proposed without any Remainder, and therefore 8 is one Affirmative value of the Root a . Again, because the Divisors of 163 the last Term of this Equation $aaa - 63a - 163 = 0$ (which was the Quotient of the said Division) are only Unity and 163, I try to divide the Equation last mentioned by $a - 1$ and $a + 1$, likewise by $a - 163$ and $a + 163$, but none of these Divisions working off just without a Fraction, and there being no second Term to be taken away, I search out one Affirmative value of a out of the said Equation $aaa - 63a - 163 = 0$; (that is, $aaa - 63a = 163$;) by the general Method in the foregoing *Chap. 10.* and thereby discover $a = 9.0055$, &c. then I divide the said Cubick Equation $aaa - 63a - 163 = 0$; by $a - 9.0055 = 0$, and the Quotient (the Remainder after the Division is ended being neglected) is $aa + 9.0055a + 18.09903025 = 0$, but this Equation cannot possibly have any affirmative Root, and therefore I conclude that the Equation first proposed to be resolved hath only two affirmative Roots or values of a , to wit, 8 and 9.0055, &c. found out as above.

By the like Operation it will appear that this Equation $a^4 - 17a^3 - 312aa + 4979a - 2113 = 0$ may be expounded by every one of these three Roots or values of a , to wit, 11, 7.1125, &c. and 15.8874, &c. but by no other affirmative Root.

When the Index of the Power of the unknown Quantity in every Term of an Equation is an even number, the Resolution of such Equation will admit of a Contraction; which will be made manifest by this following

QUEST. 7.

1. If $a^6 - 29a^4 + 244a^2 - 576 = 0$, what is $a = ?$

RESOLUTION.

2. Here because the Indices of the unknown Powers are even numbers? $b = a^2$
to wit, 6, 4 and 2, put $b = a^2$

3. Then

3. Then for $\left. \begin{array}{l} +a^6 \\ -29a^4 \\ +244a^2 \end{array} \right\}$ write $\left. \begin{array}{l} +e^3 \\ -29e^2 \\ +244e \end{array} \right\}$
4. To which Powers of e joyn -576 the last Term of the given Equation, and it makes $e^3 - 29e^2 - 244e - 576 = 0$.
5. Which last Equation being resolv'd by Division, (in like manner as in the preceding Examples of this Section,) there will be found three Affirmative values of the Root e , viz. 4, 9 and 16; then because e was put equal to aa , the Square Roots of 4, 9 and 16, that is, 2, 3 and 4, shall be three Roots or values of a in the Equation first propos'd, to wit, $a^6 - 29a^4 + 244a^2 - 576 = 0$, as may easily be proved.

I might here shew how to reduce a Biquadratick Equation not falling under any of the three Forms in *Self. 1. Chap. 15. Book 1.* to a Cubick Equation, and sometimes into two Quadratick Equations, but I shall spare that labour for these Reasons; First, that Reduction being subject to many Cases, is very tedious and troublesome; Secondly, such a Biquadratick Equation is very seldom capable of being reduced into two Quadratick Equations, and when 'tis reduced to a Cubick Equation, this may happen to be such as its Root or Roots in numbers cannot be perfectly found out by any Rules hitherto published by any Author: Thirdly, by the Method in this ninth Section, all the Roots of any Cubick, Biquadratick or other Equation of higher degrees may be found out in numbers, either exactly, if they be Rational, or as near the truth, if they be Irrational, as shall be needful for any practical use: And lastly, my undertaking (as I have before hinted,) is not to handle all, but the most useful Rules only in this profound Art.

Note. The Resolutions of the preceding Questions of this ninth Section do clearly shew, that there is no small labour in making tryals with the Divisors of the last Term of an Equation to find its Root or Roots; and therefore to lessen that work, first, it will be convenient to make some tryals by the general Method in the foregoing *Chapt. 10.* to find out limits within which the Root or Roots of an Equation do fall, or to argue the same from some things given in a Question producing the said Equation, and then to make tryals only with such Divisors of the last Term as fall within those limits; but when all Contractions are used, the work is sufficiently laborious, so that one chief scope of an Analyst in resolv'g a knotty Question must be to frame his Positions with such artifice that the Resolution may end in as simple an Equation as is possible: And although one way of Resolution may produce an Equation composed of high Powers, yet often-times by another way you may come to a more simple Equation, as may partly appear by the foregoing fourth and fifth Questions of this Section; but the skill of finding out the most simple and facil ways of Resolution, is not attainable; (as I conceive,) by any certain or constant Method, but rather by much use and exercise in the solving of Questions.

Self. X. Concerning the Resolution of certain Cubick Equations in numbers, by two Rules, the Invention whereof Cardanus attributes to Scipio Ferreus.

1. All Cubick Equations, after the second Term is taken away, when there happens to be any, (by the Rule in *Self. 8.* of this *Chapt.*) are reducible to these three following Forms, in which a represents the Root or Quantity sought, but p and q known Quantities.

$$\begin{array}{l|l} aaa = -6a + 20 & aaa = -pa + q \\ aaa = +6a + 40 & aaa = +pa + q \\ aaa = +91a - 330 & aaa = +pa - q \end{array}$$

2. Now let it be required to resolve the first of those Equations, viz.

$$\text{If } aaa = -6a + 20; \text{ or, } aaa = -pa + q;$$

What is the value of a ?

Preparation.

3. Suppose $a = e - y$
4. Suppose also $20 = eee - 3yy$
5. And $6 = 3ey$
6. Then by multiplying each part of the Equation in the third step into it self Cubically, this is produced, viz.
- $$aaa = eee - 3eey + 3eyy - yyy$$

7. And

7. And by multiplying the Equations in the third and fifth steps one into the other, it makes $6a = 3eey - 3eyy$
8. And by subtracting the Equation in the seventh step from that in the fourth, there remains $20 - 6a = eee - 3eey + 3eyy - yyy$
9. Therefore by the sixth and eighth steps 'tis manifest that $aaa = eee - 3eey + 3eyy - yyy = 20 - 6a$
10. From the premises it's evident, that if in the Equation propos'd to be resolv'd, to wit, $aaa = -6a + 20$, or $aaa = -pa + q$, we suppose the Root a sought to be equal to the difference of two unknown numbers e and y , also the Absolute number 20 (or q) to be equal to the difference of the Cubes of the same two numbers, and the Coefficient 6 (or p) to be equal to the triple Product of their multiplication; then as well aaa , as $20 - 6a$ (that is, $q - pa$) shall be equal to the Cube of the difference of those two numbers, viz. to the Cube of $e - y$; and therefore when two such numbers are found out, their difference shall be the Root or number a sought: But to find out the said two numbers (e and y) there is given the Product of their multiplication, to wit, 2, (or $\frac{1}{3}p$), that is, one third part of the Coefficient, as also 20 (or q) the difference of the Cubes of the same two numbers; and therefore the numbers themselves shall be given severally by the Canon of *Quest. 15. Chap. 16. Book 1.* and consequently the Root a sought shall be given also, as will be made manifest by this following.

Operation.

- | | | |
|--|--------------------------------------|--|
| 11. To the Square of half the given Absolute number 20 (or q) viz. to | 100 | $\frac{1}{2}qq$ |
| 12. Add the Cube of 2 (or $\frac{1}{3}p$) viz. the Cube of the Coefficient 6 (or p), which Cube is | 8 | $\frac{1}{27}PPP$ |
| 13. The sum is | 108 | $\frac{1}{2}qq + \frac{1}{27}PPP$ |
| 14. The Square Root of that sum is | $\sqrt{108}$ | $\sqrt{\frac{1}{2}qq + \frac{1}{27}PPP}$ |
| 15. To that Square Root add half the Absolute number 20 (or q), and the sum is | $10 + \sqrt{108}$ | $\frac{1}{2}q + \sqrt{\frac{1}{2}qq + \frac{1}{27}PPP}$ |
| 16. The Cubick Root of that sum is the greater number e sought, viz. | $\sqrt[3]{(3) : 10 + \sqrt{108} :}$ | $\sqrt[3]{(3) : \frac{1}{2}q + \sqrt{\frac{1}{2}qq + \frac{1}{27}PPP} :}$ |
| 17. Again, from the square Root in the fourteenth step, subtract half the Absolute number 20 (or q), and the Remainder is | $-10 + \sqrt{108}$ | $-\frac{1}{2}q + \sqrt{\frac{1}{2}qq + \frac{1}{27}PPP}$ |
| 18. Then the Cubick Root of that Remainder shall be the lesser number y sought, viz. | $\sqrt[3]{(3) : -10 + \sqrt{108} :}$ | $\sqrt[3]{(3) : -\frac{1}{2}q + \sqrt{\frac{1}{2}qq + \frac{1}{27}PPP} :}$ |

19. And then the difference of the two Cubick Roots found out in the sixteenth and eighteenth steps shall be the value of the Root a in the Equation propos'd, viz.

$$a = \sqrt[3]{(3) : 10 + \sqrt{108} :} - \sqrt[3]{(3) : -10 + \sqrt{108} :} \text{ that is,}$$

$$a = \sqrt[3]{(3) : \frac{1}{2}q + \sqrt{\frac{1}{2}qq + \frac{1}{27}PPP} :} - \sqrt[3]{(3) : -\frac{1}{2}q + \sqrt{\frac{1}{2}qq + \frac{1}{27}PPP} :}$$

20. It remains to make trial whether the Binomial $10 + \sqrt{108}$ hath a perfect Cubick Root or not; so by the Rule in *Self. 18. Chap. 9.* of this Second Book, it will appear that $1 + \sqrt{3}$ is the Cubick Root of $10 + \sqrt{108}$, and $\sqrt{3} - 1$ is the Cubick Root of $\sqrt{108} - 10$, and consequently the value of the Root a before found out in the nineteenth step is expressible by a Rational number, for if $\sqrt{3} - 1$ be subtracted from

N^o

from

In handling the following Method I shall give three principal Rules, and explain them by Examples; but to prescribe Rules for all Cases, is (as I conceive) an impossible work.

RULE I.

When many Quantities are fought by a Question, first let them be severally represented by different letters; then after you have well considered the conditions in the Question, abstract it from words, and express the tenor thereof by Equations; that done, by the help of Transposition find what the first, that is, any single letter representing a number or quantity fought in the first Equation is equal to; then wheresoever that first letter is found in the other Equations, take instead of it those Quantities to which the said first letter was found equal; so such first letter will quite vanish out of those other Equations. Again, by Transposition set a second letter alone in one of those Equations out of which the first letter was expell'd, and proceed as before; so at length one of the numbers fought will be made known, by the help whereof the rest will easily be discovered: This work will be better understood by Examples than many words, and therefore I shall proceed to Questions.

QUEST. 1.

A Factor exchanged 6 French Crowns and two Dollars, for 45 Shillings of English Money; also at another time he exchanged 9 French Crowns and 5 Dollars (each of these being of the same value with the former) for 76 Shillings: I demand the value of a French Crown, and also of a Dollar, in English Money?

Let a represent the desired value of a Crown, and e the value of a Dollar, then the Question being abstracted from words may be stated thus;

1. If $6a + 2e = 45$
2. And $9a + 5e = 76$

What are the numbers a and e ?

RESOLUTION.

3. By transposition of $2e$ in the first Equation this ariseth, $6a = 45 - 2e$
4. And by dividing each part of the third Equation by 6, it gives $a = \frac{45 - 2e}{6}$
5. The fourth Equation multiplied by 9, produceth $9a = \frac{405 - 18e}{6}$
6. Then if instead of $9a$ in the second Equation, you take the latter part of the fifth, this will arise, $\frac{405 - 18e}{6} + 5e = 76$
7. The sixth Equation, after due Reduction, discovers the value of a Dollar, viz. $e = 4\frac{1}{2}$
8. The seventh Equation multiplied by 2 gives $2e = 8\frac{1}{2}$
9. And by setting the latter part of the eighth Equation in the place of $2e$ in the first, this Equation ariseth, $6a + 8\frac{1}{2} = 45$
10. From which last Equation, after due Reduction, the value of a , or one French Crown is discovered, viz. $a = 6\frac{1}{4}$

Thus by the seventh and tenth Equations it is found that a Dollar was valued at $4\frac{1}{2}$ s. d. and a French Crown at $6\frac{1}{4}$ s. d. which numbers will satisfy the conditions in the Question, as may easily be proved.

QUEST. 2.

Three men had every one of them a certain number of Pounds in his Purse, the sum of the first and second man's money was 5 (or b) Pounds, the sum of the second and third man's money was 12 (or c) Pounds, and the sum of the third and first man's money was 11 (or d) Pounds: How many Pounds had every one in his Purse?

Let the three numbers of Pounds fought be represented by a , e and y , then respect being had to the numbers given, the Question may be stated thus, viz.

1. If $a + e = b (= 5)$
2. And $e + y = c (= 12)$
3. And $y + a = d (= 11)$

What are the numbers a , e and y ?

RESO.

RESOLUTION.

4. By transposition of a in the first Equation, there will arise $e = b - a$
5. Then by taking the latter part of the fourth Equation instead of e in the second, this Equation ariseth, $b - a + y = c$
6. And by transposition of $b - a$ in the 5th Equation it gives $y = c - b + a$
7. And by taking the latter part of the sixth Equation instead of y in the third, this ariseth, $c - b + a + a = d$
8. From which seventh Equation, after due Reduction, the number a will be made known, viz. $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$
9. Again, if instead of a in the first Equation we take the latter part of the eighth, this ariseth, $\frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c + e = b$
10. Then from the ninth, after due Reduction, the number e will be made known, viz. $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$
11. Again, if instead of a in the third Equation we take the latter part of the eighth, this ariseth, $y + \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c = d$
12. Lastly, from the eleventh Equation, after due Reduction, the number y will be made known, viz. $y = \frac{1}{2}d + \frac{1}{2}c - \frac{1}{2}b$

The eighth, tenth and twelfth Equation gives this

CANON.

From the sum of every two of the three numbers given, subtract the remaining number, then the halves of the three Remainders shall be the numbers sought. Whence the numbers sought, to wit, a and y will be found 2, 3 and 9: for $2 + 3 = 5$; also $3 + 9 = 12$; and $9 + 2 = 11$; as was required.

The foregoing Resolution of this Quest. 2. is formed according to Rule 1. but the same Canon may be more expeditiously discovered by this following Resolution, viz.

The sum of the first, second and third Equations? $2a + 2e + 2y = b + c + d$
which state the Question is

The half of that sum is $a + e + y = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$

Then from that half sum subtract the first? $y = \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}b$

Equation, and the Remainder will be $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$

Again, from the said half sum subtract the second Equation, and the Remainder is $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$

Lastly, from the said half sum subtract the third Equation, and the Remainder gives

Which three last Equations do manifestly give the same values of a , e and y , as were found out by the former Resolution.

QUEST. 3.

Three Men discourse of their moneys in this manner, the first saith to the other two, if 100 l. were added to his money, the sum would be equal to both their moneys; the second saith to the other two, if 100 l. were added to his money, the sum would be equal to the double of both their moneys; the third saith to the other two, if 100 l. were added to his money, the sum would be equal to the triple of both their moneys: The Question is, to find how many Pounds each Man had?

Let the three numbers of Pounds fought be represented by a , e and y , then the Question may be stated thus, viz.

1. If $a + 100 = e + y$
2. And $e + 100 = 2a + 2y$
3. And $y + 100 = 3a + 3e$

What are the numbers a , e and y ?

RESOLUTION.

4. From the first Equation by transposition of y , this ariseth, $a + 100 - y = e$
5. Then if instead of e in the second Equation, there be taken that which is equal to e , to wit, the first part of the fourth, this will arise, $a + 100 - y + 100 = 2a + 2y$
6. That is, after due Reduction, $200 = a + 3y$

7. Again,

7. Again, if instead of 3*e* in the third Equation, there be taken the triple of the first part of the fourth Equation, this will arise, to wit, $y + 100 = 3a + 3d + 300 - 3y$
8. Which last Equation, after due Reduction, gives $y = \frac{1}{2}a - 50$
9. Then if instead of 3*y* in the sixth Equation, there be let the triple of the latter part of the eighth; this will come forth, viz. $200 = a - \frac{1}{2}a + 150$
10. From the ninth Equation, after due Reduction, the number *a* will be discovered, viz. $a = 97\frac{1}{2}$
11. Again, if instead of *a* in the sixth Equation, there be taken $97\frac{1}{2}$, to wit, the value of *a* found out in the tenth, it will give $200 = 97\frac{1}{2} + 3y$
12. The eleventh Equation duly reduced discovers the number *y*, viz. $y = 63\frac{1}{2}$
13. From the fourth, tenth and twelfth Equations by exchange of equal Quantities, this Equation ariseth, viz. $97\frac{1}{2} + 100 - 63\frac{1}{2} = e$
14. The thirteenth reduced gives $e = 45\frac{1}{2}$
- From the 10th, 14th and 12th Equations, the three numbers sought, *a*, *e* and *y* are discovered, viz. The first Man had $97\frac{1}{2}l$. the second $45\frac{1}{2}l$. and the third $63\frac{1}{2}l$, which numbers will satisfy the Question, as may easily be proved.
- If 121 be given instead of 100 in this third Question, then the three numbers sought will be whole numbers, to wit, 11, 55, 77.

RULE II.

When the same Quantity, suppose *a*, is found in two several Equations, and equal numbers are prefixed to those Quantities, then if their signs be both +, or both -, subtract the lesser Equation from the greater; but if one of the signs be + and the other -, add those two Equations together; so the said Quantity *a* will quite vanish, as will appear by the Resolution of the following Question.

QUEST. 4.

The sum of two numbers being given 12 (or *b*), and their difference 8 (or *c*); to find the numbers.

Let *a* be put for the greater number, and *e* for the lesser, and the Question may be stated thus

1. If $a + e = b (= 12)$
2. And $a - e = c (= 8)$

What are the numbers *a* and *e*? ||

RESOLUTION.

3. For as much as *a* or $+\frac{1}{2}a$ is found in each of the Equations proposed, therefore (according to Rule 2.) I subtract the lesser Equation from the greater; whence the letter *a* quite vanisheth, and there remains $2e = b - c (= 4)$
4. Then by dividing each part of the third Equation by 2, the number *e* is made known, viz. $e = \frac{1}{2}b - \frac{1}{2}c (= 2)$
5. And by taking the latter part of the fourth Equation instead of *e* in the first, there remains $a + \frac{1}{2}b - \frac{1}{2}c = b (= 12)$
6. Lastly, the fifth Equation duly reduced discovers the number *a*, viz. $a = \frac{1}{2}b + \frac{1}{2}c (= 10)$

The 6th and 4th Equations discover a Canon to find out the numbers sought, which in this Example are 10 and 2, and the Canon is the same with that before found in Quest. 14. Book 1.

Otherwise thus;

7. For as much as $+\frac{1}{2}e$ is found in the first Equation, and $-e$ in the second, therefore by adding those two Equations together, (according to Rule 2.) the letter *e* vanisheth, and the sum is $2a = b + c (= 20)$

8. There

8. Therefore by dividing each part of the seventh Equation by 2, there ariseth the same value of *a* which was before found in the sixth Equation, viz. $a = \frac{1}{2}b + \frac{1}{2}c (= 10)$
9. And by setting the latter part of the eighth Equation in the place of *a* in the first, this ariseth, $\frac{1}{2}b + \frac{1}{2}c + e = b (= 12)$
10. Which last Equation reduced discovers the same value of *e* which was before found in the fourth Equation, viz. $e = \frac{1}{2}b - \frac{1}{2}c (= 2)$

RULE III.

When the same Quantity, suppose *a*, is found in two several Equations, but the numbers prefix to those equal Quantities are unequal, those two Equations may be reduced into two others which shall have equal numbers prefix to the said Quantity *a*, by this Rule, viz. Multiply all the Quantities in the first Equation by the number which is prefix to the said Quantity *a* in the second; multiply likewise all the Quantities in the second Equation by the number which is prefix before the same Quantity *a* in the first; so by such alternate multiplication two new Equations will be produced, wherein the numbers prefix to the said Quantity *a* will be equal to one another: and then by adding or subtracting, according to the import of Rule 2. of this Chap. that Quantity *a* will quite vanish. That done, renew the like work to expell the same Quantity out of the rest of the Equations; and proceed in like manner with a second Quantity, until at length the value of some one Quantity be made known. This I shall make plain by the Resolutions of Five Questions next following.

QUEST. 5.

To find two numbers, that if the quadruple of the greater be increased with the triple of the less, it may make 36; but if the triple of the greater be lessened by the double of the less, the Remainder may be 10.

Put *a* for the greater number, and *e* for the lesser, then the Question may be stated thus, viz.

1. If $4a + 3e = 36$
2. And $3a - 2e = 10$

What are the numbers *a* and *e*? ||

RESOLUTION.

3. The first Equation multiplied by 3, which is prefix to *a* in the second, produceth $12a + 9e = 108$
4. The second Equation multiplied by 4, which is prefix to *a* in the first, makes $12a - 8e = 40$
5. Now for as much as the Quantity $12a$ is found both in the fourth and third Equations, and is Affirmative in each; therefore according to Rule 2. I subtract the lesser Equation from the greater, so the Quantity $12a$ vanisheth, and this Equation remains, $9e + 8e = 68$
6. The fifth Equation, after due Reduction, discovers the number *e*, viz. $e = 4$
7. Then I set 12 (which by the sixth Equation is the value of $3e$) in the place of $3e$ in the first, and this Equation ariseth, $4a + 12 = 36$
8. Lastly, the seventh Equation duly reduced discovers the number *a*, viz. $a = 6$

From the 8th and 6th Equations the two numbers sought are found *a* and *e*, which will solve the Question: For four times *a* with thrice *e* makes 36; and thrice *a*, to wit, 18; lessened by twice *e* gives 10; as was required.

QUEST. 6.

1. If $2a + 3e - 2y = 50$
2. And $5a - 2e + 5y = 240$
3. And $-a + 5e - 3y = 10$

What are the numbers *a*, *e* and *y*? ||

RESO.

RESOLUTION.

4. The first Equation multiplied by 5, which is prefix to a in the second, produceth $10a + 15e - 107 = 250$
5. Likewise the second Equation multiplied by 2, which is prefix to a in the first, makes $10a - 4e + 107 = 480$
6. Then (according to Rule 2.) by subtracting the fourth Equation from the fifth, the Quantity $10a$ vanisheth, and this Equation ariseth, $-19e + 207 = 230$
7. Again, the third Equation multiplied by 5 which is prefix to a in the second, produceth $-5a + 25e - 157 = 50$
8. And the second Equation multiplied by 1, which is supposed to be prefix to a in the third, gives the same second Equation without alteration, viz. $+5a - 2e + 57 = 240$
9. Then because $-5a$ and $+5a$ by addition will destroy one another, therefore (according to Rule 2.) I add the seventh and eighth Equations together; so the letter a vanisheth, and this Equation ariseth, $+23e - 107 = 290$
10. Again, I proceed with the sixth and ninth Equations according to Rule 3. viz. I multiply the sixth Equation by 23, (which is prefix to e in the ninth,) and it makes $-437e + 4607 = 5190$
11. Also the ninth Equation multiplied by 19 (which is prefix to e in the sixth, produceth $+437e - 1907 = 5510$
12. Then (according to Rule 2.) by adding the tenth and eleventh Equations together, the letter e vanisheth, and this Equation ariseth, viz. $+2707 = 10800$
13. And by dividing each part of the twelfth Equation by 270, the number y is discovered, viz. $y = 40$
14. Then instead of 107 in the ninth Equation taking ten times 40, that is, 400, (which by the thirteenth Equation is equal to 107) the ninth will be reduced to this, $+23e - 400 = 290$
15. And from the fourteenth Equation, after due Reduction, the number e will be discovered, viz. $e = 30$
16. Then instead of $3e - 27$ in the first Equation, I take $90 - 80$, (which by the fifteenth and thirteenth Equations will be found equal to $3e - 27$.) so the first Equation will be converted into this, viz. $2a + 90 - 80 = 50$
17. Lastly, the sixteenth Equation duly reduced discovers the number a , viz. $a = 10$

From the 17th, 15th and 13th Equations the three desired numbers a, e, y , are 10, 30 and 40, which will constitute the three Equations first proposed, as may easily be proved.

QUEST. 7.

Three Men discourse of their moneys in this manner; the first saith to the other two, if you give me 100 Pounds, my money will be made equal to both your remaining moneys: the second saith to the other two, if ye give me 100 Pounds, my money will be made equal to the double of both your remaining moneys: lastly, the third saith to the other two, if ye give me 100 Pounds, my money will be equal to the triple of both your remaining moneys: I demand how many Pounds each Man had?

Let a letter be assumed to represent each Man's money; as a for the first, e for the second, and y for the third; then the Question may be stated thus, viz.

1. If $a + 100 = e + y - 100$
2. And $e + 100 = 2a + 2y - 200$
3. And $y + 100 = 3a + 3e - 300$

What are the numbers a, e, y ? ||

RESOLUTION.

4. The first Equation by transposition will be reduced to this, $-a + e + y = 200$
5. Likewise

5. Likewise the second Equation by transposition gives $+2a - e + 27 = 300$
6. And the 3d Equation by transposition produceth $+3a + 3e - y = 400$
7. Then I proceed with the fourth and fifth Equations according to Rule 3. viz. I multiply the fourth Equation by 2, (which is prefix to a in the fifth,) and it produceth $-2a + 2e + 27 = 400$
8. The sum of the fifth and seventh Equations gives $e + 47 = 700$
9. Again, I proceed with the fifth and sixth Equations according to Rule 3. viz. multiplying the fifth Equation by 3, (which is prefix to a in the sixth,) it gives $6a - 3e + 67 = 900$
10. Also the sixth Equation multiplied by 2, (which is prefix to a in the fifth) produceth $6a + 6e - 27 = 800$
11. Then by subtracting the tenth Equation from the ninth, the Remainder is $-9e + 87 = 100$
12. Again, I proceed with the eighth and eleventh Equations according to Rule 3. viz. multiplying the eighth Equation by 9, (which is prefix to e in the eleventh,) it makes $+9e + 367 = 6300$
13. Then (according to Rule 2.) the eleventh and twelfth Equations added together make $447 = 6400$
14. And by dividing the thirteenth Equation by 44, the number y is made known, viz. $y = 145\frac{1}{11}$
15. From the eighth and fourteenth, by exchange of equal Quantities, this ariseth, viz. $e + 581\frac{1}{11} = 700$
16. And from the fifteenth by subtraction of $581\frac{1}{11}$ from each part, the number e is discovered, viz. $e = 118\frac{1}{11}$
17. From the first, fourteenth and sixteenth Equations, by exchange of equal Quantities, this Equation ariseth, viz. $a + 100 = 118\frac{1}{11} + 145\frac{1}{11} - 100$
18. Lastly, the seventeenth Equation, after due Reduction, discovers the number a , viz. $a = 63\frac{7}{11}$

Thus, by the 18th, 16th and 14th Equations it is found that the first Man had $63\frac{7}{11} l$, the second $118\frac{1}{11} l$, and the third $145\frac{1}{11} l$, which three numbers will satisfy the Question, as may easily be proved.

QUEST. 8.

1. If $a + \frac{1}{2}e + \frac{1}{3}y + \frac{1}{4}u = 112$
2. And $e + \frac{1}{2}a + \frac{1}{3}y + \frac{1}{4}u = 114$
3. And $y + \frac{1}{2}a + \frac{1}{3}e + \frac{1}{4}u = 125\frac{1}{2}$
4. And $u + \frac{1}{2}a + \frac{1}{3}e + \frac{1}{4}y = 133\frac{1}{2}$

What are the numbers a, e, y and u ? ||

RESOLUTION.

5. The first Equation multiplied by 3, (the Denominator of the Fraction $\frac{1}{3}$) produceth this Equation in Integers, to wit, $3a + 2e + 27 + 2u = 336$
6. Likewise the second Equation multiplied by 4, produceth $3a + 4e + 37 + 3u = 456$
7. And the third Equation multiplied by 5 gives $4a + 4e + 57 + 4u = 628$
8. Also the fourth Equation multiplied by 6 produceth $5a + 5e + 57 + 6u = 800$
9. For as much as $3a$ is found in the fifth, and also in the sixth Equation, I subtract the lesser from the greater, so $3a$ quite vanisheth, and this Equation ariseth, $2e + 7 + u = 120$

O O

10. Then

10. Then I proceed with the fifth and seventh Equations according to *Rule 3*, viz. I multiply the fifth Equation by 4, (which is prefix to a in the seventh,) and there comes forth
- $$12a + 8e + 8y + 8u = 1344$$
11. Also I multiply the seventh Equation by 3, (which is prefix to a in the fifth,) and it produceth
- $$12a + 12e + 15y + 12u = 1884$$
12. Then by subtracting the tenth Equation from the eleventh, the quantity $12a$ quite vanisheth, and this Equation ariseth, to wit,
- $$4e + 7y + 4u = 540$$
13. The ninth Equation multiplied by 2, produceth
- $$4e + 2y + 2u = 240$$
14. Then by subtracting the thirteenth Equation from the twelfth, this ariseth, to wit,
- $$5y + 2u = 300$$
15. Again, I proceed with the fifth and eighth Equations according to *Rule 3*, viz. I multiply the fifth Equation by 5, (which is prefix to a in the eighth,) and it produceth
- $$15a + 10e + 10y + 10u = 1680$$
16. Likewise the eighth Equation multiplied by 3, (which is prefix to a in the fifth,) produceth
- $$15a + 15e + 15y + 18u = 2400$$
17. Then by subtracting the fifteenth Equation from the sixteenth, this ariseth, viz.
- $$5e + 5y + 8u = 720$$
18. Again, I proceed with the ninth and seventeenth Equations according to *Rule 3*, viz. I multiply the ninth Equation by 5, (which is prefix to e in the seventeenth,) and it produceth
- $$10e + 5y + 5u = 600$$
19. And the seventeenth Equation multiplied by 2, (which is prefix to e in the ninth,) produceth
- $$10e + 10y + 16u = 1440$$
20. Then by subtracting the eighteenth Equation from the nineteenth, there remains
- $$5y + 11u = 840$$
21. And by subtracting the 14th Equation from the 20th, (for since $5y$ is found in each of those Equations, they need no Reduction according to *Rule 3*.) there remains
- $$9u = 540$$
22. Which twenty first Equation divided by 9 discovers the number u , viz.
- $$u = 60$$
23. From the 20th and 22d Equations, by setting eleven times 60, to wit, 660 in the place of $11u$ in the 20th, there ariseth
- $$5y + 660 = 840$$
24. Therefore from the twenty third Equation, after due Reduction, the number y is discovered, viz.
- $$y = 36$$
25. And from the 9, 24 and 22 Equations, this ariseth,
- $$2e + 36 + 60 = 120$$
26. The 25th duly reduced discovers the number e , viz.
- $$e = 12$$
27. From the 5th, 26th, 24th, and 22d Equations, by exchange of equal Quantities, this Equation ariseth,
- $$3a + 24 + 72 + 120 = 336$$
28. Lastly, from the 27th, after due Reduction, the number a is discovered, viz.
- $$a = 40$$

Thus by the 28th, 26th, 24th and 22d Equations the four numbers sought, (to wit, a, e, y, u) are found 40, 12, 36 and 60, which will constitute the four Equations in *Quest. 9*.

QUEST. 9.

A Maid being at the Market is offer'd 10 Apples for a penny, and 25 Pears for two pence; now if at those rates she would lay out $9\frac{1}{2}$ pence to buy 100 Apples and Pears together, how many Apples, and how many Pears ought she to have?

1. For the number of Apples sought put a
 2. And for the number of Pears sought put e
 3. Then search out the cost of the number of Apples in the first step, and say, If 10 : 1 :: a : $\frac{a}{10}$: so the cost of the number of Apples sought is $\frac{a}{10}$

4. Search

4. Search out also the cost of the number of Pears in the second step, and say, If 25 : 2 :: e : $\frac{2e}{25}$: so the cost of the number of Pears sought is found
5. Then (according to the Question) the money laid out for all the Apples and Pears sought must be equal to $9\frac{1}{2}$ Pence; hence this Equation
- $$\frac{a}{10} + \frac{2e}{25} = 9\frac{1}{2}$$
6. But the number of Apples, together with the number of Pears bought must make 100, therefore
- $$a + e = 100$$
7. Then the Equation in the fifth step, after due Reduction, will give this Equation in Integers, to wit,
- $$50a + 40e = 4750$$
8. And the Equation in the sixth step being multiplied by 50 produceth
- $$50a + 50e = 5000$$
9. Then by subtracting the Equation in the seventh step from that in the eighth, there ariseth
- $$10e = 250$$
10. And the Equation in the ninth step divided by 10, discovers the number e , viz.
- $$e = 25$$
11. Lastly, from the sixth and tenth steps, the number a is also made known, viz.
- $$a = 75$$

By the first, second, eleventh and tenth steps it appears that there might be bought 75 Apples, and 25 Pears, which numbers will solve the Question, as may easily be proved.

QUEST. 10.

To divide 90 into four such numbers, that if the first be increased with 2; the second lessened by 2; the third multiplied by 2; and the fourth divided by 2; the Summ, Remainder, Product and Quotient may be equal between themselves.

Let b and d be put for the two given numbers, 90 and 2; also a, e, y and u for the four numbers sought, then the Question may be stated thus;

1. If $a + e + y + u = b$
 2. And $a + d = e - d$
 3. And $a + d = dy$
 4. And $a + d = \frac{u}{d}$

What are the numbers a, e, y and u ? ||

RESOLUTION.

5. The first number sought is equal to it self, viz. $a = a$
 6. From the second Equation, by transposition of $-d$, this ariseth, $a + 2d = e$
 7. And by dividing each part of the third Equation by d , this ariseth, $\frac{a+d}{d} = y$
 8. And the fourth Equation multiplied by d produceth $da + dd = u$
 9. The summ of the four last Equations gives

$$2a + 2d + \frac{a+d}{d} + da + dd = a + e + y + u = b$$

10. Which last Equation, after due Reduction; gives $a = \frac{bd - ddd - 2dd - d}{dd + 2d + 1}$
 11. Then from the tenth and sixth Equations, by exchange of equal Quantities, $e = \frac{bd + ddd + dd + d}{dd + 2d + 1}$
 12. And from the tenth and seventh Equations, $y = \frac{bd}{dd + 2d + 1}$
 13. And from the tenth and eighth Equations, $u = \frac{bdd}{dd + 2d + 1}$

The four last Equations give a Canon to find out the four numbers sought, which are 18, 22, 10 and 40, which will solve the Question. For, first, their summ is 90; then if the first number 18 be increased with the given number 2, it makes 20; and if the second number

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number 22 be lessened by 2, the Remainder is also 20: moreover, if the third number 10 be multiplied by 2, it likewise produceth 20: lastly, if the fourth number 40 be divided by 2, the Quotient is also 20. Therefore the conditions in the Question are satisfied.

But the Numerator of the Fraction in the latter part of the tenth Equation shews, That the numbers b and d must not be given at random, but so, that $add + 2dd + d$ may be subtracted from bd and leave a Remainder greater than nothing; therefore bd must be greater than $add + 2dd + d$, and consequently b must be greater than $dd + 2d + 1$. Therefore, to the end the Question may be possible, the numbers given must be subject to this

Determination.

The number given to be divided (b) must be greater than the Square of ($d + 1$) the sum of the other number given and Unity.

QUEST. 11.

There are two numbers whose sum is equal to the difference of their Squares; and if the sum of the Squares of those two numbers be subtracted from the Square of their sum, the Remainder will be 60: what are the two numbers?

Put b for the given number 60, also a for the greater number sought, and e for the lesser, then the Question may be stated thus, viz.

1. If $aa - ee = a + e$
2. And $aa + ee + 2ae - aa - ee = b$

What are the numbers a and e ?

RESOLUTION.

3. The second Equation after its first part is duly contracted is $2ae = b$
4. And the third Equation divided by 2 gives $ae = \frac{1}{2}b$
5. And if each part of the first Equation be divided by $a + e$ it will give $a - e = \frac{a + e}{a + e} = 1$
6. From the fifth Equation, by transposition of e , there ariseth $a = e + 1$
7. The sixth Equation multiplied by e produceth $ae = ee + e$
8. From the fourth and seventh Equations, by exchanging equal Quantities, $ee + e = \frac{1}{2}b$
9. Then the eighth Equation being resolved by the Canon in Sect. 6. Chap. 15. Book 1. the lesser number sought will be made known, viz. $e = \sqrt{\frac{1}{4} + \frac{1}{2}b} - \frac{1}{2} = 5$
10. And from the ninth and sixth Equations the greater number sought will also be made known, viz. $a = \sqrt{\frac{1}{4} + \frac{1}{2}b} + \frac{1}{2} = 6$

The two last Equations give a Canon to find out the two numbers sought, which are 6 and 5; as may easily be proved.

QUEST. 12.

There are two numbers, such, that if their sum be subtracted from the sum of their Squares, the Remainder is 42; but if the sum of the said two numbers be added to the Product of their multiplication, it makes 34: what are the numbers?

Let a and e represent the two numbers sought, then the Question may be stated thus, viz.

1. If $aa + ee - a - e = 42$
2. And $ae + a + e = 34$

What are the numbers a and e ?

RESOLUTION.

3. By adding the first and second Equations together, the sum will be $aa + ee + ae = 76$
4. And by adding the second Equation to the third, the sum will be $aa + ee + 2ae + a + e = 110$
5. Suppose $j = a + e$

6. Then

6. Then by squaring each part of the fifth Equation, this ariseth $jj = aa + ee + 2ae$
7. The sum of the two last Equations makes $jj + j = aa + ee + 2ae + a + e$
8. And from the seventh and fourth Equations, by exchange of equal quantities, this Equation ariseth $jj + j = 110$
9. Which eighth Equation being resolved by the Canon in Sect. 6. Chap. 15. Book 1. the number j , to wit, $a + e$ will be made known, viz. $j (= a + e) = 10$
10. Then by setting 10 (the value of $a + e$) in the place of $a + e$ in the second Equation, there ariseth $ae + 10 = 34$
11. And by subtracting 10 from each part of the tenth Equation, there remains $ae = 24$
12. And from the ninth Equation, by transposition of a , there ariseth $e = 10 - a$
13. And if a in the eleventh be multiplied by 10 instead of e , the said eleventh Equation will be reduced to this, $10a - aa = 24$
14. Wherefore the last Equation being resolved by the Canon in Sect. 10. Chap. 15. Book 1. the two numbers sought will be discovered, viz. $\begin{cases} a = 6 \\ e = 4 \end{cases}$

Thus 6 and 4 are found out, which will solve the Question proposed, as will be evident by the Proof.

QUEST. 13.

There are two numbers, such, that the sum of their Squares makes 100, and if the sum of the two numbers be added to the Product of their multiplication, it makes 62; what are the numbers?

Let a and e be put for the two numbers sought, then the Question may be stated thus, viz.

1. If $aa + ee = 100$
2. And $ae + a + e = 62$

What are the numbers a and e ?

RESOLUTION.

3. The second Equation multiplied by 2 produceth $2ae + 2a + 2e = 124$
4. The sum of the first and third Equations gives $aa + ee + 2ae + 2a + 2e = 224$
5. Suppose $j = a + e$
6. Then by squaring each part of the fifth Equation this is produced, viz. $jj = aa + ee + 2ae$
7. And by adding the double of the fifth Equation to the sixth, it gives $jj + 2j = aa + ee + 2ae + 2a + 2e$
8. And from the seventh and fourth Equations, by exchange of equal quantities, this Equation ariseth $jj + 2j = 224$
9. Which last Equation being resolved by the Canon in Sect. 6. Chap. 15. Book 1. the number j , to wit, $a + e$, will be made known, viz. $j = a + e = 14$
10. Then from the ninth and second Equations, by taking 14 instead of $a + e$, the second Equation will be reduced to this, viz. $ae + 14 = 62$
11. Which last Equation, by equal subtraction of 14, gives $ae = 48$
12. The ninth Equation by transposition of a gives $e = 14 - a$
13. Then by multiplying a in the eleventh Equation by 14 instead of e , this Equation is produced, to wit, $14a - aa = 48$
14. Wherefore the last Equation being resolved by the Canon in Sect. 10. Chap. 15. Book 1. the two numbers sought will be discovered, viz. $\begin{cases} a = 8 \\ e = 6 \end{cases}$

So the numbers sought are found 8 and 6, which will solve the Question, as will appear by the Proof.

QUEST. 14.

QUEST. 14.

There are two numbers, such, that their sum is equal to the Product of their multiplication; and if the Product or sum of the said numbers be added to the sum of their Squares, it makes $15\frac{1}{2}$: what are the numbers?

Let a and e be put for the two numbers sought, then the Question may be stated thus, viz.

1. If $ae = a + e$
2. And $aa + ee + ae = 15\frac{1}{2}$

What are the numbers a and e ?

RESOLUTION.

3. The sum of the first and second Equations is $aa + ee + 2ae = a + e + 15\frac{1}{2}$
4. And from the third Equation, by transposition of $a + e$, there ariseth $aa + ee + 2ae - a - e = 15\frac{1}{2}$
5. Suppose $y = a + e$
6. Then by squaring each part of the fifth Equation, $yy = aa + ee + 2ae$
7. And by subtracting the fifth Equation from the sixth, there remains $yy - y = aa + ee + 2ae - a - e$
8. And from the fourth and seventh Equations, by exchange of equal Quantities, there will arise $yy - y = 15\frac{1}{2}$
9. Which last Equation being resolved by the Canon in *Self. 8. Chap. 15. Book 1*, the number y , to wit, $a + e$ will be made known, viz. $y = a + e = 4\frac{1}{2}$
10. Therefore from the first and ninth Equations, $a + e = ae = 4\frac{1}{2}$
11. From the ninth Equation, by transposition of a , $e = 4\frac{1}{2} - a$
12. The eleventh Equation multiplied by a , produceth $ae = 4\frac{1}{2}a - aa$
13. And from the tenth and twelfth Equations, by exchange of equal Quantities, $4\frac{1}{2}a - aa = 4\frac{1}{2}$
14. Wherefore the last Equation being resolved by the Canon in *Self. 10. Chap. 15. Book 1*, the two numbers sought will be discovered, viz. $\begin{cases} a = 3 \\ e = 1\frac{1}{2} \end{cases}$

So the numbers sought are found 3 and $1\frac{1}{2}$, which will solve the Question; for their sum is equal to the Product of their multiplication, and if their sum $4\frac{1}{2}$ be added to $11\frac{1}{2}$ the sum of their Squares, it makes $15\frac{1}{2}$, as the Question requires.

QUEST. 15.

There are two numbers, such, that the Square of their difference is equal to the Product of their multiplication; and the sum of their Squares makes 20: what are the numbers?

Let a and e be put for the two numbers sought, and let a be the greater; then the Question may be stated thus, viz.

1. If $aa - 2ae + ee = ae$
2. And $aa + ee = 20$

What are the numbers a and e ?

RESOLUTION.

3. From the first Equation by transposition of $-2ae$, this ariseth $aa + ee = 3ae$
4. Therefore from the second and third Equations $3ae = 20$
5. And the third Equation divided by 3, gives $ae = \frac{20}{3}$
6. And by adding the double of the fifth Equation to the second, it makes $aa + ee + 2ae = 22\frac{2}{3}$
7. Therefore by extracting the Square Root of each part of the sixth Equation, the sum of the two numbers sought will be made known, viz. $a + e = \sqrt{22\frac{2}{3}}$
8. From the seventh Equation, by transposition of a , this ariseth $e = \sqrt{22\frac{2}{3}} - a$
9. The eighth Equation multiplied by a , produceth $ae = \sqrt{22\frac{2}{3}} \times a - aa$

10. And

10. And from the fifth and ninth Equations this ariseth, $\sqrt{22\frac{2}{3}} \times a - aa = \frac{20}{3}$
11. Wherefore the last Equation being resolved by the Canon in *Self. 10. Chap. 15. Book 1*, the two numbers sought will be discovered, viz. $\begin{cases} a = \sqrt{8\frac{1}{3}} + \sqrt{1\frac{2}{3}} \\ e = \sqrt{8\frac{1}{3}} - \sqrt{1\frac{2}{3}} \end{cases}$

The Proof.

The difference of the two numbers in the eleventh step is $\sqrt{1\frac{2}{3}} + \sqrt{1\frac{2}{3}} = \sqrt{2\frac{4}{3}}$
 The Square of the said difference is $2\frac{4}{3}$
 And (by the last of the three Rules in *Self. 10. Chap. 9*, of this Book) the Product of the multiplication of the same two numbers is also $2\frac{4}{3}$
 Lastly, (by the first and second of the said three Rules) the sum of the Squares of the said two numbers is 20

QUEST. 16.

There are two numbers, such, that if their sum be multiplied by their difference, the Product is 21; but if the sum of the Squares of those two numbers be multiplied by the difference of their Squares, the Product is 609: what are the numbers?

Let a and e be put for the two numbers sought, and let a represent the greater; then the Question may be stated thus, viz.

1. If $\frac{a+e}{2} \times \frac{a-e}{2} = 21$, that is, $aa - ee = 21$
2. And $\frac{aa+ee}{2} \times \frac{aa-ee}{2} = 609$, that is, $aaaa - eeee = 609$

What are the numbers a and e ?

RESOLUTION.

3. By supposition in the first Equation, $aa - ee = 21$
4. Therefore (by transposition of $-ee$) $aa = ee + 21$
5. And by squaring each part of the fourth Equation this ariseth $aaaa = eeee + 42ee + 441$
6. And by taking the latter part of the fifth Equation instead of $aaaa$ in the second, the said second Equation will be reduced to this, $eeee + 42ee + 441 - eeee = 609$
7. The sixth Equation, after due Reduction, gives $ee = 4$
8. Therefore by extracting the square Root out of each part of the seventh Equation, the lesser number sought is discovered, viz. $e = 2$
9. Then from the fourth and seventh Equations this ariseth, $aa = 4 + 21 = 25$
10. Therefore by extracting the square Root out of each part of the last Equation, the greater number sought is also made known, viz. $a = 5$

So the numbers sought are found 5 and 2, which will solve the Question, as will be evident by the Proof.

QUEST. 17.

There are two numbers, such, that if their sum be multiplied by the sum of their Squares, the Product is 272; but if the difference of the same two numbers be multiplied by the difference of their Squares the Product is 32: what are the numbers?

Put a for the greater number sought, and e for the lesser; then the Question may be stated thus, viz.

1. If $\frac{a+e}{2} \times \frac{aa+ee}{2} = 272$
2. And $\frac{a-e}{2} \times \frac{aa-ee}{2} = 32$

What are the numbers a and e ?

RESOLUTION.

3. By multiplying $a + e$ into $aa + ee$, the first Equation will be reduced to this, $aaaa + aaee + aee + eeee = 272$
4. Likewise,

4. Likewise by multiplying $a - e$ into $aa - ee$, the second Equation will be reduced to this, $aaa - aae - aee + eee = 32$
5. The sum of the third and fourth Equations gives $2aaa + 2eee = 304$
6. The half of the fifth Equation is $aaa + eee = 152$
7. The fourth Equation subtracted from the third, leaves $2aaa - 2eee = 240$
8. The half of the seventh Equation is $aaa + eee = 120$
9. The sum of the seventh and eighth Equations is $3aaa + 3eee = 360$
10. The sum of the sixth and ninth Equations is $aaa + 3aaa + 3eee + eee = 512$
11. The cubick Root of the tenth being extracted, there ariseth $a + e = 8$
12. By dividing each part of the first Equation by the respective part of the eleventh, there will arise $aa + ee = 34$

By the two last Equations, the sum of the two numbers sought is found 8, and the sum of their Squares 34; therefore by the Canon of *Quest. 7. Chap. 16. Book 1.* the numbers themselves will be found 5 and 3, which will solve the Question, as may easily be proved.

QUEST. 18.

To divide a given number 14 (or b) into three continual Proportionals, such, that if the said given number be divided severally by every one of the said three Proportionals, the sum of the three Quotients may be equal to $12\frac{1}{2}$ (or d) a number given.

RESOLUTION.

1. For the first (or least) of the three Proportionals fought put e
2. For the second (or mean) Proportional put a
3. Then the Square of the mean Proportional being divided by the first gives the third, to wit, $\frac{aa}{e}$
4. Therefore the sum of the three Proportionals is $e + a + \frac{aa}{e}$
5. Which sum must be equal to the given number 14, (or b), whence this Equation ariseth, viz. $e + a + \frac{aa}{e} = b$
6. Then by reducing that Equation to Integers, this ariseth $ee + ae + aa = be$
7. Again, (according to the Question) let the given number b be divided by every one of the three Proportionals in the fourth step, so the three Quotients added together, will give $\frac{b}{e} + \frac{b}{a} + \frac{be}{aa}$
8. But the sum of the three Quotients in the seventh step must be equal to the given sum $12\frac{1}{2}$, (or d), hence this Equation ariseth, $\frac{b}{e} + \frac{b}{a} + \frac{be}{aa} = d$
9. Which last Equation reduced to Integers will produce $baaa + baee + baee = daaa$
10. And by dividing every Term of the Equation in the ninth step by a , this ariseth $baa + baee + baee = daa$
11. The sixth Equation multiplied by b , produceth $baa + baee + baee = bbe$
12. And from the tenth and eleventh Equations, (where each of two Quantities is found equal to a common third) this ariseth, viz. $daa = bbe$
13. The twelfth Equation divided by e gives $daa = bb$
14. And the thirteenth Equation divided by d gives $aa = \frac{bb}{d}$
15. Therefore by extracting the Square Root out of each part of the fourteenth Equation, the mean Proportional fought will be made known, viz. $a = \sqrt{\frac{bb}{d}} = 4$
16. And because a is now known, to wit, 4; and $b = 14$; therefore the Equation in the sixth step may be reduced into this, viz. $ee + 4e + 16 = 14e$

17. Which

17. Which last Equation, after due Reduction, will give $10e - ee = 16$
18. Lastly, the Equation in the seventeenth step being resolved by the Canon in *Self. 10. Chap. 15. Book 1.* the first and third Proportionals will be discovered, viz. $e = \frac{2}{8}$

Thus the three Proportionals fought are found 2, 4, 8, which will satisfy the conditions in the Question: For first, 2, 4 and 8 are manifestly in continual proportion; secondly, their sum is 14; thirdly, if 14 be divided by 2, 4 and 8 severally, the sum of the Quotients $7, 3\frac{1}{2}$ and $1\frac{1}{2}$ is $12\frac{1}{2}$, as was prescribed in the Question.

It may also be observed, that those three Quotients are continual Proportionals, as will be manifest from the seventh step of the Resolution, where they are represented by $\frac{b}{e}, \frac{b}{a}$ and $\frac{be}{aa}$; for the Product made by the multiplication of the two extremes,

to wit, the Product $\frac{bbe}{aa}$, that is, $\frac{bb}{aa}$, is equal to the Square of the mean Proportional $\frac{b}{a}$.

QUEST. 19.

To find three numbers in Arithmetical Progression, such, that if the first be multiplied by 1, the second by 2, the third by 3, the sum of the Products may be 62; and that the sum of the Squares of the three numbers may make 275.

Let the three numbers sought be represented by a, e, y , and suppose a to be the smallest and first Term, then the Question may be stated thus, viz.

1. If $a - a = 7 - e$
2. And $a + 2e + 3y = 62$
3. And $aa + ee + yy = 275$

What are the numbers a, e, y ?

RESOLUTION.

4. By supposition in the first step $e - a = 7 - a$
5. Therefore by transposition of $-a$ and $-e$, there ariseth $a + y = 2e$
6. And by dividing each part of the last Equation by 2, it gives $\frac{1}{2}a + \frac{1}{2}y = e$
7. And by squaring the Equation in the sixth step, there comes forth $\frac{1}{4}aa + \frac{1}{4}ay + \frac{1}{4}yy = ee$
8. Then if instead of $2e$ in the second Equation, there be taken the first part of the fifth, the second will be converted into this, viz. $a + a + y + 3y = 62$
9. That is, $2a + 4y = 62$
10. The half of the last Equation is $a + 2y = 31$
11. And by transposition of Quantities in the tenth Equation this ariseth, viz. $31 - 2y = a$
12. And by squaring the eleventh Equation, there comes forth $961 - 124y + 4yy = aa$
13. From the seventh, eleventh and twelfth Equations this ariseth, $241 - 12y + 4yy = ee$
14. It is evident that $yy = yy$
15. And by adding the twelfth, thirteenth and fourteenth Equations into one sum, it makes $241y - 222y + 222\frac{1}{2} = aa + ee + yy$
16. But by supposition in the third step, $275 = aa + ee + yy$
17. Therefore from the fifteenth and sixteenth Equations, by exchange of equal Quantities, $241y - 222y + 222\frac{1}{2} = 275$
18. And after due Reduction the Equation in the seventeenth step gives $11\frac{1}{2}y - yy = 222\frac{1}{2}$
19. Therefore by resolving the Equation in the 18th step, (according to the Canon in *Self. 10. Ch. 15. Book 1.*) two values of y will be discovered, viz. $y = 13$, or $13\frac{1}{2}$
20. And from the 19th and 11th Equations, $a = 5$, or $3\frac{1}{2}$
21. Lastly, from the 20th, 15th and 6th Equations $e = 9$, or $8\frac{1}{2}$

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From the three last Equations 'tis evident, that the three desired numbers a, e, j may be either 5, 9, 13, or $3\frac{1}{2}, 8\frac{1}{2}$, and $13\frac{1}{2}$: For first, 5, 9, 13 are in Arithmetical Progression; and if 5 be multiplied by 1, 9 by 2, and 13 by 3, the sum of the three Products is 62; moreover, the sum of the Squares of 5, 9, 13 makes 275, as was required. The like may be proved by $3\frac{1}{2}, 8\frac{1}{2}$ and $13\frac{1}{2}$.

QUEST. 20.

To find three such numbers, that the Square of the first being added to the Product of the first multiplied into the second may make the sum 48; also, that the Square of the first being subtracted from the Product of the first multiplied into the third, the Remainder may be 32; and that the sum of the Squares of the first and third may have the same proportion to the Square of the second as 5 to 2.

Let the three numbers sought be represented by a, e, j , and then the Question may be stated thus, *viz.*

1. If $aa + ae = 48$
2. And $ay - aa = 32$
3. And $aa + jj - ee :: 5 : 2$

What are the numbers a, e, j ?

RESOLUTION.

4. From the first Equation by transposition of aa , this aritheth, *viz.* $ae = 48 - aa$
 5. And by dividing each part of the last Equation by a , it gives $e = \frac{48 - aa}{a}$
 6. And by transposition of $-aa$ in the second Equation, it makes $ay = aa + 32$
 7. And by dividing the sixth Equation by a , there aritheth $y = \frac{aa + 32}{a}$
 8. From the Analogy in the third step, by comparing the Product of the extremes to the Product of the means, this Equation aritheth, $5ee = 2aa + 2jj$
 9. The Square of the seventh Equation is $jj = \frac{1024 + 64aa + a^4}{aa}$
 10. The double of the ninth Equation is $2jj = \frac{2048 + 128aa + 2a^4}{aa}$
 11. If instead of $2jj$ in the latter part of the eighth Equation there be taken the latter part of the tenth, the eighth will be converted into this, *viz.* $5ee = \frac{2048 + 128aa + 4a^4}{aa}$
 12. The Square of the fifth Equation is $ee = \frac{2304 + 96aa + a^4}{aa}$
 13. The twelfth Equation multiplied by 5 gives $5ee = \frac{11520 + 480aa + 5a^4}{aa}$
 14. From the eleventh and thirteenth Equations, by comparing their latter parts one to the other, and reducing the Equation thereby resulting, this Equation aritheth, *viz.* $608aa - a^4 = 9472$
 15. Which Equation in the 14th step being resolved by the Canon in Sect. 10. Chap. 15. Book 1. will discover two values of a , *viz.* $a = \sqrt{592} \text{ or } 4$
 16. But the lesser of those two values of a , to wit, 4, is the first number sought by the Question, for the Square of the greater value $\sqrt{592}$ exceeds 48, but according to the supposition in the first step it ought to be less than 48; supposing then $a = 4$, it follows from the fifth step, that $e = 8$
 17. Lastly, from the 15th and 7th Equations, $j = 12$
- So three numbers are found out, to wit, 4, 8 and 12; which will satisfy the Question, as may easily be proved.

QUEST. 18.

QUEST. 21.

To find three such numbers, that the Square of the first, together with the Product of the first multiplied by the second may make 10; also, that the Square of the second with the Product of the second into the third may make 21; and lastly, that the Square of the third, with the Product of the third into the first may make 24.

Let the three numbers sought be represented by a, e, j , and then the Question may be stated thus;

1. If $aa + ae = 10$
 2. And $ee + ej = 21$
 3. And $jj + ja = 24$
- What are the numbers a, e, j ?

RESOLUTION.

4. By transposition of aa in the first Equation this aritheth, $ae = 10 - aa$
5. And by dividing each part of the fourth Equation by a , it gives $e = \frac{10 - aa}{a}$
6. And by squaring the fifth Equation it makes $ee = \frac{a^4 - 20aa + 100}{aa}$
7. And from the second, fifth and sixth Equations this aritheth, $\frac{a^4 - 20aa + 100}{aa} + \frac{10 - aa}{a} = 21$
8. And by subtracting $\frac{a^4 - 20aa + 100}{aa}$ from each part of the seventh Equation, this remains, $\frac{10 - aa}{a} = \frac{41aa - 100 - a^4}{aa}$
9. And by dividing each part of the 8th Equation by $\frac{10 - aa}{a}$, this aritheth, $j = \frac{41aa - 100 - a^4}{10a - aa}$
10. And by squaring the ninth Equation it makes $jj = \frac{a^8 - 82a^6 + 1681a^4 - 8200aa + 10000}{100aa - 20a^4 + a^8}$
11. And by multiplying the ninth Equation by a , it produceth $ja = \frac{41aaa - 100aa - a^5}{10a - aa}$
12. And by adding the eleventh Equation to the tenth, the sum makes $jj + ja = \frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^8} = 24$
13. Therefore from the third and twelfth Equations this aritheth, $\frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^8} = 24$
14. Which last preceding Equation, after due Reduction, gives this that follows, *viz.* $-a^8 + 78\frac{1}{2}a^6 - 1435\frac{1}{2}a^4 + 5800aa - 5000 = 0$
15. That is, after transposition of 5000, $-a^8 + 78\frac{1}{2}a^6 - 1435\frac{1}{2}a^4 + 5800aa - 5000 = 0$
16. Then by supposing $u = 2a$, and proceeding according to the Rule in Sect. 7. Chap. 11. of this Second Book, the Equation last above written will be reduced to this following Equation in Integers, *viz.* $-u^8 + 314u^6 - 22968u^4 + 371200u^2 - 1280000 = 0$
17. And by supposing $x = uu$, we may instead of $-u^8$ in the last preceding Equation, write $-x^2$, and instead of $+314u^6$ we may set $314x^3$, also $-22968u^4$ in the place of $-22968u^2$, and $+371200u^2$ instead of $+371200u$, and last of all the Absolute number -1280000 : whence this following Equation aritheth, and then after x is made known, its Square Root shall be the number u ; (for by supposition $x = uu$) $-x^2 + 314x^3 - 22968x^2 + 371200x - 1280000 = 0$
18. Now because the last Term -1280000 in the Equation last above written hath many Divisors which will be useless in the finding of the value of x ; it will be convenient before they be found out, to search out limits, within which such a value of the Root x doth fall as will produce a value of a capable of solving the Question proposed; to which end I proceed thus, *viz.*

19. By the latter part of the fourth Equation it's manifest that $a \supset \sqrt{10}$
 20. And by the second Equation, after transposition of ee , it will likewise appear that $e \supset \sqrt{21}$
 21. Now suppose $e = \sqrt{21}$
 22. Then by multiplying $\sqrt{21}$ instead of e by a in the first Equation, it will be reduced to this, viz. $aa + \sqrt{21} \times a = 10$
 23. Which last Equation being resolved by the Canon in Sect. 6. Chap. 15, Book 1. gives $a = 1\frac{1}{2}\sqrt{21}, \&c.$
 24. And because when e is supposed to be equal to $\sqrt{21}$, the Equation in the twenty second step gives $a = 1\frac{1}{2}\sqrt{21}$, it may easily be conceived that when e is less than $\sqrt{21}$, (as it ought to be) then the first Equation, to wit, $aa + ea = 10$ will necessarily give $a \supset 1\frac{1}{2}\sqrt{21}, \&c.$
 25. Therefore by doubling each part of the nineteenth and twenty fourth steps, it is manifest that $2a \supset \sqrt{40}$
 26. And by squaring each part in the twenty fifth step, it follows that $4aa \supset 40$
 27. But by supposition in the sixteenth step $a = 2a$, and consequently $aa = 4aa$
 28. Therefore from the two last precedent steps it's evident that $aa \supset 40$
 29. And because by supposition in the seventeenth step, $x = aa$
 30. Therefore from the twenty eighth and twenty ninth steps it follows that $x \supset 40$
 31. Having found that such a value of x in the Equation in the seventeenth step as is capable of producing a true value of the desired first number a , must be less than 40, but greater than $10\frac{1}{2}\sqrt{21}$; it is manifest that among the Divisors of 1280000, the last Term of that Equation, these three only, to wit, 16, 20, 32, are necessary to make trials in finding out the said value of x , and consequently of a , and therefore (according to the Rule in Sect. 9. Chap. 11. of this Book) I first divide the said Equation in the seventeenth step, to wit, $-x^3 + 314x^2 - 22968xx - 371200x - 1280000 = 0$ by $a - 16$, and the Quotient is exactly $-x^2 + 298xx - 18200x + 80000$, wherefore 16 shall be a true value of x in that Equation: And because by supposition $x = aa = 4aa$, it follows that $\sqrt{16}$ (that is, \sqrt{x}) $= a = 2a$, and consequently $2 = a$ the first number sought.
 32. Now since 2 is found equal to a , the first Equation, to wit, $aa + ae = 10$ will be reduced to this, viz. $4 + 2e = 10$
 33. Whence the second number e is discovered, viz. $e = 3$
 34. And consequently the second Equation will be reduced to this, $9 + 37 = 21$
 35. Whence the third number y is discovered, viz. $y = 4$

Thus the three numbers sought (to wit, a, e, y .) are found 2, 3, 4, which will solve the Question: For the Square of the first with the Product of the first and second makes 10; also the Square of the second with the Product of the second and third makes 21; and the Square of the third with the Product of the third and first makes 24, as was required.

Note, That the Quotient found out in the thirty first step, to wit, the Equation $-x^3 + 298xx - 18200x + 80000 = 0$ hath three Affirmative Roots, whose values (by the Rule in Sect. 9. Chap. 11. of this Second Book) will be found very near equal to $4\frac{1}{2}\sqrt{21}, 78\frac{1}{2}\sqrt{21}$, and $21\frac{1}{2}\sqrt{21}$; but these are without the limits of x discovered in the thirtieth step, and therefore although the Equation in the fifteenth step may be expounded by four Affirmative values of a , yet only one of them, to wit, 2, is capable of solving the Question proposed.

Note also, That if none of those Divisors which were discovered to be within the limits for the finding of a due value of x had produced an exact Quotient without a Remainder; and consequently in such case the number a had been Irrational, yet a Rational number near the true value of x , and consequently of a , might be found out by the help of the General Method in Chap. 10. of this Second Book.

CHAP.

C H A P. XIII.

Concerning the Resolution of such Arithmetical Questions as are capable of innumerable Answers.

I. After a Question is stated by Equations in such manner as hath been shewn in the foregoing twelfth Chapter, if those Equations be equal in multitude to the Quantities sought, then the Question hath a certain determinable number of Answers; but whensoever a Question affords not as many given Equations, not mutually depending upon one another, as there be Quantities required, it is capable of innumerable Answers. Questions of this latter kind are very pleasant and delightful, but oftentimes exceeding hard to be resolved, especially when all the Answers in whole numbers that a Question is capable of are desired, and therefore I suppose it will not be unacceptable to the Learner, if in this Chapter I give him a taste of that vast skill, by expounding three Propositions found out by Monsieur Bachet; the two first of which contain the substance of the eighteen and twenty-first in his ingenious little Book, entituled *Problemes plaisans & deliceables, qu'il se font par les Nombres*, (printed at Lyons in 1624.) but his Method of solving and demonstrating the same being very tedious and obscure, I shall wave it, and deliver two ways of my own finding out, which are both intelligible and demonstrative. The third Proposition (which is handled by the same Author in his Comment upon the 41. Prop. of the fourth Book of *Diophantus*.) I shall also explain at large by various Questions.

PROP. I.

Two whole numbers prime between themselves being given, to find out two others, suppose a and b , that if a be multiplied by the greater of the two given numbers, and to the Product there be added a given whole number, the sum shall be equal to the Product of b multiplied by the lesser of the two numbers first given. Moreover, to find out all the whole numbers a and b that are capable of producing the same effect.

Explication.

1. Numbers prime between themselves are such as have only Unity for their common Divisor; (per Desm. 12. Elem. 7. Euclid.) so 12 and 5 are said to be Prime between themselves, because they have no common Divisor but 1, to divide them severally, so as to leave no Remainder; the like may be said of 20 and 21, 7 and 3, &c.
 2. I call a number the Multiple of another when it exactly contains that other twice, thrice, or more times, without any Remainder: As, 6 is a Multiple of 3; because it contains 3 exactly twice, likewise 18 is a Multiple of 6, because it contains 6 just thrice without any Remainder. Moreover I take the liberty to call a number the Multiple of it self, because it contains it self just once. These things premised, I shall proceed to shew two ways of solving the preceding Prop. 1. and explain the same by Questions.

Sect. II. The first Method of solving the foregoing Prop. 1.

QUEST. I.

To find out all the values of a and b in whole numbers that may make $9a + 6 = 7b$, viz. that nine times the whole number a with 6 added make seven times the whole number b .

The Equation proposed, $9a + 6 = 7b$;

1	15	14	2
2	24	21	3
3	33	28	4
4	42	35	5
5	51	42	6
6	60		
7	69		

The Resolution :

Explic.

Explication.

1. To the number 9 prefix to a I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the sum is 24, to which I add again 9, and it gives 33; and in like manner I continue the addition of 9 to every next preceding sum until I have found out these seven numbers, 15, 24, 33, 42, 51, 60, 69, which stand (as you see in the Example) under $9a$, and on the left hand of those numbers I set 1, 2, 3, 4, 5, 6, 7. These two Columns of numbers do shew that if 1 be taken for the value of a , then $9a + 6$ makes 15; but if $2 = a$, then $9a + 6 = 24$; if $3 = a$, then $9a + 6 = 33$; and so of the rest. The addition aforesaid is in this Example continued only to the seventh sum inclusive, because (as hereafter will appear) the smallest whole number that can express the value of a , never exceeds the number prefix to b in the Equation propos'd.
2. Then under $7b$ I set the Multiples of 7 orderly one under another, viz. 14, (to wit, twice 7), 21, 28, &c. until I have found out a number equal to one of the seven numbers 15, 24, 33, &c. so at length among the Multiples of 7, I find 42, that is, six times 7, to be equal to 42 that stands among the numbers in the second Column, which later 42 (by the construction aforesaid) is compos'd of 6 and four times 9. Whence it manifest that if 4 be taken for the value of a , and 6 for the value of b , then $9a + 6 = 7b$ ($= 42$), viz. nine times 4 together with 6 is equal to seven times 6, and therefore one Answer to the Question is discovered.

Note 1. When the given whole number prefix to b in the Equation propos'd is a single figure, or some small number of two places, then this first Method will readily discover the smallest values of a and b in whole numbers; for the smallest whole number a never exceeds the given number prefix to b , as hereafter will be made manifest: But if the number prefix to b be large, then the work by this first Method will be intolerably tedious, especially in the solving of Prop. 2.

Note 2. If the two given whole numbers which are prefix to a and b in the Equation propos'd be not prime between themselves, then it will sometimes be impossible to find out any whole numbers for the values of a and b to solve the Proposition; as, if two whole numbers a and b be desired that may make $6a + 3 = 2b$, it may easily be shewn that 'tis impossible to find out two such whole numbers. For the whole number a must be either even or odd, but whether it be even or odd, if it be multiplied by the even number 6 the Product shall be even; (by Prop. 21, & 28. Elem. 9. Euclid.) to which adding 3 the sum will be odd, (for odd added to even makes odd,) which sum must be equal to $2b$, and consequently the half of that sum is the number b , but the half of an odd number cannot be a whole number, and therefore b in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefix to a and b be Prime to one another, then whatever whole number be given to be added to the desired Multiple of a , innumerable whole numbers may be found out for the values of a and b , as hereafter will be shewn.

3. After the two smallest whole numbers are found out for the values of a and b to constitute the Equation propos'd, all other pairs of whole numbers that are capable of producing the same effect, may be orderly enumerated in two Arithmetical Progressions thus formed, viz. Having found 4 for the smallest whole number a , and 6 for the smallest whole number b to constitute the Equation before propos'd, to wit, $9a + 6 = 7b$, let the said 4 be made the first Term, and 7, which is prefix to b , the common difference of the Terms of the first Progression; then let 6, the smallest whole number b , be the first Term, and 9 which is prefix to a in the said Equation, the common difference of the Terms of the latter Progression; so the Terms of those Progressions will be these, viz.

Values of a ; 4, 11, 18, 25, 32, 39, 46, 53, &c.
Values of b ; 6, 15, 24, 33, 42, 51, 60, 69, &c.

4. Now out of the first of those Progressions you may take any Term for the value of a , as 11, (the second Term,) and then the correspondent Term in the latter Progression, to wit, 15, shall be the value of b ; by which two numbers 11 and 15 the Equation $9a + 6 = 7b$ may be expounded, viz. nine times 11 with 6 added is equal to seven times 15. Likewise 18 and 24, also 25 and 33, and every pair of correspondent Terms in those two Progressions will cause the same effect, as I shall now demonstrate.

Proq.

Preparation.

5. Let e and n represent two whole numbers Prime between themselves, and a, b, d three other whole numbers, such, that all five will make this Equation, viz. $ea + d = nb$
6. Let an Arithmetical Progression be so formed that a may be the first and least Term, and n the common difference of the Terms, as, $a, a + n, a + 2n, \&c.$
7. Let another Arithmetical Progression be formed from b the first and least Term, and e the common difference of the Terms, as, $b, b + e, b + 2e, \&c.$
8. I say, if you multiply e by $a + n$, (the second Term of the first Progression,) instead of a in the Equation in the fifth step, and to the Product add d , the sum shall be equal to a Multiple of n , to wit, the Product of n multiplied into $b + e$, (the second Term of the latter Progression;) and the like may be affirmed of every following Term in each Progression.

Demonstration.

9. By supposition in the fifth step, $ea + d = nb$
10. And by adding en to each part of that Equation, this arith, $ea + en + d = nb + en$
11. Therefore from the last Equation, Which was to be shewn, $e(a + n) + d = n(b + e)$
12. Again, if to each part of the Equation first granted in the ninth step you add $2en$, it makes $e(a + 2n) + d = n(b + 2e)$
13. That is, $e(a + 2n) + d = n(b + 2e)$
14. After the same manner it may be shewn that $e(a + 3n) + d = n(b + 3e)$ And so forwards. Which was to be proved.
15. Now supposing a and b to express the smallest whole numbers that are capable of constituting the Equation in the fifth step, to wit, $ea + d = nb$, I shall demonstrate that no other whole numbers besides the Terms which follow a and b in the two Progressions formed in the sixth and seventh steps, can be taken instead of a and b to produce the same effect: If it be possible, let $a + f$ some whole number f , viz. $a + f$ be taken instead of a ; and let $b + g$ some whole number g , viz. $b + g$ be taken instead of b ; then e multiplied by $a + f$ makes $ea + ef$, to which adding d , the sum is $ea + ef + d$, which must be equal to the Product of n multiplied by $b + g$, to wit, $nb + ng$, whence $ea + ef + d = nb + ng$
16. And by supposition in the fifth step, $ea + d = nb$
17. Therefore by subtracting the last Equation from the last but one, this remains, $ef = ng$
18. And by resolving the last Equation into Proportions, this Analogy arith, viz. $n : e :: f : g$
19. Whence it is manifest that the whole numbers f and g are in the same Reason (or Proportion) as the whole numbers n and e ; and consequently, since n and e are by supposition whole numbers Prime between themselves, f must necessarily be equal either to n , or $2n$, or $3n$, &c. and g must be equal to e , or $2e$, or $3e$, &c. Wherefore $a + n, a + 2n, a + 3n, \&c.$ the Terms which follow a in the Progression in the sixth step, and $b + e, b + 2e, b + 3e, \&c.$ the Terms which follow b in the Progression in the seventh step, are the only whole numbers that can be taken instead of a and b , the least whole numbers to constitute the Equation propos'd, to wit, $ea + d = nb$. Which was to be shewn.
20. If there be two whole numbers a and b , given or found out, which will constitute the Equation before propos'd, or such like, and those two numbers be not the smallest values of a and b , you may by the help of those given find out the smallest, by this Rule, viz. Divide the given whole number a , by the given number which is prefix to b in the Equation propos'd, then after the division is finish'd there will remain either a number or nothing; if a number remain, it shall be the smallest value of a , but if 0 remain, then the number prefix to b is the smallest value of a , and consequently the correspondent value of b is easily discovered by the Equation. The Reason of this Rule is manifest by Sect. 9. Chap. 17. Book 1. For if any Term greater than the least of an Arithmetical Progression

Explication.

1. To the number 9 prefix to a I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the sum is 24, to which I add again 9, and it gives 33; and in like manner I continue the addition of 9 to every next preceding sum until I have found out these seven numbers, 15, 24, 33, 42, 51, 60, 69, which stand (as you see in the Example) under $9a$, and on the left hand of those numbers I set 1, 2, 3, 4, 5, 6, 7. These two Columns of numbers do shew that if 1 be taken for the value of a , then $9a + 6$ makes 15; but if $2 = a$, then $9a + 6 = 24$; if $3 = a$, then $9a + 6 = 33$; and so of the rest. The addition aforesaid is in this Example continued only to the seventh sum inclusive, because (as hereafter will appear) the smallest whole number that can express the value of a , never exceeds the number prefix to b in the Equation propos'd.
2. Then under $7b$ I set the Multiples of 7 orderly one under another, viz. 14, (to wit, twice 7), 21, 28, &c. until I have found out a number equal to one of the seven numbers 15, 24, 33, &c. so at length among the Multiples of 7, I find 42, that is, six times 7, to be equal to 42 that stands among the numbers in the second Column, which later 42 (by the construction aforesaid) is compos'd of 6 and four times 9. Whence 'tis manifest that if 4 be taken for the value of a , and 6 for the value of b , then $9a + 6 = 7b$ ($= 42$), viz. nine times a together with 6 is equal to seven times b , and therefore one Answer to the Question is discovered.

Note 1. When the given whole number prefix to b in the Equation propos'd is a single figure, or some small number of two places, then this first Method will readily discover the smallest values of a and b in whole numbers; for the smallest whole number a never exceeds the given number prefix to b , as hereafter will be made manifest: But if the number prefix to b be large, then the work by this first Method will be intolerably tedious, especially in the solving of Prop. 2.

Note 2. If the two given whole numbers which are prefix to a and b in the Equation propos'd be not prime between themselves, then it will sometimes be impossible to find out any whole numbers for the values of a and b to solve the Proposition: as, if two whole numbers a and b be defined that may make $6a + 3 = 2b$, it may easily be shewn that 'tis impossible to find out two such whole numbers. For the whole number a must be either even or odd, but whether it be even or odd, if it be multiplied by the even number 6 the Product shall be even; (by Prop. 21, & 28. Elem. 9. Euclid.) to which adding 3 the sum will be odd, (for odd added to even makes odd,) which sum must be equal to $2b$, and consequently the half of that sum is the number b ; but the half of an odd number cannot be a whole number, and therefore b in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefix to a and b be Prime to one another, then whatever whole number be given to be added to the desired Multiple of a , innumerable whole numbers may be found out for the values of a and b , as hereafter will be shewn.

3. After the two smallest whole numbers are found out for the values of a and b to constitute the Equation propos'd, all other pairs of whole numbers that are capable of producing the same effect, may be orderly enumerated in two Arithmetical Progressions thus formed, viz. Having found 4 for the smallest whole number a , and 6 for the smallest whole number b to constitute the Equation before propos'd, to wit, $9a + 6 = 7b$, let the said 4 be made the first Term, and 7, which is prefix to b , the common difference of the Terms of the first Progression; then let 6, the smallest whole number b , be the first Term, and 9 which is prefix to a in the said Equation, the common difference of the Terms of the latter Progression, so the Terms of those Progressions will be these, viz.

Values of a ; 4, 11, 18, 25, 32, 39, 46, 53, &c.
Values of b ; 6, 15, 24, 33, 42, 51, 60, 69, &c.

4. Now out of the first of those Progressions you may take any Term for the value of a , as 11, (the second Term,) and then the correspondent Term in the latter Progression, to wit, 15, shall be the value of b ; by which two numbers 11 and 15 the Equation $9a + 6 = 7b$ may be expounded, viz. nine times 11 with 6 added is equal to seven times 15. Likewise 18 and 24, also 25 and 33, and every pair of correspondent Terms in those two Progressions will cause the same effect, as I shall now demonstrate.

Prepa-

Preparation.

5. Let e and n represent two whole numbers Prime between themselves, and a, b, d three other whole numbers, such, that all five will make this Equation, viz. $ea + d = nb$
6. Let an Arithmetical Progression be so formed that a may be the first and least Term, and n the common difference of the Terms, as, $a, a + n, a + 2n, \&c.$
7. Let another Arithmetical Progression be formed from b the first and least Term, and e the common difference of the Terms, as, $b, b + e, b + 2e, \&c.$
8. I say, if you multiply e by $a + n$, (the second Term of the first Progression,) instead of a in the Equation in the fifth step, and to the Product add d , the sum shall be equal to a Multiple of n , to wit, the Product of n multiplied into $b + e$, (the second Term of the latter Progression;) and the like may be affirmed of every following Term in each Progression.

Demonstration.

9. By supposition in the fifth step, $ea + d = nb$
10. And by adding en to each part of that Equation, this arith, $ea + en + d = nb + en$
11. Therefore from the last Equation, $e(a + n) + d = n(b + e)$ Which was to be shewn.
12. Again, if to each part of the Equation first granted in the ninth step you add $2en$, it makes $ea + 2en + d = nb + 2en$
13. That is, $e(a + 2n) + d = n(b + 2e)$
14. After the same manner it may be shewn that $e(a + 3n) + d = n(b + 3e)$ And so forwards. Which was to be proved.
15. Now supposing a and b to express the smallest whole numbers that are capable of constituting the Equation in the fifth step, to wit, $ea + d = nb$, I shall demonstrate that no other whole numbers besides the Terms which follow a and b in the two Progressions formed in the sixth and seventh steps, can be taken instead of a and b to produce the same effect: If it be possible, let $a + f$ some whole number f , viz. $a + f$ be taken instead of a ; and let $b + g$ some whole number g , viz. $b + g$ be taken instead of b ; then e multiplied by $a + f$ makes $ea + ef$, to which adding d , the sum is $ea + ef + d$, which must be equal to the Product of n multiplied by $b + g$, to wit, $nb + ng$, whence $ea + ef + d = nb + ng$
16. And by supposition in the fifth step, $ea + d = nb$
17. Therefore by subtracting the last Equation from the last but one, this remains, $ef = ng$
18. And by resolving the last Equation into Proportions, this Analogy arith, viz. $n : e :: f : g$
19. Whence it is manifest that the whole numbers f and g are in the same Reason (or Proportion) as the whole numbers n and e ; and consequently, since n and e are by supposition whole numbers Prime between themselves, f must necessarily be equal either to n , or $2n$, or $3n$, &c. and g must be equal to e , or $2e$, or $3e$, &c. Wherefore $a + n, a + 2n, a + 3n, \&c.$ viz. the Terms which follow a in the Progression in the sixth step, and $b + e, b + 2e, b + 3e, \&c.$ viz. the Terms which follow b in the Progression in the seventh step, are the only whole numbers that can be taken instead of a and b , the least whole numbers to constitute the Equation propos'd, to wit, $ea + d = nb$. Which was to be shewn.
20. If there be two whole numbers a and b , given or found out, which will constitute the Equation before propos'd, or such like, and those two numbers be not the smallest values of a and b , you may by the help of those given find out the smallest, by this Rule, viz. Divide the given whole number a , by the given number which is prefix to b in the Equation propos'd, then after the division is finish'd there will remain either a number or nothing; if a number remain, it shall be the smallest value of a , but if 0 remain, then the number prefix to b is the smallest value of a , and consequently the correspondent value of b is easily discovered by the Equation. The Reason of this Rule is manifest by Self. 9. Chap. 17. Book 1. For if any Term greater than the least of an Arithmetical Progression

Progression be given, as also the common Difference, the least Term shall be given also, either by a continual subtraction of the common Difference, or by the *Rule* above express'd.

As, for example, If in the former of the two Arithmetical Progressions in the third step, which express values of a and b to constitute the Equation $9a + 6 = 7b$, there be given 31 for the value of a , I divide 32 by 7 which is prefix to b , and find 7 contain'd four times in 32, and there remains 4; now this Remainder 4 is the smallest value of a , whence the correspondent whole number b is easily discovered; for if $a = 4$, then $9a + 6 = 42 = 7b$. Therefore 42 divided by 7 gives 6 for the whole number b .

Again, If $a = 20$, and $b = 26$, then this will be a true Equation, *viz.* $5a + 4 = 4b$; now if you desire the smallest whole numbers a and b to constitute that Equation, divide 20 the given value of a by 4 which is prefix to b , and there remains 0, therefore (according to the *Rule* before given) the said 4 shall be the smallest value of a ; whence $5a + 4 = 24 = 4b$, and consequently $6 = b$.

Lastly, from what hath been said in the third step, all the values of a and b in whole numbers that are capable of constituting the said Equation $5a + 4 = 4b$ are the Terms of these two Arithmetical Progressions, *viz.*

Values of a ; 4, 8, 12, 16, 20, 24, 28, 32, &c.
Values of b ; 6, 10, 14, 18, 22, 26, 30, 34, 38, &c.

SECT. III. Another way of solving the foregoing Prop. 1.

In this latter Method there are four principal Cases, which I shall first explain by Questions, and then shew how the Resolution of the Proposition will always run into one of those four Cases.

QUEST. 2.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $8a + 97 = 5b$.

$$\begin{array}{lcl} \text{The Equation proposed,} & . & . & . & 1 & 8a + 97 = 5b \\ & & & & 2 & 8 + 97 = 105 \\ \text{The Resolution,} & . & . & . & 3 & 105 = 21 = b \\ & & & & 4 & 1 = a \end{array}$$

Explication.

First I add 97 (to wit, $+97$ in the Equation proposed) to 8, which is prefix to a , and it makes 105, this I divide by 5 the number prefix to b ; and because the Quotient happens to be exactly a whole number without any Remainder, it shall be the smallest whole number b sought, and the whole number a in this case is always 1. The Reason is evident, for if $a = 1$, then $8a + 97 = 8 + 97$; and if this sum happens to be a Multiple of the given number prefix to b , then b is necessarily a whole number. This is the first of the four Cases above mentioned.

Then after 1 and 21, the smallest whole numbers a and b to constitute the Equation propos'd, are found out, all the other values of a and b in whole numbers will be found in these two following Arithmetical Progressions formed according to the *Rule* in the third step of the foregoing Sect. 2. *viz.*

Values of a ; 1, 6, 11, 16, 21, 26, &c.
Values of b ; 21, 29, 37, 45, 53, 61, &c.

I say, every two correspondent numbers in those Progressions may be taken for values of a and b in this Equation, $8a + 97 = 5b$; as, for example, if 11 be taken for a , and 37 for b , then eight times 11, with 97 added shall be equal to five times 37, *viz.* $185 = 185$. And so of the rest.

QUEST. 3.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $49a + 6 = 13b$.

$$\begin{array}{lcl} \text{The Equation proposed,} & . & . & . & 1 & 49a + 6 = 13b \\ & & & & 2 & 55 = 65 - 10 \\ & & & & 3 & 49 = 39 + 10 \\ \text{The Resolution,} & . & . & . & 4 & 104 = 104 \\ & & & & 5 & 104 = 8 = b \\ & & & & 6 & 13 \\ & & & & 7 & 104 - 6 = 2 = a \\ & & & & 8 & 49 \end{array}$$

Explication.

First, I add 6 (to wit, $+6$ in the Equation proposed) to 49 which is prefix to a , and it makes 55; now if this 55 were exactly divisible by 13 which is prefix to b , the Quotient would be the whole number b sought, and 1 the number a , (as in *Quest. 2.*) But 55 not being a Multiple of 13, I proceed thus, *viz.* I seek the Multiple of 13 which is next greater than 55, by dividing 55 by 13, so I find that four times 13 is less than 55, but five times 13, that is, 65, exceeds 55 by 10; and therefore 55 is equal to 65 wanting 10, *viz.* $55 = 65 - 10$. This is the second Equation in the Example.

2. Then I divide 49 which is prefix to a , by 13 which is prefix to b , so I find that three times 13, that is, 39, is the greatest Multiple of 13 contained in 49, and there remains 10; therefore $49 = 39 + 10$: which is the third Equation.

3. Now because -10 is found in the third Equation, and $+10$ in the second, I add those Equations together, so the said 10 vanisheth; and there ariseth $104 = 104$, which is the fourth Equation.

4. Then I divide 104, that is, either part of the fourth Equation, by 13 which is prefix to b in the Equation propos'd, and the Quotient 8 is the whole number b sought.

5. Then from the said 104 in the fourth Equation, I subtract 6, (to wit, $+6$ in the Equation propos'd) and divide the Remainder 98 by 49 which is prefix to a , so the Quotient gives 2 for the whole number a sought.

I say $2 = a$ and $8 = b$ will make $49a + 6 = 13b$, as was required in *Quest. 3.* and all the values of a and b in whole numbers that are capable of producing the same effect, are the Terms of these two following Arithmetical Progressions whose construction hath been shewn before.

Values of a ; 2, 15, 28, 41, 54, 67, &c.

Values of b ; 8, 57, 106, 155, 204, 253, &c.

Note. That the manner of forming the second and third Equations in the foregoing Resolution of *Quest. 3.* must be diligently observed; because the like work is constantly used in the following fourth, fifth, sixth, seventh, eighth and ninth Questions: But by accident, that the same number 10 follows the signs $-$ and $+$ in the said second and third Equations, and therefore the adding them together to produce the fourth Equation, is an Operation peculiar only to this and the like accident, which I call the second of the four Cases before mentioned.

But that in this second Case, the Resolution infallibly produceth whole numbers for the values of a and b , I prove thus: First by Construction, $65 - 10$ (the latter part of the second Equation) wants 10 of a Multiple of 13, and $39 + 10$ (the latter part of the third Equation) exceeds a Multiple of 13 by 10; therefore the sum of the said $65 - 10$ and $39 + 10$, to wit, 104 (the latter part of the fourth Equation) shall be a Multiple of 13; and consequently 104 divided by 13 will exactly give a whole number, to wit, 8, for the value of b . Secondly, because 104 (the first part of the fourth Equation) is by construction compos'd of a Multiple of 49 together with 6; by subtracting 6 from 104, the Remainder 98 shall be a Multiple of 49, and consequently 98 divided by 49 will give the Quotient an exact whole number, to wit, 2, for the value of a . Whence it is manifest, that if after the second and third Equations are formed out of the first, (to wit, the Equation proposed) according to the preceding directions for solving *Quest. 3.* it happens that the number following $+$ in the latter part of the third Equation, is the same with the number following $-$ in the latter part of the second, there will certainly arise two whole numbers for the values of a and b .

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QUEST. 4:

QUEST. 4.

To find all the whole numbers a and b that may make $82a + 66 = 13b$;

The Equation propos'd,	1	$82a + 66 = 13b$
	2	$148 = 156 - 8$
	3	$82 = 78 + 4$
	4	$164 = 156 + 8$
	5	$312 = 312$
	6	$312 = 24 = b$
	7	$112 - 66 = 3 = a$

The Resolution,

Explication.

1. The second and third Equations are formed out of the first in such manner as before hath been explain'd in the Resolution of *Quest. 3.*

2. Because the number 4 which follows the sign $+$ in the latter part of the third Equation, happens to be an Aliquot part, to wit, $\frac{1}{2}$ of 8 which follows the sign $-$ in the latter part of the second Equation, I multiply each part of the third Equation by 2 (the Denominator of the said Aliquot part,) to the end there may be $+8$ in the Equation made by that Multiplication; so there is produced $164 = 156 + 8$, which is the fourth Equation.

3. Now since $+8$ is found in the fourth Equation, and -8 in the second, I add those Equations together, so the said 8 vanisheth, and there ariseth $312 = 312$, which is the fifth Equation.

4. Then I divide 312, (to wit, either part of the fifth Equation) by 13 which is prefix to b in the Equation propos'd, and the Quotient 24 is the whole number b sought.

5. Lastly, from the said 312, (in the fifth Equation) I subtract 66, to wit, $+66$ in the Equation propos'd, and divide the Remainder 246 by the given number 82, (which is prefix to a), so the Quotient 3 is the whole number a sought.

I say, 3 = a and 24 = b will make $82a + 66 = 13b$, as was required in *Quest. 4* and all the values of a and b in whole numbers that are capable of producing that Equation, are the Terms of these two Arithmetical Progressions, (whose Construction hath been shewn before in the third step of *Self. 2.*) viz.

Values of a : 3, 16, 29, 42, 55, 68, &c.

Values of b : 24, 106, 188, 270, 352, 434, &c.

Note. That it was by meer chance that the number following the sign $+$ in the third Equation happened to be an Aliquot part of the number following the sign $-$ in the second, and therefore the multiplying of the third Equation by the Denominator of the Aliquot part, is an Operation peculiar only to that and the like accident, which is the third of the four Cases before-mentioned. The reason of the Operation in this fourth Question (of third Case,) may be easily discerned by the Demonstration before given in *Quest. 3.* but for further illustration I shall add another Example of *Case 3.*

QUEST. 5.

To find all the whole numbers that may be values of a and b in this Equation, viz. $601a + 9 = 200b$.

The Equation propos'd,	1	$601a + 9 = 200b$
	2	$610 = 800 - 190$
	3	$601 = 600 + 1$
	4	$114190 = 114000 + 190$
The Resolution,	5	$114800 = 114800$
	6	$114800 = 574 = b$
	7	$114800 - 9 = 191 = a$
	8	601

Explication.

Explication.

The Resolution of this Question is like that in the foregoing *Quest. 4.* for since $+1$ in the latter part of the third Equation happens to be an Aliquot part of 190 which followeth — in the second Equation, I multiply each part of the third by 190, to the end that $+190$ may be found in the Product, as you see in the fourth Equation; then by adding the fourth Equation to the second, the sum makes the fifth, which is free from the signs $+$ and $-$; lastly, from the fifth Equation the whole numbers 574 and 191 expressing the values of b and a are discovered, in like manner as in the preceding third and fourth Questions; which numbers will constitute the Equation propos'd: For 601 times 191 together with 9 is equal to 200 times 574, that is, 114800; and all the rest of the values of a and b in whole numbers to make that Equation will be found in these two following Arithmetical Progressions formed by the Rule before given in the third step of *Self. 2.*

Values of a : 191, 391, 591, 791, 991, &c.

Values of b : 574, 1175, 1776, 2377, 2978, &c.

QUEST. 6.

If	1	$121a + 5 = 93b$	{	What are a and b in whole numbers?
	2	$126 = 186 - 60$		
Out of 1.	3	$121 = 92 + 28$	{	$e = ? \quad d = ?$
	4	$93e + 60 = 28d$		
Suppose	5	$153 = 168 - 15$	{	$e = ? \quad f = ?$
	6	$93 = 84 + 9$		
Out of 4.	7	$28e + 15 = 9f$	{	Here the Regressive work begins.
	8	$43 = 45 - 2$		
Suppose	9	$28 = 27 + 1$	{	
	10	$56 = 54 + 2$		
Eq. 9 $\times 2$.	11	$99 = 99$	{	
	12	$22 = 21 = f$		
Eq. 8 $+ 10$.	13	$11 \times 93 + 153 = 1176$	{	
	14	$1176 = 42 = d$		
Out of 11 and 7.	15	$42 \times 121 + 126 = 5208$	{	
	16	$5208 = 56 = b$		
12, 6 and 5.	17	$5208 - 5 = 43 = a$	{	
	18	121		

Explication.

1. The second and third Equations are formed out of the first in like manner as before in the Explication of *Quest. 3.*

2. But because 28 which follows $+$ in the third Equation, is not equal to, nor an Aliquot part of 60 which follows — in the second, the process cannot be made like that in the third, fourth and fifth Questions; so that now a fourth Case takes rise, and the scope of a new search is to find out a number d , such, that if it multiply the said $+28$, the Product may exceed a Multiple of 93 (which is prefix to b) by 60; for then it will be evident, that if the third Equation be multiplied by that number d , an Equation will be produced whose first part shall be a Multiple of 121, and the latter part shall exceed a Multiple of 93 by 60, and then the rest of the work will be like that in *Case 2.* in *Quest. 3.* In the search therefore of the number d , the fourth Equation is assumed, to wit, $93e + 60 = 28d$.

3. The fifth and sixth Equations are formed out of the fourth, in like manner as the second and third out of the first.

4. Because 9 which follows $+$ in the sixth Equation, is neither equal to, nor an Aliquot part of 15 which follows the sign — in the fifth, the next scope (for the like reason before given

Q. 2

given

given concerning the number d) is to find out a number f , such, that if it multiply the said $+9$, the Product may exceed a Multiple of 28 which is prefix to d , by the said 15; to which end the seventh Equation is assumed, to wit, $28a + 15 = 9f$.

5. The eighth and ninth Equations are formed out of the seventh, in like manner as the second and third out of the first.

6. Because 1 which follows $+$ in the ninth Equation, is an Aliquot part of 2 which stands next after $-$ in the eighth, the ninth is multiplied by 2 the Denominator of the said part; (according to the Rule in Case 3. *Quest.* 3.) whence the tenth Equation is produced, to wit, $56 = 54 + 2$.

7. The eleventh Equation, to wit, $99 = 99$ is the sum of the eighth and tenth; and since the said eleventh is free from the signs $+$ and $-$, a Regressive work now begins, to find out the whole numbers f, d, b and a ; in this manner, *viz.*

8. By dividing either part of the eleventh Equation, to wit, 99 , by 9 which is prefix to f in the seventh, there ariseth $11 = f$, as in the twelfth Equation.

9. Then multiplying the number f to wit, 11, by 93, that is, either part of the fifth Equation, and to the Product adding 153, that is, either part of the fifth Equation, the sum makes 1176, (as you see in the thirteenth Equation,) which 1176 is a Multiple of 28, to wit, that which is represented by $28d$ in the fourth Equation; Therefore,

10. By dividing the said 1176 by 28, the Quotient 42 is the number d , as in the fourteenth Equation.

11. Then multiplying the number d , to wit, 42, by 121, that is, either part of the third Equation, and to the Product adding 126, that is, either part of the second Equation, the sum makes 5208, as you see in the fifteenth Equation, which 5208 is a Multiple of 93, to wit, that which is represented by $93b$ in the first Equation; Therefore,

12. By dividing either part of the fifteenth Equation, to wit, 5208, by 93, the Quotient 56 is the number b found.

13. Then from the said 5208 subtracting 5, to wit, $+5$ in the first Equation; and dividing the Remainder 5203 by 121 which is prefix to a in the first Equation, the Quotient gives 43 for the number a sought, as in the seventeenth and last Equation. Therefore, if 43 be taken for a , and 56 for b , then $121a + 5 = 93b$, which is the Equation proposed in *Quest.* 6. and all the values of a and b in whole numbers that are capable of constituting that Equation are the Terms of these two following Arithmetical Progressions, whose Construction hath been shewn before in the third step of *Self.* 2.

Values of a ; 43, 136, 229, 322, 415, 508, &c.

Values of b ; 56, 177, 298, 419, 540, 661, &c.

14. After the numbers f and d in the foregoing Resolution of *Quest.* 6. are known, the numbers a and c in the seventh and fourth Equations may easily be discovered; but there is no need of their help in the finding out of the desired numbers a and b .

15. But me-thinks I hear the Reader make this Objection, *viz.* How doth it appear, that from every three whole numbers given in such sort as before is declared in *Prop.* 1. there may infallibly be found out two whole numbers a and b to solve the said Proposition, by the Operation before explained in the four Cases before mentioned: For Answer to this Objection, I shall here shew how far the Process need be continued at the farthest, to find out an Equation having $+1$ in its latter part; for when such Equation ariseth, 'tis manifest by the Operation in the third Case explain'd in *Quest.* 4. and 5. that two whole numbers a and b will infallibly be discovered to satisfy the Proposition, and consequently innumerable other pairs of whole numbers to produce the same effect. First then in the foregoing *Quest.* 6. the given number 121 which is prefix to a , being divided by the given number 93 which is prefix to b , after the Division is finish'd there remains 28, to wit, $+28$ in the latter part of the third Equation: Secondly, the said Divisor 93 being divided by the said Remainder 28, after the Division is ended there remains 9, to wit, $+9$ in the latter part of the sixth Equation: Again, the last Divisor 28 being divided by the last Remainder 9, after this Division is ended there remains 1, that is, $+1$ in the latter part of the ninth Equation, which Remainder 1 you will always infallibly come unto by a continued Division in that manner, because the two given numbers prefix to a and b are (as the Proposition requires) Prime between themselves; and that continued Division is nothing else but the Method of finding out the greatest common Divisor unto two numbers; 10

so that you may at first (if you please) discover unto what letter at the farthest, the process need be continued before you return backward according to the Operation explain'd in *Quest.* 6. But oftentimes before you come to the said Remainder 1, the Resolution will run into one of the three Cases explain'd in *Quest.* 2, 3, 4, and 5. as will appear by the following seventh, eighth, and ninth Questions.

QUEST. 7.

If	1	$97a + 1 = 26b$,	What are a and b in whole numbers?
	2	$98 = 104 - 6$	
Out of 1.	3	$97 = 78 + 19$	$c = ?$ $d = ?$
	4	$26c + 6 = 19d$	
Suppose	5	$32 = 38 - 6$	$c = ?$ $f = ?$
	6	$26 = 19 + 7$	
Out of 4.	7	$19c + 6 = 7f$	$g = ?$ $h = ?$
	8	$25 = 28 - 3$	
Suppose	9	$19 = 14 + 5$	Here the Regressive work begins.
	10	$7g + 3 = 10$	
Out of 10.	11	$\frac{10}{5} = 2 = h$	
	12	$\frac{10}{5} = 2 = h$	
Out of 10, & 11.	13	$2 \times 19 + 25 = 63$	
	14	$\frac{63}{7} = 9 = f$	
Out of 12, 9, 8.	15	$9 \times 26 + 32 = 266$	
	16	$\frac{266}{19} = 14 = d$	
14, 6, and 5.	17	$14 \times 97 + 98 = 1456$	
	18	$\frac{1456}{26} = 56 = b$	
15, and 4.	19	$\frac{1456 - 1}{97} = 15 = a$	
	20		

Explication.

In this seventh Question the process is formed like that in the foregoing sixth, and the last letter in the work is b , whose value is discovered in the twelfth Equation by the help of the tenth and eleventh, according to the Operation in *Quest.* 2. and then by the help of the number b , the work returns backward to find out the numbers f, d, b and a , in like manner as in *Quest.* 6. But in this seventh Question the last letter in the process, to wit, b , is made known before an Equation ariseth which hath $+1$ in its latter part; and the like effect happens in the following eighth and ninth Questions.

Now in answer to this seventh Question, all the values of a and b in whole numbers that are capable of constituting the Equation proposed, to wit, $97a + 1 = 26b$, are the Terms of the two following Arithmetical Progressions, which are deduced from the two smallest values of a and b . (to wit, 15 and 56 found out as above,) according to the Rule in the third step of *Self.* 2.

Values of a ; 15, 41, 67, 93, 119, 145, &c.

Values of b ; 56, 153, 250, 347, 444, 541, &c.

QUEST. 8.

QUEST. 8.

If	1	$119a + 6 = 57b$	What are the whole numbers a and b ?
Out of 1.	2	$125 = 171 - 46$	
Suppose	3	$119 = 114 + 5$	$c = ?$ $d = ?$
Out of 4.	4	$57a + 46 = 5d$	
5 + 6.	5	$103 = 105 - 2$	Regrefs.
7, 4.	6	$57 = 55 + 2$	
8, 3, 2.	7	$160 = 160$	$c = ?$ $d = ?$
9, 1.	8	$160 = 32 = d$	
9, 1.	9	$32 \times 119 + 125 = 3933$	Regrefs.
	10	$1933 = 69 = b$	
	11	$57 = 57$	$c = ?$ $d = ?$
	12	$3933 - 6 = 33 = a$	

Values of a ; 33, 99, 147, 204, 261, 318, &c.

Values of b ; 69, 188, 307, 426, 545, 664, &c.

In which Progressions, every two correspondent Terms may be taken for values of a and b to constitute the Equation in Quest. 8.

QUEST. 9.

If	1	$173a + 1 = 71b$	What are the whole numbers a and b ?
Out of 1.	2	$174 = 213 - 39$	
Suppose	3	$173 = 142 + 31$	$c = ?$ $d = ?$
Out of 4.	4	$71a + 39 = 31d$	
Suppose	5	$110 = 124 - 14$	$c = ?$ $f = ?$
Out of 7.	6	$71 = 62 + 9$	
8, and 7.	7	$31a + 14 = 45$	Regrefs.
	8	$45 = 9$	
9, 6, 5.	9	$5 \times 71 + 110 = 465$	$c = ?$ $f = ?$
10, 4.	10	$465 = 15 = d$	
11, 3, 2.	11	$15 \times 173 + 174 = 2769$	$c = ?$ $f = ?$
12, 1.	12	$2769 = 39 = b$	
12, 1.	13	$2769 - 1 = 16 = a$	$c = ?$ $f = ?$
	14	$173 = 173$	

Values of a ; 16, 87, 158, 229, 300, 371, &c.

Values of b ; 39, 212, 385, 558, 731, 904, &c.

SECT. 4. PROP. II.

Two whole numbers Prime between themselves being given, to find out two others, suppose a and b , that if a be multiplied by the lesser of those two numbers given, and to the Product there be added a whole number given, the sum shall be equal to the Product of b multiplied by the greater of the two numbers first given. Moreover, to discover all the whole numbers a and b that are capable of producing the same effect.

When each of the two given numbers which are Prime between themselves is a single figure, or some small number consisting of two Characters, then the first of the two ways of solving the foregoing Prop. 1. will readily solve this second; but waving that Method, I shall shew two other ways by the help of the latter of those two Methods.

The

The first Method of solving Prop. 2.

QUEST. 10.

If	1	$71a + 3 = 173b$	What are a and b in whole numbers?
Out of 1.	2	$145 = 173 - 28$	
By Prop. 1.	3	$2769 = 2768 + 1$	$c = ?$ $d = ?$
Eq. 3×28 .	4	$77532 = 77504 + 28$	
2 + 4.	5	$27677 = 77677$	true Values.
Out of 5, 1.	6	$77677 = 449 = b$	
5, 1.	7	$173 = 173$	$c = ?$ $d = ?$
By the Rule in Sect. 2. num. 20.	8	$77677 - 3 = 1094 = a$	
	9	$71 = 71$	the least Values.
	10	$56 = a$	
	11	$23 = b$	

Explication.

1. I multiply 71 which is prefix to a in the Equation proposed, by such a number, that when 3, to wit, $+3$ in the same Equation is added to the Product, the sum may be either equal to, or less than some Multiple of 173; so multiplying 71 by 2, the Product 142 increased with 3 makes 145, which is equal to 173 wanting 28, viz. $145 = 173 - 28$, which is the second Equation.

2. Then by Prop. 1. of this Chap. I seek two such numbers a and b , that if a be multiplied by 173, and the Product increased with $+1$, the sum may be equal to the Product of b multiplied by 71; viz. Supposing $173a + 1 = 71b$, and proceeding according to the foregoing Quest. 9. I find 16 for the value of a , and 39 for b , therefore $173 \times 16 + 1 = 71 \times 39$, or $71 \times 39 = 173 \times 16 + 1$; that is, $2769 = 2768 + 1$, which is the third Equation.

3. Because $+1$ in the latter part of the third Equation is an Aliquot part of 28 in the second, I multiply the third Equation by 28 the Denominator of the said part, and it makes the fourth Equation, to wit, $77532 = 77504 + 28$.

4. Then by adding the fourth Equation to the second the sum gives the fifth, which is free from the signs $+$ and $-$, and from the fifth Equation the whole numbers 449 and 1094 are discovered for values of b and a , in like manner as in Quest. 4, and 5. and by the help of those the smallest values of a and b , to wit, 56 and 23 are found out by the Rule in the twentieth step of Sect. 2.

5. Lastly, by the help of the two smallest values of a and b , and the Rule in the third step of Sect. 2. all that are capable of solving Quest. 10. will be found in the two following Arithmetical Progressions, which may be continued as far as you please.

Values of a ; 56, 229, 402, 575, 748, 921, 1094, &c.

Values of b ; 23, 94, 165, 236, 307, 378, 449, &c.

QUEST. 11.

If	1	$22a + 5000 = 65b$	What are a and b in whole numbers?
Out of 1.	2	$5022 = 5070 - 48$	
By Prop. 1.	3	$66 = 65 + 1$	$c = ?$ $d = ?$
Eq. 3×48 .	4	$3168 = 3120 + 48$	
2 + 4.	5	$8190 = 8190$	true Values.
Out of 5, and 1.	6	$8190 = 116 = b$	
5, 1.	7	$65 = 65$	$c = ?$ $d = ?$
By the Rule in Sect. 2. num. 20.	8	$8190 - 5000 = 145 = a$	
	9	$22 = 22$	the least Values.
	10	$15 = a$	
	11	$82 = b$	

Explic.

Explication.

1. I add 22 to 5000 and it makes 5022, which is not exactly divisible by 65, for 77 times 65 is less than 5022, but 78 times 65, that is, 5070, exceeds 5022 by 48; therefore $5022 = 5070 - 48$, which is the second Equation.

2. Then by Prop. 1. of this Chap. I seek two such whole numbers a and b , that if a be multiplied by 65, and to the Product there be added 1, the sum may be equal to the Product of b multiplied by 22, viz. Supposing $65a + 1 = 22b$, and proceeding according to the latter Method of resolving the foregoing Prop. 1. I find 1 and 3 to be values of a and b ; therefore, $65 \times 1 + 1 = 22 \times 3$; or $22 \times 3 = 65 \times 1 + 1$; that is, $66 = 65 + 1$, which is the third Equation.

3. By prosecuting the work as before in the Explication of Quest. 10. all the defined values of a and b in whole numbers that are capable of constituting the Equation first proposed in this eleventh Question will be found to be the Terms of these two following Arithmetical Progressions, viz.

Values of a ; 15, 80, 145, 210, 275, 340, &c.

Values of b ; 82, 104, 126, 148, 170, 192, &c.

Another way of solving Prop. 2.

QUEST. 12.

If	1	$71a + 3 = 173b$	What are a and b in whole numbers?
	2	$145 = 173 - 28$	
	3	$213 = 173 + 40$	
Out of 1.	4	$173c + 28 = 40d$	$c = ?$ $d = ?$
Suppose	5	$201 = 240 - 39$	
Out of 4.	6	$173e = 160 + 13$	
6×3	7	$519 = 480 + 39$	Regress.
$5 + 7$	8	$720 = 720$	
8, 4.	9	$40 = 18 = d$	
9, 3, 2.	10	$18 \times 213 + 145 = 3279$	
10, 1.	11	$3279 = 23 = b$	
10, 1.	12	$3279 - 3 = 56 = a$	

Explication.

1. In this Question, which is the same with the foregoing tenth, the second Equation is formed as is there directed.

2. The third Equation is thus formed: For as much as the given number 71 is less than 173 which is prefix to b , I multiply 71 by such a number that the Product may exceed 173, and be also Prime to it; so multiplying 71 by 3, the Product 213 exceeds 173, also 213 and 173 are Prime to one another; then I divide the said 213 by 173, and find that 213 contains 173 once, and 40 over and above; therefore $213 = 173 + 40$, which is the third Equation.

3. The fourth, fifth, and sixth Equations here, are formed like the fourth, fifth and sixth Equations in the foregoing Quest. 6.

4. Then because 13 which follows $+1$ in the sixth Equation is an Aliquot part of 39 which follows $-$ in the fifth, I multiply the sixth Equation by 3: the Denominator of the said part, (for 13 is $\frac{1}{3}$ of 39) and it produceth the seventh Equation, to wit, $519 = 480 + 39$.

5. The eighth Equation is the sum of the fifth and seventh, (according to the Operation in Case 2.) and then in the ninth Equation the Regressive work begins, to find out the values of d , b and a in such manner as hath been shewn in divers preceding Questions of this Chapter: So at length all the values of a and b in whole numbers to solve this twelfth Question will by this latter Method be found the same as before in Quest. 10.

Soll. 5.

Soll. 5. PROP. III.

To divide a given number into three or more numbers, such, that if every one of them be multiplied by a different number given, the sum of the Products may be equal to a given number. But the sum of those Products must fall between the two Products made by multiplying the given Dividend into the greatest and least of the given Multipliers.

The Solution of this Problem is explain'd by the following Questions of this Chapter, and oftentimes requires the help of the two preceding Propositions, as will partly appear by the fifteenth Question.

QUEST. 13.

To divide 24 into three such whole numbers, that if the first be multiplied by 36, the second by 24, and the third by 8, the sum of the three Products may make 516.

Let the numbers sought be represented by a , e and y , then the Question may be stated thus;

$$\begin{array}{l} 1. \text{ If } \dots \dots \dots a + e + y = 24 \\ 2. \text{ And } \dots \dots \dots 36a + 24e + 8y = 516 \end{array}$$

What are the whole numbers a , e and y ?

RESOLUTION.

3. The first Equation multiplied by 36, which is prefix to a in the second, produceth $36a + 36e + 36y = 864$

4. The second Equation subtracted from the third, leaves $12e + 28y = 348$

5. The fourth Equation by transposition of $-28y$, gives $12e = 348 - 28y$

6. The fifth Equation divided by 12 gives $e = 29 - \frac{7y}{3}$

7. If instead of e in the first Equation there be taken the latter part of the sixth, this ariseth, $a + 29 - \frac{7y}{3} + y = 24$

8. That is, $a + 29 - \frac{4y}{3} = 24$

9. From the eighth Equation by transposition of $29 - \frac{4y}{3}$, this ariseth, $a = 24 - 29 + \frac{4y}{3}$

10. That is, $a = \frac{4y}{3} - 5$

11. By the latter part of the tenth Equation 'tis evident that $\frac{4y}{3} = 5$

12. Therefore by multiplying each part in the eleventh step by 3, it follows that $4y = 15$

13. And by dividing each part in the twelfth step by 4, $y = 3\frac{3}{4}$

14. And from the latter part of the sixth Equation, by arguing in like manner as in the eleventh, twelfth and thirteenth steps, it will be manifest that $y = 12\frac{3}{4}$

15. Now if Fractions or mixt numbers were admitted to be the values of a , e and y , then by the thirteenth, fourteenth, tenth and sixth steps 'tis evident that

$$y = \text{any number between } 3\frac{3}{4} \text{ and } 12\frac{3}{4};$$

$$a = \frac{4y}{3} - 5;$$

$$e = 29 - \frac{7y}{3}.$$

16. But to find out whole numbers to solve the Question, the limits in the thirteenth and fourteenth steps do shew that y must be some whole number greater than 3; but not greater than 12; yet every whole number within those limits will not serve the turn, for the values of a and e before discovered will not be whole numbers unless $\frac{4y}{3}$ and $\frac{7y}{3}$ be whole numbers; but

$\frac{4y}{3}$ and $\frac{7y}{3}$ cannot be whole numbers unless y be 3, or some Multiple of 3; and because 3 is without the limits, y may be 6, or 9, or 12, and consequently

R r

a	e	y
3	15	6
7	8	9
11	1	12

from the fifteenth step a shall be 3, or 7, or 11; and e , 15, or 8, or 1. Now in answer to the Question, 3, 15 and 6, (to wit, a , e and y) are three such whole numbers, that their sum is 24; and if the first be multiplied by 36, the second by 24, and the third by 8, the sum of the three Products makes 516, as was required. The like may be said of each of the two other Answers. But if Fractions or mixt numbers were admitted, innumerable Answers might be given to the Question, as before hath been shewn in the fifteenth step.

Note. When one part of an Equation consists of an Affirmative letter and some Negative Absolute number, a limit may thence be infer'd, above which the number signified by that letter ought to be taken. But if one part of an Equation consists of a Negative letter and of an Affirmative Absolute number, it will give a limit beneath which the number represented by that letter must be chosen. Sometimes also two limits will be discovered, (as in this thirteenth Question for the choice of the number y ;) and sometimes but one, (as in divers of the following Questions.)

QUEST. 14.

To find three such whole numbers that their sum may make 100; and that if the first be multiplied by 4, the second by 3, and the third by $1\frac{1}{2}$, the sum of the three Products may make 300.

For the three numbers sought put a , e and y , then the Question may be stated thus,

$$\begin{array}{l} 1. \text{ If } \dots \dots \dots a + e + y = 100 \\ 2. \text{ And } \dots \dots \dots 4a + 3e + 1\frac{1}{2}y = 300 \end{array}$$

What are the whole numbers a , e and y ? ||

RESOLUTION.

3. The first Equation multiplied by 4, (which is prefix'd to a in the second Equation,) produceth $4a + 4e + 4y = 400$.
4. The second Equation subtracted from the third, leaveth $e + \frac{11}{2}y = 100$.
5. The fourth Equation by transposition of $-\frac{11}{2}y$ gives $e = 100 - \frac{11}{2}y$.
6. If instead of e in the first Equation there be taken the latter part of the fifth, this will arise, $a + 100 - \frac{11}{2}y + y = 100$.
7. That is, after due Reduction, $a = \frac{6}{2}y$.
8. From the latter part of the fifth Equation it's manifest that $\frac{11}{2}y = 100$.
9. And consequently by multiplying each part in the eighth step by 2, $11y = 200$.
10. And by dividing each part in the ninth step by 11, it follows that $y = 18\frac{2}{11}$.

Whence 'tis manifest, that if the three numbers sought were not restrained to whole numbers, any number less than $45\frac{1}{11}$ might be taken for the number y , and then the numbers a and e would be discovered from the seventh and fifth steps. But to have the

Question solved by whole numbers, the number y must be some whole

number not greater than $45\frac{1}{11}$, and such as may cause $\frac{11}{2}y$ and $\frac{6}{2}y$ to

be whole numbers, for otherwise the values of e and a in the fifth and seventh steps will not be expressible by whole numbers; but $\frac{11}{2}y$ and

$\frac{6}{2}y$ cannot be whole numbers unless y be 5, or some Multiple of 5,

and therefore y may be 5, or 10, or 15, or any of the rest of the numbers in the third Column of this Table, and consequently, from the fifth and seventh steps of the Resolution, the whole numbers e and a will be such as stand under e and a . Thus you see that the Question

receives nine Answers in whole numbers, which are all that it's capable of. So that if you take 6 for a , 89 for e , and 5 for y , their sum is 100; and if 6 be multiplied by 4,

a	e	y
6	89	5
12	78	10
18	67	15
24	56	20
30	45	25
36	34	30
42	23	35
48	12	40
54	1	45

by 4; 89 by 3; and 5 by $1\frac{1}{2}$, the sum of the three Products maketh 300; as the Question requires. The like may be proved of every one of the other eight Answers.

Note. When three numbers are sought by a Question of this nature that is capable of many Answers in whole numbers, all the values of every one of the letters in whole numbers are in Arithmetical Progression, and therefore when two of those Answers are found out, all the rest within the limits discovered by the Resolution are consequently given by Addition or Subtraction of the common Difference in each Rank; as may easily be perceived by the values of a , e , y in the Table above-written. But when four numbers are sought, the values of a letter are oftentimes found in several Arithmetical Progressions; as in the following Quest. 20.

QUEST. 15.

To divide 1533 into three whole numbers, such, that $\frac{1}{2}$ of the first, together with $\frac{1}{3}$ of the second and $\frac{1}{11}$ of the third may make 167.

For the three whole numbers sought put a , e and y , then the Question may be stated thus,

$$\begin{array}{l} 1. \text{ If } \dots \dots \dots a + e + y = 1533 \\ 2. \text{ And } \dots \dots \dots \frac{1}{2}a + \frac{1}{3}e + \frac{1}{11}y = 167 \end{array}$$

What are the whole numbers a , e and y ? |

RESOLUTION.

3. The first Equation multiplied by $\frac{1}{2}$ produceth $\frac{1}{2}a + \frac{1}{2}e + \frac{1}{2}y = 766\frac{1}{2}$.
4. The second Equation subtracted from the third, leaveth $-\frac{1}{6}e + \frac{10}{11}y = 12\frac{1}{2}$.
5. The fourth Equation by transposition of $-\frac{1}{6}e$ gives $\frac{10}{11}y = 12\frac{1}{2} + \frac{1}{6}e$.
6. The fifth Equation divided by $\frac{10}{11}$ gives $y = \frac{2261}{10} + \frac{226e}{110}$.
7. If instead of y in the first Equation there be taken the latter part of the sixth, this ariseth, $a + e + \frac{2261}{10} + \frac{226e}{110} = 1533$.
8. The seventh Equation, after due Reduction, gives $a = \frac{126440}{110} - \frac{226e}{110}$.
9. By the eighth Equation it's manifest that $323e = 126440$.
10. And consequently by dividing each part of the last step by 323, $e = 391\frac{2}{11}$.
11. Now to find out the values of a , e and y in whole numbers, (if there be a possibility,) I multiply the sixth Equation by the Denominator 97, and it makes $97y = 2261 + 226e$.
12. That is, $226e + 2261 = 97y$.
13. Then by the foregoing Prop. 1. of this Chapter, I search out all such whole numbers as may be values of e and y to constitute the last Equation, that is, $226e + 2261 = 97y$; but with this condition, viz. That the greatest whole number among those that are found out for the values of e may not exceed 391, as the preceding tenth step requires; so I find four values of e , to wit, 47, 144, 241, 338; and four values of y , to wit, 339, 565, 791 and 1017: Then the sum of every two correspondent values of e and y being subtracted from 1533 the number first given to be divided, the Remainders shall be the desired values of a , to wit, 1147, 824, 501 and 178; so there are only four Answers to the Question in whole numbers, to wit, those inserted in the Table in the Margin.

The Proof of the first Answer.

The sum of 1147, 47 and 339 is 1533 .

$\frac{1}{2}$ of 1147 is $573\frac{1}{2}$.

$\frac{1}{3}$ of 47 is $15\frac{2}{3}$.

$\frac{1}{11}$ of 339 is $30\frac{6}{11}$.

Lastly, the sum of those three Products is 167 .

Therefore

a	e	y
1147	47	339
824	144	565
501	241	791
178	338	1017

Therefore all the conditions in the Question are satisfied, and the like may be proved by every one of the other three Answers in whole numbers; but if Fractions were admitted, innumerable Answers might be given by the tenth eighth, and sixth steps of the Resolution.

QUEST. 16.

To find three numbers, that their sum may make 300; and that if the first be multiplied by 6, the second by 5, and the third by $\frac{11937}{300}$, the sum of the three Products may make 1496.

Let a, e, y be put for the three numbers sought; then by forming the Resolution in like manner as in the preceding thirteenth, fourteenth and fifteenth Questions, it will appear that

$$y = \text{any number between } 1\frac{451}{5} \text{ and } 76\frac{113}{195};$$

$$e = 304 - \frac{11937}{300};$$

$$a = \frac{8937}{300} - 4.$$

Whence 'tis evident that there cannot be three whole numbers found out to solve this Question, for 300 is the smallest whole number that can be taken for y to cause $\frac{11937}{300}$ and $\frac{8937}{300}$ to be whole numbers; but 300 exceeds the greater of the two limits above discovered for choosing of the number y .

QUEST. 17.

If one would lay out 98 pence to buy 40 Birds, suppose Partridges, Larks and Quails; how many of each kind may be bought when Partridges are at 3 pence a piece, Larks at an half-penny a piece, and Quails at 4 pence a piece?

Let a represent the number of Partridges, e the number of Larks, and y the number of Quails; then according to the Question, $a + e + y = 40$; and because the number of all the Partridges multiplied by the price of one of them produceth the full cost of all, it's manifest that $3a$ is the full cost of all the Partridges; and for the like reason $4e$ signifies the full cost of all the Larks; likewise $4y$ the full cost of all the Quails: But those three particular sums of money must be equal to 98 pence; therefore $3a + 4e + 4y = 98$; so that the Question may be stated thus;

$$\begin{array}{l} 1. \text{ If } \dots \dots \dots a + e + y = 40 \\ 2. \text{ And } \dots \dots \dots 3a + 4e + 4y = 98 \end{array}$$

What are the whole numbers a, e and y ? ||

RESOLUTION.

3. The first Equation multiplied by 3 (which is prefixt to a in the second,) produceth $3a + 3e + 3y = 120$
4. The second Equation subtracted from the third, leaves $\frac{5e}{2} - y = 22$
5. From the fourth Equation, after due transposition, this ariseth, $y = \frac{5e}{2} - 22$
6. Then instead of y in the first Equation, if there be set the latter part of the fifth, the first will be reduced to this, $a + e + \frac{5e}{2} - 22 = 40$
7. The sixth Equation, after due Reduction, gives $a = 62 - \frac{7e}{2}$
8. By the latter part of the fifth Equation it's evident that $\frac{5e}{2} \leq 22$
9. And consequently by multiplying each part in the eighth step by 2, $5e \leq 44$
10. Whence by dividing each part by 5, it follows that $e \leq 8\frac{4}{5}$
11. Again, from the latter part of the seventh Equation, by arguing in like manner as in the eighth, ninth and tenth steps, it will appear that $e \geq 17\frac{2}{5}$

12. Now

12. Now since the nature of this Question requires that the desired values of a, e and y be whole numbers, it's evident from the fifth and seventh steps that e must be an even number, otherwise $\frac{5e}{2}$ and $\frac{7e}{2}$ will not be whole numbers; for if e be an odd number,

the Dividends $5e$ and $7e$ will be odd, (for odd multiplied by odd produceth odd,) and therefore their halves cannot be whole numbers. Since then e must be an even number,

it's manifest by the tenth and eleventh steps, that e may be 10, or 12, or 14, or 16, but no other even number whatever; and consequently from the fifth step y shall be 3, or 8, or 13, or 18; and from the seventh step, a shall be 27, or 20, or 13, or 6. Thus it appears that the Question may be solved by four several Answers (and not more) in whole numbers, viz. First, 27 Partridges, 10 Larks, and 3 Quails, which are in multitude

Partr.	Larks.	Quails.
a	e	y
27	10	3
20	12	8
13	14	13
6	16	18

40, may be bought for 98 pence at their respective prices given in the Question; or 20 Partridges, 12 Larks, and 8 Quails, which are likewise 40 in multitude, and the like may be affirmed of the other two Answers inserted in the Table in the Margin.

But if a Question of the same nature be desired that hath but one Answer in whole numbers, the following Epigram (cited by Monsieur Bachet in his Comment upon the one and fourtieth Question of the fourth Book of Diophantus,) will be satisfactory.

QUEST. 18.

Ut tot emantur aves, bis denis pectus nummis;

Perdix, Anser, Anas empta vocetur avis.

Sit simplex obolus pretium Perdici, ematur

Sex obolis Anser, bisque duobus Anas.

Ut tua procedat in lucem questio, mentem

Consule, sit laqueus pectoris arca mihi.

Sint Anates tres atque dua, simplex oris Anser,

Accipe Perdices quatuor atque decem.

The sense is this: If the price of a Partridge be an half-penny; a Goose 3 pence; and a Duck 2 pence; how many of each kind may be bought at those rates; if it be desired that all the Birds bought may be 20 in number, and cost 20 pence?

Let a represent the number of Partridges, e the number of Geese, and y the number of Ducks, then this Question (like the preceding seventieth,) may be stated thus;

$$\begin{array}{l} 1. \text{ If } \dots \dots \dots a + e + y = 20 \\ 2. \text{ And } \dots \dots \dots \frac{3}{4}a + 3e + 2y = 20 \end{array}$$

What are the whole numbers a, e and y ? ||

RESOLUTION.

3. The first Equation multiplied by $\frac{1}{4}$ produceth $\frac{3}{4}a + \frac{1}{4}e + \frac{1}{4}y = 5$
4. The third Equation subtracted from the second, leaves $\frac{5e}{2} + \frac{3y}{2} = 10$
5. By transposition of $\frac{3y}{2}$ in the fourth Equation, this ariseth, $\frac{5e}{2} = 10 - \frac{3y}{2}$
6. The fifth Equation divided by $\frac{1}{2}$ gives $e = 4 - \frac{3y}{5}$
7. By setting the latter part of the sixth Equation in the place of e in the first, this ariseth, $a + 4 - \frac{3y}{5} + y = 20$
8. Which last Equation, after due Reduction, gives $a = 16 - \frac{2y}{5}$
9. From the latter part of the sixth Equation it may be inferred, (in like manner as in divers of the preceding Questions,) that $y \leq 6\frac{2}{3}$
10. But the sixth and eighth steps do shew, that to the end the values of e and a may be whole numbers, as the nature of this Question requires, it is requisite that $\frac{3y}{5}$ and $\frac{2y}{5}$ be

be whole numbers; but $\frac{37}{5}$ and $\frac{27}{5}$ cannot be whole numbers unless y be 5 or some Multiple of 5; and by the ninth step y must be less than 62, therefore 5 is the only whole number that can be taken for y ; or the number of Ducks; and consequently the sixth step gives 1 for the value of e , that is, 1 Goose; and by the eighth step, the value of a is 14, that is, 14 Partridges; which three numbers will solve the Question, as may easily be proved.

The Resolutions of the following nineteenth and twentieth Questions do shew how to find out innumerable Answers to any Question belonging to the Rule of Alligation alternate in Vulgar Arithmetick, when three or more things are to be mixed together, according to the import of that Rule.

QUEST. 19.

A Vintner having three sorts of Wines, the prices whereof per Gallon are 24 pence, 22 pence, and 18 pence, desires to make a Mixture out of them that may contain 60 Gallons, in such manner, that the total Mixture being sold at some mean price per Gallon between 24 pence and 18 pence, suppose at 20 pence, may make the same sum of money, as all the particular quantities of Wine in the Mixture at their own prices. The Question is, to find what Quantity of each sort of Wine may be taken to make that Mixture.

For the desired number of Gallons of the first sort of Wine to make the Mixture, put a , for the number of the second sort e , and of the third y : Then $a + e + y = 60$, (the total number of the Gallons in the Mixture;) and because every Gallon of the mix'd quantity must be sold for 20 pence, the 60 Gallons mix'd are worth 1200 pence, and so much also must all the Products of the particular Quantities of each sort of Wine multiplied by their peculiar prices amount unto; therefore, $24a + 22e + 18y = 1200 = 60 \times 20$. So that the Question may be stated thus;

1. If $a + e + y = 60$
2. And $24a + 22e + 18y = 1200 (= 60 \times 20)$

What are the numbers a, e, y ? ||

RESOLUTION.

3. The first Equation multiplied by 24, (which is prefix to a in the second Equation,) produceth $24a + 24e + 24y = 1440$
4. The second Equation subtracted from the third, leaveth $2e + 6y = 240$
5. The fourth Equation by transposition of $6y$ gives $2e = 240 - 6y$
6. The fifth Equation divided by 2, gives $e = 120 - 3y$
7. By taking the latter part of the sixth Equation instead of e in the first, this ariseth, $a + 120 - 3y + y = 60$
8. The seventh Equation, after due Reduction, discovers the value of a , viz. $a = 2y - 60$
9. From the eighth Equation it's evident that $y \leq 30$
10. And from the sixth Equation, $y \geq 40$
11. By the 10th, 9th, 8th and 6th steps it's manifest that innumerable Answers may be given to the Question proposed; for since Fractions are not here excluded from being Answers, you may esteem . . . $y =$ any number between 30 and 40;
 $a = 2y - 60$;
 $e = 120 - 3y$.

12. Whence nine Answers in whole numbers are discovered, to wit, those express'd in this Table. But the Rule of Alligation in Vulgar Arithmetick finds out only one Answer to this Question, to wit, the sixth. And because innumerable numbers may be taken between 30 and 40 for values of y , you may find out as many Answers as you please in Fractions, (which are not excluded in Questions of this nature;) so if for y , you take $30\frac{1}{2}$, then $a = 1$, ($= 2y - 60$;) and $e = 28\frac{1}{2}$, ($= 120 - 3y$.)

a	e	y
2	27	31
4	24	32
6	21	33
8	18	34
10	15	35
12	12	36
14	9	37
16	6	38
18	3	39

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The Proof of the first Answer.

Two Gallons of Wine at 24 pence per Gallon, together with 27 Gallons at 22 pence per Gallon, and 31 Gallons at 18 pence per Gallon, amount to 1200 pence; which is also the value of 60 Gallons at 20 pence per Gallon.

QUEST. 20.

A Vintner having four sorts of Wines, whose prices per Quart are 16 pence, 10 pence, 8 pence, and 6 pence, desires to make a Mixture out of them that may contain 100 Quarts, so as this mix'd Quantity being sold at some mean price per Quart between 16 pence and 6 pence, suppose at 12 pence, may produce the same sum of money, as all the particular quantities of Wine in the Mixture if they were sold at their own prices. The Question is, to find what quantity of Wine of each sort may be taken to make that Mixture?

Let a, e, y and u be put for the unknown Quantities of Wine that are sought to make the Mixture, then $a + e + y + u = 100$, (the total number of Quarts in the Mixture,) and by multiplying those Quantities severally into their peculiar prices, the sum of the Products is $16a + 10e + 8y + 6u$, which sum must be equal to the Product of 100 multiplied into 12, that is, 1200 pence: So that the Question may be stated thus;

1. If $a + e + y + u = 100$
2. And $16a + 10e + 8y + 6u = 1200$

What are the numbers a, e, y and u ? ||

The given Equations being fewer in multitude than the numbers sought, it's a sign that the Question is capable of innumerable Answers; now that you may find out as many of them as you please, the first scope in the Resolution must be to discover limits to direct your choice of some one of the numbers sought, and accordingly; the drift in the eight Equations next following is to search out limits for the first number a .

RESOLUTION.

3. From the first Equation by transposition of a , this ariseth, $e + y + u = 100 - a$
4. And from the second Equation by transposition of 16a, this ariseth, $10e + 8y + 6u = 1200 - 16a$
5. The third Equation multiplied by 6, to wit, the least of the known numbers which are prefix to the letters in the first part of the fourth Equation, produceth $6e + 6y + 6u = 600 - 6a$
6. Again, the third Equation multiplied by 10, that is, the greatest of the known numbers which are prefix to the letters in the first part of the fourth Equation, produceth $10e + 10y + 10u = 1000 - 10a$
7. It is manifest that the first part of the fifth Equation is less than the first part of the fourth, therefore also the latter part of the fifth shall be less than the latter part of the fourth, viz. $600 - 6a < 1000 - 10a$
8. Therefore from the seventh step, after due Reduction, it follows, that $a > 40$
9. Again, for as much as the first part of the fifth Equation is greater than the first part of the fourth, therefore also the latter part of the fifth shall be greater than the latter part of the fourth, viz. $1000 - 10a < 1200 - 16a$
10. Therefore from the ninth step, after due Reduction, it follows that $a < 33\frac{1}{3}$

Now since it is found by the eighth and tenth steps, that a the number of Quarts sought of the first sort of Wine to make the Mixture must be less than 60, but greater than 33, let some number within those limits be taken for the value of a , viz.

11. Suppose

11. Suppose $47 = a$
 12. Then by setting 47 in the place of a in the first Equation, this ariseth, *viz.* $47 + e + y + u = 100$
 13. Whence by equal subtraction of 47 there remains $e + y + u = 53$
 14. And by multiplying the Equation in the eleventh step by 16, (the number prefix to a in the second,) it gives $752 = 16a$
 15. Then by setting 752 in the place of $16a$ in second Equation, this ariseth, $752 + 10e + 8y + 6u = 1200$
 16. And by subtracting 752 from each part of the Equation in the fifteenth step, this remains, *viz.* $10e + 8y + 6u = 448$
 17. The Equation in the thirteenth step multiplied by 10, (which is prefix to e in the sixteenth,) produceth $10e + 10y + 10u = 530$
 18. Then by subtracting the Equation in the sixteenth step from that in the seventeenth, the letter e vanisheth, and this Equation remains, *viz.* $2y + 4u = 82$
 19. From the eighteenth step, by transposition of $-4u$, this Equation ariseth, $2y = 82 - 4u$
 20. And by dividing each part of the Equation in the nineteenth step by 2, it gives $y = 41 - 2u$
 21. Then by setting the latter part of the Equation in the twentieth step in the place of y in the thirteenth step, it makes $e + 41 - 2u + u = 53$
 22. Whence, after due Reduction, $e = u + 12$
 23. By the latter part of the Equation in the twentieth step, it's evident that $2u = 41$, therefore $u = 20\frac{1}{2}$

And because the known number 12 which follows $-4u$ in the twenty-second step, (expressing the value of e) is Affirmative, there is not any limit to shew above which the number u ought to be taken; and therefore, according to the three and twentieth step, u may be any number less than $20\frac{1}{2}$. Therefore,

24. Suppose $u = 20$
 25. Then from the twentieth and twenty-fourth steps it follows, that $y = 1, (= 41 - 2u)$
 26. And from the twenty-second and twenty-fourth steps, $e = 32, (= u + 12)$

Thus by the eleventh, twenty-sixth, twenty-fifth and twenty-fourth steps, four whole numbers are discovered, to wit, 47, 32, 1 and 20 for the values of a, e, y and u , which numbers will solve the Question. For if 42 quarts of the first sort of Wine, 37 quarts of the second, 1 quart of the third, and 20 of the fourth be mixed together, the sum makes 100 quarts, which at 12 pence per Quart yields 1200 pence; and the same number of pence will be produced by selling 47 quarts at 16 pence per Quart, 32 quarts at 10 pence, 1 quart at 8 pence, and 20 quarts at 6 pence, which was required.

But because (by the twenty-third step) u may be any whole number less than $20\frac{1}{2}$, nineteen Answers more in whole numbers may be found out by repeating the Process in the twenty-fourth, twenty-fifth and twenty-sixth steps; so that 47 being taken for a , there will be twenty Answers in whole numbers, which are inserted in the following Table. And by putting a equal to every whole number severally between $33\frac{1}{2}$ and 60, which are the limits discovered in the eighth and tenth steps, for the choosing of the number a , after a due repetition of the Process with every one of those whole numbers, in like manner as before with 47 from the eleventh step to the end of the Resolution, two hundred ninety four Answers more in whole numbers will be discovered, which with those twenty in the Table make three hundred and fourteen Answers in whole numbers to this twentieth Question.

Question, to which the Rule of *Alligation* in Vulgar Arithmetick gives only one Answer, which consists partly of Fractions too; but by the Method above deliver'd, innumerable Answers may be found out in Fractions. The Table follows.

a	e	y	u
47	32	1	20
47	31	3	19
47	30	5	18
47	29	7	17
47	28	9	16
47	27	11	15
47	26	13	14
47	25	15	13
47	24	17	12
47	23	19	11
47	22	21	10
47	21	23	9
47	20	25	8
47	19	27	7
47	18	29	6
47	17	31	5
47	16	33	4
47	15	35	3
47	14	37	2
47	13	39	1

QUEST. 21.

Forty-one persons consisting of Men, Women and Children spent in the whole at a Feast 40 shillings; whereof every Man paid 4 shillings, every Woman 3 shillings, and every Child 4 pence, or $\frac{1}{3}$ of a shilling: It's desired to find the number of Men, likewise of the Women and Children.

The nature of this Question not admitting Fractions in the Answer, the scope of the Resolution must be to divide 41 into three such whole numbers, that if the first be multiplied by 4, the second by 3, and the third by $\frac{1}{3}$, the sum of the three Products may make 40: To which purpose, let a, e and y be put for the desired numbers of Men, Women and Children, and then the Question may be stated thus, *viz.*

1. If $a + e + y = 41$
 2. And $4a + 3e + \frac{1}{3}y = 40$

What are the whole numbers a, e, y ? ||

RESOLUTION.

3. By forming the Resolution in like manner as in the foregoing thirteenth, fourteenth and fifteenth Questions it will appear, that
$$\begin{cases} y = 31\frac{1}{3} \\ y = 33\frac{2}{3} \\ e = 124 - \frac{11y}{3} \\ u = \frac{8y}{3} - 83. \end{cases}$$

Whence 'tis manifest that 32 and 33 are the only whole numbers within the limits for the choosing of the number y , but this must necessarily be a Multiple of 3, otherwise $\frac{11y}{3}$ and $\frac{8y}{3}$ will not be whole numbers, and consequently the values of e and u above-express'd cannot be whole numbers; therefore 33 is the sole whole number that can be taken for the value of y , to wit, the number of Children, and consequently the values of e and u above-express'd will give 3 for the number of Women, and 5 for the number of Men: which three numbers 5, 3 and 33 will solve the Question, for their sum is 41; and if the first be multiplied by 4, the second by 3, and the third by $\frac{1}{3}$, the sum of the three Products is 40, as was required.

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QUEST. 22.

QUEST. 22.

Twenty persons, consisting of Men, Women, Boys and Girls spent at a Feast in the whole 94 shillings; whereof every man paid 6 shillings, every Woman 4 shillings, every Boy 3 shillings, and every Girl 1 shilling: It's desired to find out the number of Men, likewise of Women, Boys and Girls.

The scope of this Question is to find out four such whole numbers that their sum may make 20; and that if the first be multiplied by 6, the second by 4, the third by 3, and the fourth by 1, the sum of the four Products may make 94; therefore by putting a, e, γ, n to represent those four whole numbers, the Question may be stated thus;

1. If $a + e + \gamma + n = 20$
 2. And $6a + 4e + 3\gamma + n = 94$

What are the whole numbers a, e, γ, n ?

RESOLUTION.

The first Scope is to search out limits for the number a in like manner as before in the twentieth Question, viz.

3. By transposition of a in the first Equation, this arithet, $e + \gamma + n = 20 - a$
 4. Likewise by transposition of $6a$ in the second Equation, there comes forth $4e + 3\gamma + n = 94 - 6a$
 5. The third Equation multiplied by 1, (to wit, the smallest of the numbers prefix to the letters in the first part of the fourth Equation, where 1 is supposed to be prefix to n), doth produce the same third, viz. $e + \gamma + n = 20 - a$
 6. Again, the third Equation multiplied by 4, to wit, the greatest of the numbers prefix to the letters in the first part of the fourth Equation, doth produce $4e + 4\gamma + 4n = 80 - 4a$
 7. It is manifest that the first part of the fifth Equation is less than the first part of the fourth, therefore also the latter part of the fifth shall be less than the latter part of the fourth, viz. $20 - a > 94 - 6a$
 8. Therefore from the seventh step, after due Reduction, it follows that $a > 14\frac{1}{2}$
 9. Again, for as much as the first part of the sixth Equation is greater than the first part of the fourth, therefore also the latter part of the sixth shall be greater than the latter part of the fourth, viz. $80 - 4a < 94 - 6a$
 10. Therefore from the ninth step, after due Reduction, it follows, that $a < 7$

Now since 'tis found by the tenth and eighth steps, that a , (or the number of Men,) is greater than 7, but less than $14\frac{1}{2}$, let some whole number within those limits be taken for the value of a , viz.

11. Suppose $12 = a$
 12. Then by setting 12, in the place of a in the first Equation, this arithet, $12 + e + \gamma + n = 20$
 13. Whence by equal subtraction of 12, there remains $e + \gamma + n = 8$
 14. And by multiplying the Equation in the eleventh step by 6, it makes $72 = 6a$
 15. Then by setting 72 in the place of $6a$ in the second Equation, it gives $72 + 4e + 3\gamma + n = 94$
 16. And by subtracting 72 from each part of the last Equation, the Remainder is $4e + 3\gamma + n = 22$
 17. The Equation in the thirteenth step being multiplied by 4, (which is prefix to e in the sixteenth,) gives $4e + 4\gamma + 4n = 32$
 18. Then by subtracting the Equation in the sixteenth step from that in the seventeenth, the letter e vanisheth, and this Equation remains, $\gamma + 3n = 10$

19. Whence

19. Whence by transposition of $3n$, this Equation arithet, $\gamma = 10 - 3n$
 20. Then by setting the latter part of the Equation in the nineteenth step in the place of γ in the thirteenth, this arithet, $e + 10 - 3n + n = 8$
 21. Whence, after due Reduction, this Equation arithet, $e = 2n - 2$
 22. From the latter part of the nineteenth Equation, it may be infer'd that $n = 3\frac{1}{2}$
 23. And from the latter part of the twenty-first Equation, $n = 1$

Now since by the twenty-second and twenty-third steps, n (or the number of Girls) is found to fall between 1 and $3\frac{1}{2}$, let 2 be taken for the value of n , viz.

24. Suppose $n = 2$
 25. Then from the nineteenth and twenty-fourth steps, $\gamma = 4$ ($= 10 - 3n$)
 26. And from the twenty-first and twenty-fourth steps, $e = 2$ ($= 2n - 2$)

Thus by the eleventh, twenty-sixth, twenty-fifth and twenty-fourth steps, four whole numbers are discovered, to wit, 12, 2, 4 and 2, for the values of a, e, γ and n .

Again, by taking 3 for the value of n , (which is within the limits before discovered) the nineteenth and twenty-first steps will discover 1 and 4 for the values of γ and e , (a being 12, as before). Wherefore two Answers to the Question are found out, for the number of Men being put 12, the number of Women will be 2, the number of Boys 4, and the number of Girls 2; or the number of Men being 12 as before, there will be four Women, 1 Boy and 3 Girls. Again, if 11 be put equal to a , (or the number of Men,) and the process be repeated from the eleventh step to the end of the Resolution, there will be found two Answers more in whole numbers. In like manner, if 9, 10 and 13 be severally put equal to a , three Answers more will be discovered; But if 8 and 14 be severally put equal to a , although they be within the limits in the eighth and tenth steps, yet the work being repeated as before will not succeed to find e, γ and n in whole numbers, so that there are only seven Answers, to wit, those inserted in the Table, but that every one of them will solve the Question may easily be proved.

If a Question of this nature be desired that hath but one Answer in whole numbers; let the number of persons be 60, and 100 the number of shillings spent; also let every Man spend 3 shillings, every Woman $\frac{1}{2}$ of a shilling, every Boy $\frac{1}{2}$ of a shilling, and every Girl $\frac{1}{2}$ of a shilling; then by forming the Resolution as before, the number of Men will be found 46, the number of Women 3, the number of Boys 5, and the number of Girls 6.

QUEST. 23.

To divide 200 into five such whole numbers, that if the first be multiplied by 12, the second by 3, the third by 1, the fourth by $\frac{1}{2}$, and the fifth by $\frac{1}{3}$, the sum of the Products may also make 200.

This Question may be resolved like the foregoing twentieth and twenty-second, but I shall leave it as an exercise to the industrious Analyst, who, (if he thinks it to be worth his pains,) may find out 6639 Answers to it in whole numbers, (as Monsieur Bachet, in the two last pages of his little Book before cited in Sect. 1. of this Chapter, doth affirm.

Nicholas Tartaglia handling this very Question, (which is the last of the seventeenth Book of the first Part of his Arithmetick,) thought it a great matter that he had found out one single Answer to it in these five whole numbers, to wit, 6, 12, 34, 52, 96, and asserted, That Questions of this sort could not be perfectly solved, either by the Algebraical Art, or any certain Rule; but the contents of this Chapter do manifestly shew, that the Imperfection was in the Artist, and not in the Art.

The End of the Second BOOK.

THE
THIRD & FOURTH
BOOKS
OF THE
ELEMENTS
OF
ALGEBRA.

Compiled by
JOHN KERSET,

*Si quid novisti recte, tibi
Candidus imperti; si non, his utere mecum.*



LONDON:

Printed by **WILLIAM GODDID**, for **Thomas Passinger**
at the Three Bibles on *London-Bridge*. c15. 10c. LXXIV.

The C O N T E N T S of the Third Book.

THE Third Book is an Analysis in Species of the choicest of Diophantus's much admired Questions concerning Squares, Cubes, and right-angled Triangles in rational numbers; with other Questions of like nature. To which is also added a brief Exposition upon Monsieur Fermat's Analytical Invention, prefix'd to Monsieur Bachet's Comment upon Diophantus, Printed at Toloze, Ann. 1670. The number of Questions in the Third Book is 130.

The C O N T E N T S of the Fourth Book.

THE Fourth Book is an Introduction to Mathematical Resolution and Composition; where, the excellent use of the Algebraical or Analytical Art is evidently shewn, both in finding out the Solutions of Geometrical Plane Problems, (viz. of such whose Delineations require only the drawing of Right and Circular lines;) as also in discovering Synthetical Demonstrations from the Analytical steps, to prove the truth of those Solutions by a Series of Consequences deduced altogether from things really given or granted. All which are clearly expounded, and copiously exemplified both Geometrically and Arithmetically, by a choice Collection of useful and delightfull Problems. The Fourth Book is divided into Ten Chapters, the Contents whereof are these, viz.

CHAP.

1. The Explication of Characters, &c.
2. The Explication of Axioms.
3. The Explication of Definitions concerning the usual ways of arguing to deduce one Analogy from another.
4. Various Fundamental Theorems demonstrated.
5. A Collection of Canonical Geometrical Effections.
6. Algebraical Fractions Geometrically expounded.
7. } Four Classes of Examples of the Resolution and Composition of
8. } Plane Problems.
9. }
10. }

*Faults to be Corrected in the Third and Fourth Books,
before they be read.*

Pag.	Lin.	Faults,	thm to be Corrected.
7	27	part,	pair.
23	24	To the given difference 60,	The given difference is 60.
55	38	adding 2,	adding 3.
63	10	Chapt.	Book.
67	37	upon the fourth Book,	upon Quest. 2. Book IV.
71	48		
104	54	$+ \frac{228}{729}$	$+ \frac{228}{729}$
109	49	16098918400,	160989184000.
214	4	right-angle,	right line.
221	2	greater,	lesser.
228	16	36,	26.
228	19	39,	40.
229	51	and the extremes of,	of the extremes and.

THE

THE ELEMENTS OF THE ALGEBRAICAL ART.

Book III.



Among all the Writers upon the *Algebraical Art*, there hath none been hitherto known more antient, nor any more famous for shewing the admirable force of that Art in solving Arithmetical Problems, than *Diophantus of Alexandria*, who lived, (as Authors compute,) above thirteen hundred years ago. He wrote thirteen Books of *Arithmetick*, and one concerning *Multangular numbers*; but of those thirteen, fix only are extant, which contain two hundred and eight Questions, many of which are so knotty and abstruse, that they can hardly be solved without the help of *Diophantus* his peculiar Method, whose Speculations are so sublime, that where there seems to be an impossibility of finding out a single Answer to a Question, he shews how to find out innumerable Answers in rational (or ordinary) numbers. Now to give the ingenious Reader of these *Elements* a delightful Prospect of the rare Inventions of that Prince of Arithmeticians, I have with no small labour framed this Third Book, and therein explain'd the Resolutions of the hardest and choicest of his Questions, in the second, third, fourth, fifth and sixth Books of his *Arithmetick* before-mentioned; as also of divers other subtil Questions invented upon his grounds by *Vieta*, *Bachet*, (the best Commentator hitherto upon *Diophantus*;) *Fermat* and others; among which also divers Questions of my own are inserted, to wit, those which have no citation referring to any Author.

Note. That Δ stands for the word *Triangle*, and \square for a *Square number*; but as to the rest of the Characters used in this third Book, they have already been explain'd in Chap. 1. Book 1. of these *Elements*.

QUEST. 1. (This is the 9th of the second Book of *Diophantus*.)

To divide a given square number into two Squares.

RESOLUTION 1.

1. Let the square number given to be divided be 16
2. The Root or side thereof is 4
3. For the Root or side of the first of the two Squares sought put a
4. Therefore the first Square is aa
5. And consequently, (by the first and fourth steps,) the second Square must be equal to $16 - aa$
6. Now let the side of the second Square be feigned to be $2a - 4$, or $4 - 2a$
7. Therefore the Square of the said feigned side is $4aa - 16a + 16$
8. Which Square must be equal to $16 - aa$ in the fifth step, hence this following Equation ariseth, *viz.*

$$4aa - 16a + 16 = 16 - aa$$

Δ

9. From

9. From which Equation, after due Reduction, the side of the first Square will be made known, *viz.*
 10. And by the ninth and sixth Steps, the side of the second Square will likewise be discovered, *viz.*

So the sides of the two Squares fought are found $\frac{14}{3}$ and $\frac{22}{3}$; which will solve the Question, for the Square of $\frac{14}{3}$ is $\frac{196}{9}$, and the Square of $\frac{22}{3}$ is $\frac{484}{9}$, both which Squares added together make $\frac{680}{9}$, that is, 16, as was required.

Note. That which is most remarkable in the foregoing Resolution of *Diophantus*, is, his ingenious and peculiar way of feigning the side of a Square to be equated to $16 - aa$ in such manner, that after the Equation is duly reduced, the number represented by a will necessarily be Rational. Now because he makes great use of the like manner of feigning the sides of Squares to be equal to Algebraical quantities, in resolving divers hard Questions, (as will copiously appear in this third Book,) the Learner must endeavour to be very well acquainted with the said Method; for his ease therefore I shall explain it in the following Observations.

Observations upon Quest. 1.

1. Concerning the feigned side $2a - 4$ in the sixth step of the foregoing Resolution, it may be asked, why -4 , and not -5 , -6 , or some other number? to this I answer, There is a necessity that this number be always the side of the Square given to be divided into two Squares; so *Diophantus* feigns the second side to be $2a - 4$, (4 being the side or Square Root of 16 the given Square,) to the end that in the Square of $2a - 4$ there may be found the Absolute number 16, to wit, the Square given to be divided, for the Square of $2a - 4$ being equated to $16 - aa$, (as in the eighth step of the Resolution,) there will be found -16 on each part of the Equation, whence by subtracting 16 from each part, there ariseth $5aa = 16a$; and consequently each part of this Equation being divided by a , the Quotients give $5a = 16$, wherefore $a = \frac{16}{5}$.

2. One of the parts of the said feigned side of the second Square must (in this *Quest.* 1.) necessarily have the sign $-$ prefix to it, so *Diophantus* feigns the said second side to be $2a - 4$, for if it were $2a + 4$, all the parts or terms of its Square would be Affirmative, and consequently no possible Equation would arise; as will easily appear by comparing the Square of $2a + 4$, that is, $4aa + 16a + 16$ to $16 - aa$, whence $5aa + 16a = 0$.

3. The Learner may also demand, why in the sixth step of the foregoing Resolution, the side of the second Square fought is feigned to be $2a - 4$, or $4 - 2a$, and not $a - 4$, or $4 - a$, *viz.* why is 2, and not 1 or some other number prefix to a ? to this I answer, If the side of the first Square fought be assumed or supposed to be a or $1a$, (as it is in the third step of the Resolution,) then the side of the second Square cannot be $a - 4$, or $4 - a$, as will be evident by a due process upon that supposition, for the Square of $a - 4$, or $4 - a$, that is, $aa - 8a + 16$ being equated to $aa - 16$, there will arise, after due Reduction, $a = 4$, and consequently, $a - 4$, or $4 - a$, which was put for the second Square, will be equal to nothing: The like absurdity will follow as often as the numbers prefix to a in the feigned sides of the two Squares fought are equal to one another, *viz.* if the first side be feigned to be $5a$, and the second $5a - 4$, or $4 - 5a$; or if the first side be $8a$, and the second $8a - 4$, or $4 - 8a$, &c. from such suppositions a fruitless Equation will ensue, for the side of the second Square will be found equal to nothing: Now for the avoiding of such absurdity, the Learner may take this for a Rule, (the reason whereof will hereafter appear by *Observat.* 1. of the following Resolution 2. of this *Quest.* 1.) Let the numbers prefix to a in the feigned sides of the two Squares fought be any two unequal numbers, *viz.* if a or $1a$ be put for the first side, the second may be $2a - 4$, or $3a - 4$, &c. Again, if we put $3a$ for the first side, the Square thereof will be $9aa$, and consequently because the Square given to be divided into two Squares is 16, the second Square shall be equal to $16 - 9aa$, whose side we may feign to be $7a - 4$, or $4 - 7a$, the Square whereof is $4aa - 16a + 16$; this equated to the said $16 - 9aa$, gives after due Reduction $a = \frac{16}{5}$; therefore $3a$ which was put for the side of the first Square shall be $\frac{48}{5}$, and $4 - 2a$ which was put for the side of the second Square will be found $\frac{22}{5}$, and consequently, the two Squares fought shall be $\frac{2304}{25}$ and $\frac{484}{25}$, whose sum makes $\frac{2788}{25}$, as the Question requires.

In which last Example (which is worthy of the Learner's observation) it happens, that in resolving the Positions, the second side is expounded by $4 - 2a$, not by $2a - 4$, although the

the Resolution be justly formed from either of them; for the Square $4aa - 16a + 16$, whilst a is unknown, may have for its side either $2a - 4$, or $4 - 2a$, and which so ever of these sides be feigned, the same Equation will arise to find out the number a , which, after it is discovered, is to be compared to such of those two feigned sides as will produce a number greater than nothing; so the number a being before found out to be $\frac{16}{5}$, it is manifest that $2a - 4$ is less than nothing, but $4 - 2a$ gives $\frac{22}{5}$, which is the true side of the second Square. So likewise in the Example of *Diophantus*, the side of the second Square cannot be expounded by $4 - 2a$, (although the value of a may be rightly found out from that supposition, as well as from $2a - 4$;) for a being found equal to $\frac{16}{5}$, the said $4 - 2a$ is less than nothing.

4. Here I shall recommend to the Learner one general Observation, *viz.* The principal scope in feigning the side of a Square to be equated to some Algebraick Quantity wherein the highest unknown Power is aa , must be to feign the said side in such manner, that after due Reduction, either the Absolute numbers may vanish, and consequently an Equation remain between some number of aa , and some number of a , whence by Division the number a will necessarily be Rational, or else, (as hereafter will fully appear in this Book,) that aa may vanish out of each part; and consequently an Equation remain between some number of a , and some Absolute number; hence also the number a will be Rational.

Having explain'd *Diophantus* his Resolution of *Quest.* 1. by Numeral Algebra, I shall in the next place resolve the same by Literal Algebra, whence divers useful Canons will be brought to light.

RESOLUTION 2. of *Quest.* 1. which is here repeated, *viz.*

To divide a given square number into two Squares.

1. For the side of the given Square put d
2. Therefore the said Square is dd
3. Take any two unequal numbers, suppose s the greater, and r the lesser, (which s and r are to be used instead of the numbers prefix to a in the foregoing Resolution,) then for the side of the first Square fought put $sa - d$, or $d - sa$
4. And for the side of the second Square fought, put raa
5. Therefore, from the third step, the first Square is $ssaa - 2sda + dd$
6. And from the fourth step, the second Square is $rraa$
7. Therefore the sum of those Squares is $ssaa + rraa - 2sda + dd$
8. Which sum must be equal to the given Square dd , hence this Equation ariseth, *viz.* $ssaa + rraa - 2sda + dd = dd$
9. Which Equation, after due Reduction, gives $a = \frac{2sd}{ss + rr}$
10. Therefore out of the ninth and third steps, the side of the first Square fought is now made known, for it is equal to $\frac{ss - rr}{ss + rr}$
11. And from the ninth and fourth steps the side of the second Square is also known, for it's equal to $\frac{ssd - rrd}{ss + rr}$

Observations upon the preceding Resolution 2. of *Quest.* 1.

1. The eleventh step of the said Resolution discovers that the known numbers s and r must be unequal, to the end the difference of their Squares may be greater than nothing.

2. After the number a is made known, (as in the ninth step,) it will be manifest that the side of the second Square is to be expounded by $sa - d$, not by $d - sa$; for since a is found equal to $\frac{2sd}{ss + rr}$, (as appears by the ninth step of *Resolut.* 2.) it follows that $ss = \frac{2sd}{ss + rr}$, and $d - sa = \frac{rrd - ssd}{ss + rr}$, which is less than nothing, for s is greater than r by supposition: But whilst a is unknown, the side of the second Square may be feigned $d - sa$ as well as $sa - d$, for each of these produceth the same Square $ssaa - 2sda + dd$.

3. The side of the first Square may be feigned sa , and the side of the second $ra - d$, or $d - ra$, for from these Positions the true sides of the two Squares fought will be found the

the same as before are express'd in the tenth and eleventh steps of *Resolut.* 2. but in this latter way of feigning the sides, the side of the second Square will be expounded by $d - ra$, not by $ra - d$, for this will be found less than nothing.

4. The tenth and eleventh steps of the foregoing *Resolution* 2. give this

CANON.

Take any two unequal numbers; multiply severally the double of the Product of their multiplication, and the difference of their Squares by the side of the given Square, lastly, divide those Products severally by the sum of the Squares of the two numbers first taken, and the Quotients shall be the sides of the two Squares sought.

An Example in Numbers.

Let the side of the given Square be 4, then take two unequal numbers at pleasure, as 1 and 2; the double Product of their multiplication is 4, the difference of their Squares is 3, then by multiplying the said 4 and 3 severally by 4, (the side of the given Square) the Products are 16 and 12, these divided severally by 5, (that is, by the sum of the Squares of 1 and 2 the numbers first taken,) give the Quotients $\frac{16}{5}$ and $\frac{12}{5}$, which are the sides of the two Squares sought, for the Squares of $\frac{16}{5}$ and $\frac{12}{5}$ added together make 16, which was given to be divided into two Squares.

5. For as much as (by *Prop.* 47. *Elem.* 1. Euclid.) when a Square is equal to two Squares, the sides of those three Squares will make a right-angled Triangle, the preceding *Quest.* 1. may be thus stated, *viz.*

A Rational number being given for the Hypotenusal of a right-angled Triangle, to find Rational numbers to express the Base and Perpendicular, *viz.* the sides about the right-angle.

This may be solved by the preceding Canon; for if d be put to represent the given Hypotenusal, and s and r any two unequal numbers, r being the lesser, these three following numbers will constitute a right-angled Triangle having d for its Hypotenusal, *viz.*

Hypotenusal,	Base,	Perpendicular.
d	$\frac{ssd - rrd}{ss + rr}$	$\frac{2rsd}{ss + rr}$

6. Moreover, if those three sides of a right-angled Triangle be severally multiplied by the Denominator $ss + rr$, the Products shall also be the sides of a right-angled Triangle, to wit, these following;

Hypoten.	Base,	Perp.
$ssd + rrd$	$ssd - rrd$	$2rsd$

7. And by dividing every one of the three sides last express'd, by their common Factor 4, the Quotients will give these three following sides of a right-angled Triangle, *viz.*

Hypoten.	Base,	Perp.
$ss + rr$	$ss - rr$	$2rs$

8. Which three sides last above express'd are in words the following useful Canon, to form a right-angled Triangle in numbers by the help of any two unequal numbers given.

CANON.

Take any two unequal numbers, (suppose s the greater, and r the lesser,) then the sum of their Squares shall be the Hypotenusal, the difference of the same Squares shall be one of the sides about the right-angle, and the double Product of the multiplication of the said two numbers, the other side.

The Proof of this Canon.

The Square of $ss + rr$ is	$s^4 + 2srr + r^4$
The Square of $ss - rr$ is	$s^4 - 2srr + r^4$
The Square of $2rs$ is	$+ 4srr$

The first of those three Squares is manifestly equal to the sum of the other two; and therefore the sides of those three Squares, if they be express'd by numbers, shall be the measures of the sides of a right-angled Triangle.

An Example of the said Canon in Numbers.

Take two unequal numbers at pleasure, as 1 and 2; then the sum of their Squares is 5 for the Hypotenusal, the difference of the Squares of the same two numbers is 3 for the Base, (that is, either of the sides about the right-angle,) and the double Product of the two numbers

numbers is 4 for the Perpendicular; but that the numbers 5, 3, 4 may be taken for the measures of the sides of a right-angled Triangle is evident, for the Square of the first is equal to the Squares of the two latter.

9: Three Corollaries deduced from the last preceding Canon.

First, in every right-angled Triangle in such whole numbers which are Prime between themselves, the sum of the Hypotenusal ($ss + rr$) and ($2rs$) one of the sides about the right-angle is a square number, to wit, ($ss + rr + 2rs$) the Square of the sum of (s and r) the two numbers by which the said Triangle may be formed according to the last preceding Canon.

Secondly, the sum of the Hypotenusal ($ss + rr$) and ($ss - rr$) the other of the sides about the right-angle is the double of a square number, to wit, the double of (ss) the Square of (s) the greater of the two numbers by which the Triangle may be formed. And the excess of the Hypotenusal ($ss + rr$) above the said side ($ss - rr$) is the double of the Square of (r) the lesser of the same two numbers; therefore.

Thirdly, the three sides of any right-angled Triangle in such Rational whole numbers as are Prime between themselves being severally given, we may find two whole numbers by which the said Triangle may be formed according to the Canon in *Observat.* 8. As, for example, to find two numbers to form these three sides of a right-angled Triangle, to wit, 65, 33, 56, (which are Prime between themselves, for they have no common Divisor but Unity,) I add the Hypotenusal 65 to 33 and 56 severally, and it makes 98 and 121; which latter sum is a Square, and therefore (per *Coroll.* 1.) its Root 11 is the sum of the two numbers sought, and the first sum 98 is the double of the Square 49, whose Root 7 shall be the greater of the two numbers sought, (per *Coroll.* 2.) lastly, by subtracting 7 from 11, the Remainder 4 is the lesser number sought; whence I conclude, that the right-angled Triangle proposed may be formed out of 7 and 4; for the sums of their Squares makes the Hypotenusal 65; the difference of the same Squares is 33 one of the sides about the right-angle, and the double Product of 7 and 4, to wit, 56 is the other side.

10. From the two preceding Canons (in *Observat.* 4. and 8.) another may be deduced to solve *Quest.* 1. *viz.* to divide a given square number into two Squares, or a Rational number being given for the Hypotenusal of a right-angled Triangle, to find the Base and Perpendicular in Rational numbers.

CANON.

By the foregoing Canon in *Observat.* 8. let a right-angled Triangle be formed out of any two unequal numbers, and call this Triangle the first; then it shall be, as the Hypotenusal of the said first Triangle is to its Base, so is the given Hypotenusal of a second Triangle desired to its Base; and as the Hypotenusal of the first Triangle is to its Perpendicular, so is the Hypotenusal of the second to its Perpendicular.

An Example in Numbers.

Let it be required to find the Base and Perpendicular of a right-angled Triangle in numbers whose Hypotenusal shall be 7.

First, by the Canon in *Observat.* 8. I form a right-angled Triangle in numbers, as, $5, 3, 4$
Then, by the Rule of Three, I find $4\frac{1}{2}$ for the desired Base, $5 : 3 :: 7 : 4\frac{1}{2}$
Likewise $5\frac{1}{2}$ for the desired Perpendicular, thus, $5 : 4 :: 7 : 5\frac{1}{2}$
Therefore 7, $4\frac{1}{2}$, $5\frac{1}{2}$ will constitute a right-angled Triangle whose Hypotenusal is 7; as was desired.

11. After the same manner, as many right-angled Triangles in numbers as shall be desired may be found out, which shall have one common Hypotenusal given: As, for example, if three right-angled Triangles in Rational numbers be desired, that 2 may be a Hypotenusal to every one of them, they may be found out thus;

First, by the Canon in the foregoing *Observat.* 8. let three right-angled Triangles be formed, suppose these, $5, 3, 4$; $13, 5, 12$; $17, 15, 8$
Then,

Then by the Rule of Three, the Bases and Perpendiculars of the three right-angled Triangles sought may be found out thus;

$$\begin{array}{l} \text{I. } \left\{ \begin{array}{l} 5 \cdot 3 :: 2 \cdot 1\frac{1}{2} \text{ (Base.)} \\ 5 \cdot 4 :: 2 \cdot 1\frac{2}{3} \text{ (Perp.)} \end{array} \right. \\ \text{II. } \left\{ \begin{array}{l} 13 \cdot 5 :: 2 \cdot 1\frac{2}{3} \text{ (Base.)} \\ 13 \cdot 12 :: 2 \cdot 1\frac{1}{3} \text{ (Perp.)} \end{array} \right. \\ \text{III. } \left\{ \begin{array}{l} 17 \cdot 15 :: 2 \cdot 1\frac{1}{3} \text{ (Base.)} \\ 17 \cdot 8 :: 2 \cdot 1\frac{4}{17} \text{ (Perp.)} \end{array} \right. \end{array}$$

Whence the desired sides of the three right-angled Triangles having 2 for a common Hypothenusal are found to be these, viz.

Hypoth.	Base.	Perp.
2	$1\frac{1}{2}$	$1\frac{2}{3}$
2	$1\frac{2}{3}$	$1\frac{1}{3}$
2	$1\frac{1}{3}$	$1\frac{4}{17}$

12. But note well, that in the search of the Triangles last mentioned, the preparatory right-angled Triangles first found out by the Canon in the preceding *Observat.* 8. must not be like, (that is, such as have proportional Sides,) for it will not be difficult to apprehend, that if from them, other Triangles be deduced by the Rule of Three in such manner as before hath been shewn, there will be but one right-angled Triangle found out, when many are desired to have a common Hypothenusal: That your labour therefore may not be in vain, the preparatory right-angled Triangles must be unlike, to which end they must be formed from pairs of numbers expressing different Reasons, and such, that the two numbers by which any one of the preparatory Triangles is formed, must not be in such proportion to one another as the sum is to the difference of two numbers by which any one of the preparatory Triangles is formed. As, for example, if a right-angled Triangle be formed from 1 and 2, then another right-angled Triangle must not be formed from 2 and 4, 3 and 6, &c. because each of these pairs of numbers expressing the same Reason as 1 and 2 will produce a right-angled Triangle like to the first; nor from 3 and 1, 6 and 2, &c. because 3 having such proportion to 1, likewise 6 to 2, as the sum of 1 and 2 to their difference, those pairs also will produce right-angled Triangles like to the first. But that two right-angled Triangles formed from pairs of numbers expressing the same Reason, or from two such pairs, that one number of the one pair hath such proportion to its yolk-fellow, as the sum of the two numbers of the other pair hath to their difference, are like, I prove thus;

First, let a right-angled Triangle be formed from two numbers s and r , so the three sides will be these, viz.

$$\text{Hyp. } ss + rr, \text{ Base } ss - rr, \text{ Perp. } 2sr$$

Then let a second right-angled Triangle be formed from ds and dr , which have the same proportion to one another as s and r , so the three sides will be these, viz.

$$ddss + ddr, \text{ Base } ddss - ddr, \text{ Perp. } 2ddr$$

Again, let a third right-angled Triangle be formed from $s+r$ and $s-r$, viz. the sum and difference of the two numbers by which the first Triangle was formed, so the three sides will be these, viz.

$$2ss + 2rr, \text{ Base } 4rr, \text{ Perp. } 2ss - 2rr$$

Now I say, that the second Triangle is like to the first, for the sides of the second are the Products of the sides of the first multiplied by the common Factor dd . The third Triangle is also like to the first, for the sides of the third are the doubles of the sides of the first, and consequently Proportionals to them, but in this order, viz. As the Hypothenusal of the first is to its Base, so is the Hypothenusal of the third to its Perpendicular; and As the Hypothenusal of the first is to its Perpendicular, so is the Hypothenusal of the third to its Base.

13. By the help of the preceding Canon in *Observat.* 8. as many right-angled Triangles in whole numbers as shall be desired, and which shall have a common Hypothenusal, may be found out in manner following, viz.

Let it be required to find out three right-angled Triangles in whole numbers, which shall have one common Hypothenusal.

First,

First, by the Canon in the foregoing *Observat.* 8. with respect also to the Note in the last preceding Observation, let three unlike right-angled Triangles be formed, suppose these,

$$\left\{ \begin{array}{l} 5 \cdot 3 \cdot 4 \\ 13 \cdot 5 \cdot 12 \\ 17 \cdot 15 \cdot 8 \end{array} \right.$$

Secondly, multiply severally the three sides of the first Triangle 5, 3, 4, by 22, that is, the Product of the second and third Hypothenusals 13 and 17; so the three Products shall be the sides of a right-angled Triangle, to wit,

$$1105, 663, 884$$

Thirdly, multiply severally the three sides of the second right-angled Triangle 13, 5, 12, by 85, that is, the Product of the first and third Hypothenusals 5 and 17; so the three Products shall be the sides of this right-angled Triangle, viz.

$$1105, 425, 1020$$

Lastly, multiply severally the three sides of the third right-angled Triangle 17, 15, 8, by 65, that is, the Product of the first and second Hypothenusals 5 and 13; so the three Products shall be also the sides of a right-angled Triangle, to wit,

$$1105, 975, 520$$

From the premisses it is manifest that three right-angled Triangles are found out in whole numbers, having 1105 for a common Hypothenusal, and by the same Method you may find out as many as you please.

14. But the smaller the numbers are that express the sides of those preparatory right-angled Triangles the better, and therefore I think it not amiss in this place to shew, how to find out all the unlike right-angled Triangles in whole numbers orderly enumerated, according as their Hypothenusals increase in greatness, so, as that the greatest Hypothenusal may not exceed a given number, suppose 180: To which end,

First, I extract the Square Root of 180, and find it falls between 13 and 14, and consequently a right-angled Triangle formed from 14 and 1, will have its Hypothenusal greater than 180; therefore all the pairs of whole numbers, which have the greater number of each pair, either 14 or greater than 14, will be unfit for our present search.

Secondly, I subtract 169 the Square of the said 13 from 180 the given limit, and the Remainder is 11, whose Square Root falls between 3 and 4; whence 'tis evident that a right-angled Triangle formed from 13 and 4 will have a Hypothenusal greater than 180; but 13 and 3 will give an Hypothenusal less than 180; and therefore I proceed to make an orderly choice of pairs of whole numbers, from the first pair 2 and 1, until I come to 13 and 3 inclusive, and no farther, in this manner, viz.

Thirdly, I write in the first Column of the following Table a Series of whole numbers proceeding from 1, according to the natural order of numbers, as, 1, 2, 3, 4, 5, &c. then at the top of the second Column I set 2 and 1 for the first pair, that done, I combine every number following or standing underneath 2 in the first Column, with every one of the numbers that stands above such following number, except in these two Cases, viz. First, when two numbers so combined are such, that their sum and difference have the same proportion to one another as the two numbers of any pair already set in Column 2. then the two numbers so combined are to be cast out of Column 2. As, for example, because the sum of 3 and 1, to wit, 4, is to their difference 2, as 2 to 1, which 2 and 1 make the first pair already set in Column 2; I omit the writing of the pair 3 and 1 in the second Column: And for the same Reason the pair 5 and 1 is not inserted in the second Column, for the sum of 5 and 1, to wit, 6, is to their difference 4, as 3 to 2, which 3 and 2 made the second pair before written in Column 2. and in like manner all other pairs causing that effect are to be excluded out of the second Column. Again, when two numbers combined as aforesaid happen to be in the same proportion as the two numbers of any pair already set in the second Column, then also the two numbers so combined are to be excluded out of the said Column 2; so 4 and 2 having the same Reason as the first pair 2 and 1, are not inserted in the second Column: the reason of excluding all pairs in those two Cases is, for that they would produce right-angled Triangles like to others before produced, which is contrary to the import of the Proposition. So at length I find only thirty-two pairs of numbers that are fit to be inserted in the said second Column.

Fourthly, from every one of those thirty-two pairs of numbers in the second Column, (the last of which pairs is 13 and 2,) I form a right-angled Triangle (by the Canon in the foregoing *Observat.* 8.) and insert those Triangles into Column 3; among which I find five, to wit, those formed from the pairs 10 and 9; 11 and 8; 11 and 10; 12 and 7;

12 and 7;

12 and 7; and 12 and 11, whose Hypotenuses exceed 180 the prescribed limit, and therefore I cast away those five Triangles, and transfer the rest, which are 27 in multitude, into the fourth Column, in such order as the Hypotenuses do increase in greatness. So 27 unlike right-angled Triangles are found out, which are all that can be given in whole numbers, so as that the greatest Hypotenuse may not exceed 180, as was required. But for further illustration of the premises view the following Table.

A Table whose fourth Column contains 27 unlike right-angled Triangles in numbers, orderly enumerated according as their Hypotenuses increase in greatness.

	H. B. P.	H. B. P.	H. B. P.	
1	2, 1	5. 3. 4	5. 3. 4	8
2	3, 2	13. 5. 12	13. 5. 12	12
3	4, 1	17. 15. 8	17. 15. 8	
4	4, 3	25. 7. 24	25. 7. 24	16
5	5, 2	29. 21. 20	29. 21. 20	
6	5, 4	41. 9. 40	41. 9. 40	20
7	6, 1	37. 35. 12	41. 9. 40	
8	6, 5	61. 11. 60	53. 45. 28	24
9	7, 2	53. 45. 28	61. 11. 60	
10	7, 4	65. 33. 56	65. 33. 56	
11	7, 6	85. 13. 84	65. 63. 16	28
12	8, 1	65. 63. 16	73. 55. 48	
13	8, 3	73. 55. 48	85. 13. 84	
14	8, 5	89. 39. 80	85. 77. 36	
15	8, 7	113. 15. 112	89. 39. 80	32
16	9, 2	85. 77. 36	97. 65. 72	
17	9, 4	97. 65. 72	101. 99. 20	
18	9, 8	145. 17. 144	109. 91. 60	36
19	10, 1	101. 99. 20	113. 15. 112	
20	10, 3	109. 91. 60	125. 17. 144	
21	10, 7	149. 51. 140	137. 105. 88	
22	10, 9	181. 19. 180	145. 17. 144	40
23	11, 2	125. 117. 44	145. 143. 24	
24	11, 4	137. 105. 88	149. 51. 140	
25	11, 6	157. 85. 152	157. 85. 152	
26	11, 8	185. 57. 176	169. 119. 120	
27	11, 10	231. 21. 220	173. 165. 52	44
28	12, 1	145. 143. 24		
29	12, 5	169. 119. 120		
30	12, 7	193. 95. 168		
31	12, 11	265. 23. 264		
32	13, 2	173. 165. 52		

15. By inspecting the preceding Table we may perceive that the unlike right-angled Triangles in Column 5, which are formed from 2 and 1, 3 and 2; 4 and 3; 5 and 4, &c. viz. from pairs of such whole numbers as differ by Unity, have these properties, namely,

First, their Bases 3, 5, 7, 9, 11, 13, &c. which you see standing under B in the fifth Column are in an Arithmetical Progression proceeding from the Base 3 of the Primitive right-angled Triangle 3, 4 by the common difference 2.

Secondly, if an Arithmetical Progression be formed from 8 as the first and least Term, and the common difference of the Terms be 4; as this Progression 8, 12, 16, 20, &c. (which is placed in the last Column of the Table,) then 8 the first Term added to the first Hypotenuse 5, makes the second Hypotenuse 13 standing under H in the fifth Column; also 12 the second Term of the same Progression added to 13 the second Hypotenuse, gives 25 the third Hypotenuse in the same Column; and 16 the third Term added to 25 the third Hypotenuse, gives 41 the fourth Hypotenuse, and so forwards continually.

continually. In like manner, 8 the first Term of the same Progression added to 4 the first Perpendicular, gives 12 the second Perpendicular, standing under P in the said fifth Column; also 12 the second Term added to 12 the second Perpendicular, gives 24 the third Perpendicular; and 16 the third Term added to 24 the third Perpendicular, makes 40 the fourth Perpendicular; and so forwards perpetually. So that by the help of the Primitive right-angled Triangle 3, 4, and the said Progression 8, 12, 16, 20, 24, &c. innumerable unlike right-angled Triangles may be found out by Addition only.

Thirdly, the difference between the Hypotenuse and Perpendicular of every one of the said Triangles in Column 5, is Unity.

Fourthly, the Base is equal to the sum of the two numbers forming the Triangle.

Fifthly, the sum of the Hypotenuse and Perpendicular is a Square, whose side is equal to the Base, or sum of the two numbers forming the Triangle; therefore,

Sixthly, if the sum of the Hypotenuse and Perpendicular be multiplied into the Base, the Product shall be a Cube, whose side is equal to the Base.

Seventhly, the difference of the Hypotenuse and Base is equal to the double of the Square of the lesser of the two numbers forming the Triangle.

The certainty of all the said Properties will be apparent, if you form right-angled Triangles from these following pairs of numbers, and compare those Triangles to one another, according to the import of the said Properties.

$a, a+1$, the first pair; || $a+1, a+2$, the second pair; || $a+2, a+3$, the third pair; || $a+3, a+4$, the fourth pair, &c.

QUEST. 2. (Quest. 10. Lib. 2. Diophant.)

To divide (13) a number given, which is compos'd of two Squares, (9 and 4) into two other Squares.

RESOLUTION 1.

1. The side or Square Root of 9 the greater Square given, is 3
2. The side of the lesser Square 4 is 2
3. Let the side of the first of the two Squares sought be $a+1$ assumed to be
4. And let the side of the second Square sought be feigned $2a-3$; or, $3-2a$
5. Therefore from the third step the first Square desired is a^2+2a+1
6. And from the fourth step the second Square sought is $4a^2-12a+9$
7. Therefore the sum of the two Squares sought is $5a^2-8a+13$
8. Which sum last express'd must be equal to the given number 13, hence this Equation ariseth, viz. $5a^2-8a+13=13$
9. And that Equation, after due Reduction, gives $a^2-\frac{8}{5}a=0$
10. Therefore from the ninth and third steps, the side of the first Square sought is made known, viz. $a=\frac{8}{5}$
11. And from the ninth and fourth steps, the side of the second Square sought is likewise discovered, viz. $3-\frac{16}{5}$

So the sides of the two Squares sought are found $\frac{8}{5}$ and $\frac{1}{5}$; for $\frac{124}{5}$ the Square of $\frac{8}{5}$, added to $\frac{1}{5}$ the Square of $\frac{1}{5}$, makes $\frac{124}{5}$; that is, 13; as was required.

This Question is of the same nature with the foregoing, and deserves to be ranked among the most excellent Problems; for it affords divers admirable Canons concerning the construction of Right-angled Triangles, and is of great use for the understanding of many of Diophantus's Questions, especially in his fifth Book; I shall therefore first explain the preceding Numeral Resolution of the Question, and afterwards resolve the same by Literal Algebra.

Observations upon Quest. 2.

1. It is evident by the foregoing Resolution of Diophantus, That after $a+1$ and $2a-3$, or $3-2a$ are feigned to be the sides of the two Squares sought, the sum of those Squares, that is, $5a^2-8a+13$, is equated to the given number 13, viz. $5a^2-8a+13=13$; which Equation, if there were not the same Absolute number 13 in each part, could not be reduced to an Equation between some number of a and some number of a , and consequently the number a would not be Rational, unless by meer chance.

B.

Whence

Whence then comes it to pass, that the same Absolute number 13 is found in each part of the said Equation? If the Operation be well examin'd, it will appear that the numbers 2 and 3 in the feigned sides of the two Squares fought are the sides of the two given Squares 4 and 9; which 2 and 3 are the only numbers that can be used in the said feigned sides, to cause the number 13 to be found in the sum of their Squares.

2. As to the Signs to be prefixt before the given sides 2 and 3 in the feigned sides of the two Squares fought, they must necessarily be either both —, or one of them —, and the other +; to the end that in the sum of the feigned Squares there may be some number of a with the sign — prefixt; whence it will follow that the said number of a may be transfer'd to the other part of the Equation with the sign +, and then the Absolute numbers vanishing by Subtraction, because they are one and the same number as hath been shewn in the preceding *Observat.* 1. there will remain an Equation between some number of aa and some number of a ; whence by due Division the number a will be Rational.

3. The numbers to be prefixt before a in the feigned sides of the two Squares fought, may be variously chosen according to divers particular Rules that might be given, among which I shall recommend but two to the Learner's practice: The first Rule is this;

Let two unequal numbers be taken to be prefixt before a in the feigned sides, but with this Caution, *viz.* That the greater of the two numbers taken may not have the same proportion to the lesser as the sum of the sides of the two Squares given in the Question hath to their difference: As, if the two Squares given be 4 and 9, whose sides are 2 and 3, the greater of the two numbers taken must not be to the lesser as 5 to 1, because 5 is the sum, and 1 the difference of the said 2 and 3. Suppose therefore that 5 and 1 be taken; then let the first feigned side be $3a + 2$, (3 being the lesser of the said two numbers taken, and 2 the lesser of the sides of the two Squares given,) and let the second feigned side be $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers taken, and 3 the side of the greater Square given.) Now if from those feigned sides the Operation be prosecuted like as in the preceding Resolution of *Quest.* 2. an Equation will rightly ensue to find out two Squares different from those given, but such as being added together shall make the same sum as those given.

The second Rule is this; Let two unequal numbers be taken with this Caution, *viz.* That they be not in the same Reason (or Proportion) as the sides of the two Squares given: As, if the two Squares given in the Question be 9 and 4, whose sides are 3 and 2; then the two numbers taken must not be 3 and 2, 6 and 4, 9 and 6, nor any numbers in the same Reason: Suppose therefore that 5 and 4 be chosen, then for the side of the first Square fought put $4a - 2$, or $2 - 4a$, (4 being the lesser of the said two numbers chosen, and 2 the lesser of the sides of the two Squares given,) and for the side of the second Square fought put $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers before chosen, and 3 the greater of the sides of the two Squares given;) then if from the said feigned sides the Operation be prosecuted like as in the foregoing Resolution of *Quest.* 2. an Equation will ensue, to find out two Squares different from those given, but such as being added together shall make the same sum as those given. The reason of these two Cautions will hereafter appear.

The preceding Observations may suffice for explication of the Resolution of *Quest.* 1. by Numeral Algebra; I shall in the next place shew how to resolve the same by Literal Algebra, and among various ways that might be used, I shall chuse but two, which correspond with the Rules before given in *Observat.* 3. and do produce divers excellent Canons.

RESOLUTION 2. of *Quest.* 2. which is here repeated, *viz.*

To divide a number given which is compos'd of two known Squares, into two other Squares

1. For the side of the greater Square given, put a
2. And for the side of the lesser Square given, put b
3. Therefore the greater Square is aa
4. And the lesser Square is bb
5. Take two unequal numbers, s the greater, and r the lesser, with this Caution, *viz.* that s be not in such proportion to r , as $a + b$ to $a - b$; which numbers s and r are to be used instead of the numbers that were prefixt before the unknown number a in the

Quest. 2.

Diophantus's Algebra explain'd.

the foregoing Numeral Resolution of this Question, and the reason of the Caution will be shewn in the sixteenth step of this Resolution.

6. For the side of the first of the two Squares fought, put $ra - b$
7. And for the side of the second Square fought put $sa - d$, or $d - sa$
8. Therefore from the sixth step the first Square fought is $rraa + 2rba - bb$
9. And from the seventh step the second Square fought is $ssaa - 2sda + dd$
10. Therefore the sum of those two Squares is $ssaa + rraa + 2rba - 2sda + bb + dd$
11. But the said sum must be equal to $dd + bb$ the sum of the two Squares given in the Question, and before prefixt in the third and fourth steps; hence the following Equation ariseth, *viz.*

$$ssaa + rraa + 2rba - 2sda + bb + dd = bb + dd.$$

12. Which Equation, after due Reduction, gives $a = \frac{2sd - 2rb}{ss + rr}$

13. Therefore from the twelfth and sixth steps, the side of the first Square fought is now made known, and found equal to this following Quantity,

$$\frac{2rsd + ssb - rrb}{ss + rr}$$

14. And from the twelfth and seventh steps, the side of the second Square fought is likewise known, and found equal to

$$\frac{ssd - rrd - 2rsb}{ss + rr}, \text{ or, } \frac{2rsb - rrd - ssd}{ss + rr}$$

That is to say, The former of those two Quantities expresseth Fraction-wise shall be the side of the second Square when $ssd - rrd$ is greater than $2rsb$, but the latter of those Quantities shall be the said side when $ssd - rrd$ is less than $2rsb$. For if $ssd - rrd$ be greater than $2rsb$, then by subtracting $2rsb$ from $ssd - rrd$, the Remainder is the same with the Numerator of the first of the two Fractions above expresseth; but if $2rsb$ be greater than $ssd - rrd$, then by subtracting $ssd - rrd$ from $2rsb$, the Remainder is the same with the Numerator of the latter of the said Fractions; therefore the side of the second Square may be expresseth thus;

$$\frac{ssd - rrd \text{ or } 2rsb}{ss + rr}$$

That is to say, If the difference between $ssd - rrd$ and $2rsb$ be divided by $ss + rr$, the Quotient shall be the side of the second Square fought.

From the premises ariseth this following

CANON 1.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the same proportion to the lesser, as the sum of the sides of the two Squares given hath to the difference of the same sides: Multiply the double Product of the multiplication of those two numbers first taken by each of the said two sides given, and reserve the Products; multiply also the difference of the Squares of the said two numbers first taken by each of the said two sides given, and reserve these Products; then add the greater of the two first reserved Products to the lesser of the two latter, and reserve the sum for a Dividend; take also the difference between the lesser of the two first Products and the greater of the two latter for a second Dividend; lastly, divide severally the said Dividends by the sum of the Squares of the two numbers first taken, to shall the Quotients be the sides of the two Squares fought.

Example 1. Where the number given is compos'd of two unequal Squares.

Let it be required to divide 13 which is compos'd of two Squares, 9 and 4, into two other Squares.

- The side of the greater Square given is 3
 The side of the lesser Square given is 2
 Take two unequal numbers, with respect to the Caution in the Canon, 2 and 1
 Then, by using those four numbers as the Canon doth direct, the sides of the two Squares fought will be found these, $\frac{18}{5}$ and $\frac{1}{5}$

The Squares of which sides being added together make 13, as was required.

Example 2.

Let it again be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares different from those found out in *Example 1.*

The sides of the two Squares given are

Take two unequal numbers with respect to the Caution in the fore-
going *Canon 1.* as these, $\left. \begin{array}{l} 3 \text{ and } 2 \\ 4 \text{ and } 3 \end{array} \right\}$
Then by using the four numbers last before express'd, as the said *Canon*
doth direct, the sides of the two Squares sought will be found these, $\left. \begin{array}{l} 86 \\ 25 \end{array} \right\}$ and $\frac{27}{13}$

The Proof.

The Square of $\frac{86}{25}$ is $\frac{7396}{625}$
The Square of $\frac{27}{13}$ is $\frac{729}{169}$
The sum of those Squares is $\frac{13000}{10405}$, or 13.

Example 3. Where the given number is compos'd of two equal Squares.

Let it be required to divide 2, which is compos'd of two equal Squares, 1 and 1, into two unequal Squares.

The side of either of the Squares given is $\left. \begin{array}{l} 1 \\ 1 \end{array} \right\}$
Take in this Case any two unequal numbers, as $\left. \begin{array}{l} 1 \text{ and } 2 \\ 2 \text{ and } 1 \end{array} \right\}$
Then by working with those three numbers according to the direction
of *Canon 1.* the sides of the two Squares sought will be found these, $\left. \begin{array}{l} 7 \\ 5 \end{array} \right\}$ and $\frac{1}{5}$

The Squares of which sides being added together make 2, as may easily be proved.

16. Now that the necessity of the Caution prescribed in the foregoing *Canon 1.* about choosing the unequal numbers s and r may appear, I shall prove, That if s the greater of them hath the same proportion to r the lesser, as $d-b$ the sum of the sides of the two unequal Squares given in *Quest. 2.* hath to $d-b$ the difference of the same sides, then the said Canon will produce the same sides d and b for the sides of the two Squares sought, and consequently the Operation in such Case will be in vain. For it is manifest by the thirteenth step, that one of the sides found out by the Canon is $\frac{2rsd + sib - rrb}{ss + rr}$, so that if we prove this side to be equal to d the side of the greater

of the two Squares given, then consequently the other side found out by the Canon, that is, the side express'd by the fourteenth step, shall be equal to the side of the lesser of the two Squares given, for the sum of the Squares found out is equal to the sum of those given.

17. Let it therefore be supposed that $\left. \begin{array}{l} d+b \\ d-b \end{array} \right\} :: s:r$

18. And then, we are to demonstrate that $\left. \begin{array}{l} 2rsd + sib - rrb \\ ss + rr \end{array} \right\} = d$

Demonstration.

19. By supposition in the seventeenth step, $\left. \begin{array}{l} d+b \\ d-b \end{array} \right\} :: s:r$

20. Therefore by comparing the Rectangle of the extremes $\left. \begin{array}{l} rd + rb \\ rd + rb \end{array} \right\} = sd - b$

21. And by adding sb to each part of the last Equation, this
aristeth, $\left. \begin{array}{l} sb + rd + rb \\ sb + rb \end{array} \right\} = sd - rd$

22. And by subtracting rd from each part, it makes $\left. \begin{array}{l} sb + rb \\ sb + rb \end{array} \right\} = sd - rd$

23. And by resolving the last Equation into Proportionals, $\left. \begin{array}{l} s+r \\ s-r \end{array} \right\} :: d:b$
this Analogy aristeth, *viz.*

24. And by drawing $s-r$ as a common Factor into the two first Terms of that Analogy, this aristeth,

$ss - rr :: ss + rr - 2rs :: d \cdot b$

25. Therefore, by comparing the Product of the extremes in the last Analogy to the Product of the means, this Equation aristeth, *viz.*

$sib - rrb = sid - rrd - 2rsd$

26. Whence by equal Addition of $2rsd$, this Equation aristeth, *viz.*

$2rsd + sib - rrb = sid + rrd$

27. Wherefore by dividing each part of the last Equation by $ss + rr$, this aristeth, *viz.*

$$\frac{2rsd + sib - rrb}{ss + rr} = d \text{ Which was to be demonstrated.}$$

Resolution 3.

RESOLUTION 3. of Quest. 2. which is here repeated, viz.

To divide a given number which is compos'd of two known Squares, into two other Squares.

1. For the side of the greater Square given, put d
2. And for the side of the lesser Square given, put b
3. Therefore the greater of those Squares shall be dd
4. And the lesser bb
5. Take two unequal numbers, s the greater, and r the lesser, with this Caution, *viz.* That s may not be in such proportion to r , as d to b , which s and r do represent the numbers to be prefix to the unknown number a , according to the second Rule before mentioned in *Observat. 3. Resolut. 1. Quest. 2.* and the reason of the Caution will be shewn in the sixteenth step of this Resolution.
6. Then for the side of the first Square sought, put $rs - b$, or $b - ra$
7. And for the side of the second Square sought, put $sa - d$, or $d - sa$
8. Therefore from the sixth step the first Square sought is $rsaa - 2rba + bb$
9. And from the seventh step the second Square sought is $saaa - 2sda + dd$
10. Therefore the sum of the Squares in the eighth and ninth steps is $rsaa + rsaa - 2rba - 2sda + bb + dd$

11. But the said sum must be equal to the two Squares given, to wit, dd and bb , hence therefore aristeth the following Equation, *viz.*

$$rsaa + rsaa - 2rba - 2sda + bb + dd = bb + dd$$

12. Which Equation, after due Reduction, gives $\left. \begin{array}{l} 2rs \\ ss + rr \end{array} \right\} a = \frac{2rb - 2sd}{ss + rr}$

13. Therefore from the twelfth and sixth steps the side of the first Square sought is now made known, and equal to one of these two Quantities, to wit,

$$\frac{2rsd + sib - rrb}{ss + rr}, \text{ or, } \frac{sib - rrb - 2rsd}{ss + rr}$$

That is to say, the former of those two Quantities express'd Fraction-wise shall be the side of the first Square sought, when $sib - rrb$ is less than $2rsd$, but the latter shall be the side sought when $sib - rrb$ is greater than $2rsd$. For if $sib - rrb$ be less than $2rsd$, then by subtracting $sib - rrb$ from $2rsd$, the Remainder will be the same with the Numerator of the first of the two Quantities above express'd Fraction-wise, but if $sib - rrb$ be greater than $2rsd$, then by subtracting $2rsd$ from $sib - rrb$, the Remainder will be the same with the Numerator of the latter of the said Quantities: Therefore the side of the first Square sought may be express'd thus,

$$\frac{sib - rrb \text{ or } 2rsd}{ss + rr}$$

That is to say, If the difference between $sib - rrb$ and $2rsd$, be divided by $ss + rr$ the Quotient shall be the side of the first Square sought.

14. But from the twelfth and seventh steps the side of the second Square sought is found equal to this known Quantity, *viz.*

$$\frac{2rsb + ssa - rrd}{ss + rr}$$

From the premises aristeth this following

CANON 2.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the same proportion to the lesser, as the side of the greater of the two Squares given hath to the lesser side: Multiply the double Product of the multiplication of the two unequal numbers first taken by each of the said two sides given, and reserve the Products; multiply also the difference of the Squares of the two numbers first taken, by each of the said two sides given, and reserve these Products; then take the difference between the greater of the two first reserved Products and the lesser of the two latter for a Dividend; take also the sum of the lesser of the two first Products and the greater of the two latter for a second Dividend; lastly, divide each of those Dividends by the sum of the Squares of the two numbers first taken, so shall the Quotients be the sides of the two Squares sought.

Example 1.

Example 1. Where the number given to be divided is compos'd of two unequal Squares.

Let it be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares.

The side of the greater Square given is 3
The side of the lesser Square given is 2
Take two unequal numbers with respect to the Caution in Canon 2. 2 and 1
as these,
Then by using those four numbers according to the direction of Canon 2. 6 and 17
the sides of the two Squares sought will be found these, viz.
The Squares of which sides $\frac{6}{2}$ and $\frac{17}{2}$ being added together make $\frac{113}{4}$ or 13, as was required.

Example 2.

Let it be again required to divide 13, which is compos'd of 9 and 4 into two other Squares different from those found out in *Example 1.*

The sides of the two given Squares, 9 and 4, are 3 and 2
Take two unequal numbers with respect to the Caution in Canon 2. 4 and 1
as these,
Then by working with those four numbers as the said Canon 2. doth 6 and 6
direct, the sides of the two Squares sought will be found these, 17 and 17
The Squares of which sides are $\frac{17}{2}$ and $\frac{17}{2}$, whose sum makes $\frac{289}{4}$, that is, 11, as was required.

Example 3. Where the number given to be divided is compos'd of two equal Squares.

Let it be required to divide 18, which is compos'd of two equal Squares, 9 and 9, into two unequal Squares.

The side of either of the Squares given is 3
Take in this Case any two unequal numbers, as 1 and 1
Then by using those three numbers according to the direction of the foregoing Canon 2, the sides of the two Squares sought will be found 3 and 21
these, viz.
The Squares of which sides are $\frac{9}{2}$ and $\frac{441}{2}$, whose sum makes $\frac{450}{2}$, that is, 11, as was required.

16. Now that the necessity of the Caution prescribed in the foregoing Canon 2, choosing the unequal numbers s and r may appear, I shall prove, That if s the greater of them hath the same proportion to r the lesser, as d the side of the greater of the two Squares given in *Quest. 2.* hath to b the side of the lesser of the same Squares, the said Canon will produce the same sides d and b for the sides of the two Squares sought, and consequently the Operation in such Case will be in vain: First, it is manifest by the fourteenth step, that one of the sides found out by the Canon is $\frac{ssd - rrd + 2rb}{ss + rr}$

so that if we prove this side to be equal to d the side of the greater of the two Squares given, then consequently the other side found out by the Canon, that is, the side expressed in the thirteenth step shall be equal to the side of the lesser of the two Squares given, for the sum of the Squares found out is equal to the sum of those given.

17. Let it therefore be supposed that $d \cdot b :: s \cdot r$
18. And then we are to demonstrate that $\frac{ssd - rrd + 2rb}{ss + rr} = d$

Demonstration.

19. By supposition in the sixteenth step, $d \cdot b :: s \cdot r$
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, $sb = rd$
21. And by drawing $2r$ into each part of the last Equation, this ariseth, viz. $2rsb = 2rrd$
22. And by adding ssd to each part of the last Equation, it gives $ssd + 2rsb = ssd + 2rrd$
23. And by subtracting rrd from each part of the last Equation, there remains $ssd - rrd + 2rsb = ssd - rrd + 2rrd$

24. Which

24. Wherefore by dividing each part of the last Equation } $\frac{ssd - rrd + 2rsb}{ss + rr} = d$
by $ss + rr$, there ariseth
Which was to be proved.

Observations upon the preceding Resolutions 2, and 3. of Quest. 2. by Literal Algebra.

1. If z be put equal to $\sqrt{bb + dd}$: that is, the square Root of the number compos'd of two Squares given in *Quest. 2.* that Question may be stated thus, viz.

Two Rational numbers, b and d , being given for the Base and Perpendicular of a right-angled Triangle whose Hypotenuse is z , Rational or Irrational, to find out other Rational numbers to express the Base and Perpendicular of a second right-angled Triangle whose Hypotenuse shall be z likewise.

The Base and Perpendicular of the Triangle sought shall be given either by Canon 1. in the fifteenth step of *Resolution 2.* of *Quest. 2.* or by Canon 2. in the fifteenth step of *Resolution 3.* and may be express'd by Letters, as before in the thirteenth and fourteenth steps of *Resolution 2.* or by the thirteenth and fourteenth steps of *Resolution 3.* viz.

	By Canon 1.		By Canon 2.	
ΔK.	Hyp. z , (or, $\sqrt{bb + dd}$.)		Hyp. z , (or, $\sqrt{bb + dd}$.)	ΔL.
	Perp. $\frac{2rsd + srb - rrb}{ss + rr}$		Perp. $\frac{srb - rrb + 2rsd}{ss + rr}$	
	Base, $\frac{ssd - rrd + 2rsb}{ss + rr}$		Base, $\frac{2rsb + ssd - rrd}{ss + rr}$	

2. If the Bases, and Perpendiculars of those two right-angled Triangles above-express'd, which I call ΔK and ΔL, be well examined, another way will be discovered to find out the same Bases and Perpendiculars by the help of the Bases and Perpendiculars of two like right-angled Triangles whose Hypotenuses are b and d . For,

First, it is manifest by *Observat. 5.* *Resolut. 2.* *Quest. 1.* of this Book, that these three following numbers will constitute a right-angled Triangle, which hath b for an Hypotenuse, viz.

Hyp.	Base,	Perp.
b	$\frac{srb - rrb}{ss + rr}$	$\frac{2rsb}{ss + rr}$
		(ΔM.)

Likewise these three following numbers will constitute a right-angled Triangle, having d for an Hypotenuse, viz.

Hyp.	Base,	Perp.
d	$\frac{ssd - rrd}{ss + rr}$	$\frac{2rsd}{ss + rr}$
		(ΔN.)

Which two Triangles last before express'd, to wit, ΔM and ΔN, are like, for each of them is like to a right-angled Triangle whose three sides are $ss + rr$, $ss - rr$, and $2rs$; Now I say, if the Perpendiculars and Bases of the two right-angled Triangles K and L before express'd in *Observat. 1.* be well viewed, it will be evident, that they are deduced from the two like right-angled Triangles M and N before express'd in this *Observat. 2.* which have b and d for Hypotenuses. For, first, the Perpendicular of ΔK is compos'd of the Base of ΔM and the Perpendicular of ΔN; secondly, the Base of ΔK is equal to the difference between the Perpendicular of ΔM and the Base of ΔN; thirdly, the Perpendicular of ΔL is equal to the difference between the Base of ΔM and the Perpendicular of ΔN; lastly, the Base of ΔL is compos'd of the Perpendicular of ΔM and the Base of ΔN.

3. Hence therefore another Canon comes to light, to solve as well the preceding *Quest. 2.* as also the following Proposition, (which is *Prop. 47.* in *pag. 35.* of *Viete's Works.*) viz.

From two right-angled Triangles given to deduce a third right-angled Triangle, such that the Square of the Hypotenuse of the third may be equal to the Squares of the Hypotenuses of the first and second.

This Proposition may be solved by the following

CANON 3.

C A N O N 3.

First, (by the Canon in *Observat.* 10. *Resolut.* 2. of the preceding *Quest.* 1.) find out two like right angled Triangles in numbers, such, that their Hypothensals may be the sides of the two Squares given in the foregoing *Quest.* 2. then, for the Perpendicular of the third right-angled Triangle sought, take the sum of the Base of the first and Perpendicular of the second; for the Base of the third, take the difference between the Perpendicular of the first and Base of the second; and the Hypothensal of the third shall be the square Root of the sum of the Squares of the Hypothensals of the first and second.

Or thus:

For the Perpendicular of the third right-angled Triangle sought, take the difference between the Base of the first and Perpendicular of the second; for the Base of the third, take the sum of the Perpendicular of the first and Base of the second; and the Hypothensal of the third shall be the same as before is express'd.

Example 1. in Numbers.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two other Squares.

The sides of 4 and 9 the two Squares given are . . . 2 and 3
Find a first right-angled Triangle in numbers whose Hypothensal shall be 2, the side of 4 the lesser of the two Squares given, as,

$$\begin{array}{lcl} \text{Hyp.} & \text{Base} & \text{Perp.} \\ 2 & \frac{4}{5} & \frac{3}{5} \end{array}$$

Find likewise a second right-angled Triangle like to the first, and such, that its Hypothensal may be 3 the side of the greater Square given, as,

$$\begin{array}{lcl} 3 & \frac{9}{5} & \frac{12}{5} \end{array}$$

Then by using the Bases and Perpendiculars of those two like right-angled Triangles as *Canon* 3. doth direct, this third right-angled Triangle will be found out, whose Hypothensal is equal to $\sqrt{4+9}$: that is, $\sqrt{13}$, and consequently the Base and Perpendicular are the sides of the two Squares sought.

$$\sqrt{13} \quad \frac{1}{5} \quad \frac{12}{5}$$

Or, according to the latter part of *Canon* 3. the sides of the two Squares sought will be found the Base and Perpendicular of this third right-angled Triangle whose Hypothensal is $\sqrt{13}$; that is, $\sqrt{4+9}$: as before . . .

$$\sqrt{13} \quad \frac{17}{5} \quad \frac{6}{5}$$

Example 2.

Let it be required to divide 25, which is compos'd of two Squares, 9 and 16, into two other Squares.

The sides of 9 and 16 the two given Squares are . . . 3 and 4
Find a first right-angled Triangle in numbers, whose Hypothensal shall be 3 the side of the lesser Square given, as,

$$\begin{array}{lcl} \text{Hyp.} & \text{Base} & \text{Perp.} \\ 3 & \frac{16}{5} & \frac{12}{5} \end{array}$$

Find likewise a second right-angled Triangle like to the first, and such, that its Hypothensal shall be 4 the side of the greater Square given, as,

$$\begin{array}{lcl} 4 & \frac{48}{13} & \frac{20}{13} \end{array}$$

Then by using the Bases and Perpendiculars of the two like right-angled Triangles last found out according to the direction of *Canon* 3. this third right-angled Triangle will be discovered, whose Hypothensal is 5, that is, $\sqrt{9+16}$: and consequently the Base and Perpendicular are the sides of the two Squares sought.

$$5 \quad \frac{33}{13} \quad \frac{56}{13}$$

Or, according to the latter part of *Canon* 3. the sides of the two Squares sought will be found the Base and Perpendicular of this third right-angled Triangle, whose Hypothensal is 5, that is, $\sqrt{9+16}$: as before, . . .

$$5 \quad \frac{63}{13} \quad \frac{16}{13}$$

4. If every one of the three sides of the two right-angled Triangles K and L before express'd in *Observat.* 1. having x for a common Hypothensal, Rational or Irrational, be multiplied by $ss+rr$, the Products shall be also the sides of two right-angled Triangles like to the two former respectively; which Products or sides shall be these, viz.

Hyp.

Quest. 4.

Diophantus's Algebra explain'd.

$$\begin{array}{l} \text{Hyp.} \quad xss+rrr \\ \text{Perp.} \quad 2rsd+ssb-rrb \\ \text{Base,} \quad ssd-rrd \cup 2rsb \end{array}$$

$$\begin{array}{l} \text{Hyp.} \quad xss+2rr \\ \text{Perp.} \quad ssb-rrb \cup 2rsd \\ \text{Base,} \quad 2rsb+ssd-rrd \end{array}$$

Now if the two right-angled Triangles last express'd be well examined, it will appear, that each of them may be deduced from two right-angled Triangles, one of which hath for its Hypothensal $ss+rr$, Base $ss-rr$, and Perpendicular $2rs$; (or $2rs$ may be called the Base, and $ss-rr$ the Perpendicular;) but of the other the Hypothensal is $x = \sqrt{bb+dd}$: Rational or Irrational, the Base is b , and the Perpendicular is d ; (or d may be called the Base, and b the Perpendicular;) I say, from these two last mentioned Triangles each of the two former may be deduced in such manner as is directed in the following *Canon*, which is the same with that rais'd by *Viete* in solving *Prop.* 46. in pag. 34. of his Works, viz.

From two right-angled Triangles given, to form a third right-angled Triangle.

C A N O N.

For the Hypothensal of the third right-angled Triangle, take the Product of the multiplication of the Hypothensals of the two right-angled Triangles given: for the Perpendicular, the sum of the Product of the Base of the first into the Perpendicular of the second, and the Product of the Base of the second into the Perpendicular of the first: and for the Base, take the difference between the Product of the Bases of the first and second, and the Product of their Perpendiculars.

Or thus:

For the Hypothensal of the third right-angled Triangle, take (as before) the Product of the multiplication of the Hypothensals of the first and second right-angled Triangles given: for the Perpendicular, the difference between the Product of the Base of the first into the Perpendicular of the second, and the Product of the Base of the second into the Perpendicular of the first: lastly, for the Base, take the sum of the Product of the Bases of the first and second, and the Product of their Perpendiculars.

An Example in Numbers.

$$\begin{array}{lcl} \text{Hyp.} & \text{Base} & \text{Perp.} \\ \text{Let there be two right-angled Triangles given in numbers, } & 5 & 3 & 4 \\ \text{suppose these, } & 13 & 5 & 12 \\ \text{Then from those Triangles, these two are deduced by the } & 63 & 33 & 56 \\ \text{two Canons last before express'd, viz. } & 63 & 63 & 16 \end{array}$$

Note 1. If the two right-angled Triangles given be unlike, then either of those Canons will form a third right-angled Triangle; but if like, then the first only will take place: for when the two right-angled Triangles given are like, then the difference of the Products mentioned in the latter *Canon* are equal to nothing, as will be evident to every diligent Reader:

Note 2. If from any right-angled Triangle taken $\begin{array}{ccc} H & B & P \\ \text{twice, suppose from these two, } & H & B & P \end{array}$

A third right-angled Triangle be deduced according to $\begin{array}{l} HH \cdot BB \cup PP \cdot 2BP \\ \text{the first Canon, as this, } \end{array}$

Then the angle at the Base of the third right-angled Triangle so deduced, viz. the angle opposite to the side BP shall be equal to the double of the angle at the Base of the first right-angled Triangle, viz. of the angle opposite to the side P ; or else equal to the Complement of the said double angle unto two right-angles, when the said double exceeds a right-angle.

Likewise, if from two unlike right-angled Triangles, $\begin{array}{ccc} H & B & P \\ \text{suppose from these, } & h & b & p \end{array}$

A third right-angled Triangle be deduced according to $\begin{array}{l} Hb \cdot Bb \cup Pp \cdot Bp - Pb \\ \text{the first Canon, as these, } \end{array}$

Then the angle at the Base of this third right-angled Triangle; viz. the angle opposite to the side $Bp-Pb$ shall be equal to the sum of the angles at the Bases of the first and second right-angled Triangles, viz. of the angles opposite to the sides P and p ; or else equal to the Complement of the said sum unto two right-angles, when that sum exceeds a right-angle.

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The converse of this rare Speculation is demonstrated by *Anderfonus*, in *Theorem. 2. of Vieta's* mysterious-Doctrine of Angular Sections; and likewise by *Herigonius* at the latter end of the First Tome of his *Cursus Mathematicus*.

QUEST. 3.

To divide a given Square number into two such Squares, that one of them may consist within given limits.

Let it be required to divide 16 into two such Squares, that one of them may be greater than 10, but less than 11.

Or thus:

A Rational number 4 being given for the Hypothensal of a right-angled Triangle, to find the Base and Perpendicular in such Rational numbers, that one of them may be greater than $\sqrt{10}$, but less than $\sqrt{11}$.

RESOLUTION.

- For the given Hypothensal 4, (which is the side of the given Square 16) put d
- For $\sqrt{10}$ the lesser of the prescribed limits, put f
- For $\sqrt{11}$ the greater of the prescribed limits, put g
- Let two unequal numbers be represented by s and r
- Then the sides about the right-angle of a right-angled Triangle whose Hypothensal is d , will be found equal to these Quantities, (by the Canon in *Observat. 5. Resolut. 2. Quest. 1.*) viz. $\frac{2rsd}{ss+rr}$ and $\frac{ssd-rrd}{ss+rr}$
- But since this Question requires that one of those sides, suppose $\frac{2rsd}{ss+rr}$, may be greater than f , yet less than g , the said numbers s and r cannot be any two unequal numbers, and therefore I shall here shew a way to chuse them, so as that they may cause the said side to agree with the said limits: To which end, first, a number at pleasure may be taken for one of the said numbers s and r , as, $1=r$; and then to search out limits for the chusing of s , I proceed in this manner, viz. I put a instead of s while it is unknown, and then since $r=1$, the before-mentioned side $\frac{2rsd}{ss+rr}$ will be express'd thus, $\frac{2da}{aa+1}$ where the number a only is unknown: Now,
- Let it be supposed (according to the import of the Question) that $\frac{2da}{aa+1} = f$
- Let it also be supposed that $\frac{2da}{aa+1} = g$
- Then by multiplying each part of the supposition in the seventh step by $aa+1$, it follows that $2da = faa + f$
- Therefore by comparing the latter part of the ninth step to the former, $faa + f = 2da$
- And by dividing each part in the last step by f , it follows, that $aa+1 = \frac{2da}{f}$
- And by subtracting 1 from each part, $aa = \frac{2da}{f} - 1$
- Likewise by equal subtraction of $\frac{2da}{f}$ it follows, that $aa - \frac{2da}{f} = -1$
- And by adding the Square of half the Coefficient $\frac{2d}{f}$ to each part in the thirteenth step, $aa - \frac{2da}{f} + \frac{d^2}{f^2} = \frac{d^2}{f^2} - 1$
- And by extracting the square Root out of each part in the fourteenth step, $a - \frac{d}{f} = \sqrt{\frac{d^2}{f^2} - 1}$
- Wherefore by adding $\frac{d}{f}$ to each part in the fifteenth step, it follows that $a = \frac{d}{f} + \sqrt{\frac{d^2}{f^2} - 1}$

Again, 17

- Again, because $\frac{d}{f} = a$, (as well as $a = \frac{d}{f}$), may be the side of the Square in the first part of the fourteenth step, it thence follows that $\frac{d}{f} = a + \sqrt{\frac{d^2}{f^2} - 1}$
- And by adding a to each part in the seventeenth step, $\frac{d}{f} = a + \sqrt{\frac{d^2}{f^2} - 1}$
- And by equal subtraction of $\sqrt{\frac{d^2}{f^2} - 1}$ it follows that $\frac{d}{f} - \sqrt{\frac{d^2}{f^2} - 1} = a$
- Wherefore by comparing the latter part of the nineteenth step to the first, it's evident that $a = \frac{d}{f} - \sqrt{\frac{d^2}{f^2} - 1}$
- Again, by supposition in the eighth step, $\frac{2da}{aa+1} = g$
- And consequently, $g = \frac{2da}{aa+1}$
- Whence by arguing in like manner as before, with f , from the ninth step to the sixteenth, it will appear that $a = \frac{d}{g} + \sqrt{\frac{d^2}{g^2} - 1}$
- Again, by arguing with g in like manner as before with f , from the seventeenth step to the twentieth, it will be evident that $a = \frac{d}{g} - \sqrt{\frac{d^2}{g^2} - 1}$
- Now because a was put instead of s , the sixteenth and twenty-third steps give a Canon for limiting the number s , when $r=1$; viz.

CANON 1.

$$s = \frac{d + \sqrt{\frac{d^2}{f^2} - 1}}{f} \quad (2.039, \text{ \&c.})$$

$$s = \frac{d - \sqrt{\frac{d^2}{f^2} - 1}}{f} \quad (1.880, \text{ \&c.})$$

- Again, the twentieth and twenty-fourth steps give another Canon for limiting the number s , (which was represented by a in the preceding argumentation,) when $r=1$, viz.

CANON 2.

$$s = \frac{d - \sqrt{\frac{d^2}{f^2} - 1}}{f} \quad (0.490, \text{ \&c.})$$

$$s = \frac{d - \sqrt{\frac{d^2}{g^2} - 1}}{g} \quad (0.531, \text{ \&c.})$$

- Therefore, if 1 be put for r , and there be given (as before in the first, second and third steps,) $4=d$, $\sqrt{10}=f$, and $\sqrt{11}=g$, then by Canon 1. s may be any number less than $2\frac{1}{10}$, but greater than $1\frac{1}{10}$, and consequently, if $1=r$, and $s=2$, (which is within the last mentioned limits of s), then the sides of the two Squares sought (being expounded according to the two Quantities in the fifth step of the Resolution of this Quest. 3.) shall be $1\frac{1}{2}$ and $1\frac{1}{2}$, viz.

$$\frac{16}{5} = \frac{2rsd}{ss+rr}; \text{ and, } \frac{12}{5} = \frac{ssd-rrd}{ss+rr}.$$

Therefore the two Squares sought are $1\frac{1}{2}$ and $1\frac{1}{2}$, whose sum is 3 ; that is, 16 g and one of those Squares, to wit, $1\frac{1}{2}$, or $10\frac{1}{2}$, is greater than 10, but less than 11, as was required.

Again, if $1=r$, $4=d$, $\sqrt{10}=f$, $\sqrt{11}=g$, (as before,) then by Canon 2. s may be any Fraction greater than $1\frac{1}{10}$, but less than $2\frac{1}{10}$; and consequently, if $1=r$, and $s=\frac{1}{2}$, (which is within the last mentioned limits of s), then the sides of the two Squares sought will be found the same as before, viz.

$$\frac{16}{5} = \frac{2rsd}{rr+ss}; \text{ and, } \frac{12}{5} = \frac{rrd-ssd}{rr+ss}.$$

Again, if $1=r$, and $s=\frac{1}{2}$, (which is also within the limits of s discovered by Canon 2.) then the sides of the two Squares sought will be found the same, to wit, $1\frac{1}{2}$ and $1\frac{1}{2}$, whose

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whose Squares are $\frac{16}{17} \times \frac{22}{23}$ and $\frac{11}{12} \times \frac{13}{14}$, which added together make 16; and the fifth of those Squares is greater than 16, but less than 17; as was required.

Note, That the manner of searching our limits in this and divers following Questions, is agreeable to the method of resolving Quadratic Equations in *Sett.* 5, 7, 9. *Chap.* 15. *Book.* 1.

QUEST. 4.

(This is the fifth of the fourth Book of Vieta's *Zeteticæ*; 'tis also resolved by Bachet in his Comment upon the twelfth of the fifth Book of Diophantus, but I shall use their ways of Resolution, and deduce one from Canon 1. in the fifteenth step of Resolution of the preceding second Question of this Book.)

To divide a given number which is compos'd of two Squares, into two other Squares, that one of the Squares sought may consist within given limits.

Preparation.

Because the following Resolution of this Question presupposeth each of the prescribed limits to be greater than the lesser of the two Squares given, I shall in the first place shew how from the given limits, when they are not qualified as aforesaid, to infer others, each of which shall be greater than the lesser of the two Squares given, and then the following Resolution will solve the Question propos'd according to any possible limits whatever.

Case 1. When the lesser of the given limits is equal to the lesser of the given Squares.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 4, but less than 5: Here instead of 4 the lesser limit, (which is equal to the lesser Square given,) we may take $4\frac{1}{2}$ or any number between 4 and 5, (5 being the greater limit given;) then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose sum shall be 13; and one of them shall be greater than 4, but less than 5, and consequently greater than 4, but less than 5; as was required.

Case 2. When the lesser limit is less than the lesser Square given, but the greater limit exceeds the same, viz. When the lesser Square given falls between the given limits.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 5: Here instead of 3 we may take $4\frac{1}{2}$, or any number between 4 the lesser Square given, and 5 the greater limit: then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose sum shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 3, but less than 5; as was required.

Case 3. When the greater of the two limits given is equal to the lesser of the two Squares given.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 4: First, subtract 3 and 4 severally from 13, so each of the Remainders 10 and 9 is greater than 4 the lesser Square given, and therefore by the following Resolution two Squares may be found out whose sum shall be 13; and one of them less than 10, but greater than 9, and consequently the other Square shall be greater than 3; but less than 4; as was required.

Case 4. When each of the two limits given is less than the lesser Square given.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 1, but less than 2: First, subtract the said limits 1 and 2 severally from 13 the number given to be divided, so each of the Remainders 12 and 11 is greater than 4 the lesser Square given, and therefore by the following Resolution two Squares may be found out whose sum shall be 13; and one of them less than 12, but greater than 11, and consequently the other Square shall be greater than 1, but less than 2; as was required.

Now let it be desired to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 6, but less than 7.

RESOL.

Quest. 4.

Diophantus's Algebra explain'd.

RESOLUTION.

- For 2, the side or square Root of 4 the lesser of the two Squares given, put $\frac{1}{2}$
- For 3, the side of 9 the greater Square given, put $\frac{3}{2}$
- For $\frac{1}{2}$, that is, the square Root of the lesser of the two limits given, put $\frac{1}{2}$
- For $\frac{3}{2}$, that is, the square Root of the greater of the two limits, put $\frac{3}{2}$
- Let two unequal numbers be represented by $\frac{1}{2}$ and $\frac{3}{2}$
- Now if $\frac{1}{2}$ be greater than $\frac{3}{2}$, and be to $\frac{3}{2}$ in any Reason (or Proportion) except that which $\frac{1}{2} + \frac{3}{2}$ hath to $\frac{1}{2} - \frac{3}{2}$, then by Canon 1. in the fifteenth step of Resolution of the preceding Quest. 2. the sides of two Squares different from dd and bb , but such, whose sum is equal to $bb + dd$, shall be equal to these two following Quantities, (which are exact also in the thirteenth and fourteenth steps of the said Resolution. 2. Quest. 2.) viz.

$$\frac{2rsd + ssb - rrb}{ss + rr} \quad \text{and} \quad \frac{ssd - rrd + 2rsb}{ss + rr}$$

- But because it's desired that one of those two Quantities or sides last above express, suppose, $\frac{2rsd + ssb - rrb}{ss + rr}$ may be greater than $\frac{1}{2}$, but less than $\frac{3}{2}$, the two unequal numbers $\frac{1}{2}$ and $\frac{3}{2}$ must be chosen so as that they may cause the said side to agree with the said limits. To which end, first, a number at pleasure may be taken for one of the said numbers $\frac{1}{2}$ and $\frac{3}{2}$, as $1 = \frac{1}{2}$, and then to search out limits for the chusing of $\frac{3}{2}$, I proceed in this manner, viz. I put a instead of $\frac{1}{2}$ while it is unknown, and then since $1 = \frac{1}{2}$, the before-mentioned side $\frac{2rsd + ssb - rrb}{ss + rr}$ will stand thus, $\frac{2da + baa - b}{da + 1}$, where the number a only is unknown: Now,

- Let it be supposed that $\frac{2da + baa - b}{da + 1} = f$
- Let it also be supposed that $\frac{2da - baa - b}{da + 1} = g$
- Then by multiplying each part in the eighth step by $da + 1$, it follows that $2da + baa - b = faa + f$
- And by adding b to each part in the tenth step, $2da + baa = faa + f + b$
- And by subtracting baa from each part in the eleventh step, it follows that $2da = faa + f + b$
- By supposition in the first and third steps, (agreeable to the Preparation to the Resolution of this Question,) f is greater than b , suppose $f = b$ therefore
- Then from the twelfth and thirteenth steps it follows, that $2da = caa + f + b$
- And by subtracting $f + b$ from each part in the fourteenth step, it's manifest that $2da - f - b = caa$
- And by dividing each part of the fifteenth step by c , $\frac{2da}{c} - \frac{f + b}{c} = aa$
- And by subtracting $\frac{2d}{c}$ from each part of the sixteenth step, $\frac{f + b}{c} = aa - \frac{2d}{c}$
- And by adding the Square of half the Co-efficient $\frac{2d}{c}$ to each part of the seventeenth step, it follows that $\frac{dd - fc - bc}{cc} = aa - \frac{2d}{c}$
- And by extracting the square Root out of each part of the eighteenth step, $\sqrt{\frac{dd - fc - bc}{cc}} = a - \frac{d}{c}$
- And by adding $\frac{d}{c}$ to each part of the nineteenth step, it follows that $\frac{d}{c} + \sqrt{\frac{dd - fc - bc}{cc}} = a$
- Wherefore by comparing the quantity a in the latter part of the twentieth step to the sum of the quantities in the first part, it is found that

22. Again]

22. Again, because $\frac{d}{c} - a$ (as well as $a - \frac{d}{c}$)
in the latter part of the nineteenth step,) may
be the Square Root of the Square which is the
latter part of the eighteenth step, it follows, that
23. And by adding a to each part of the twenty-
second step, $\sqrt{\frac{dd-fc-bc}{cc}} \mp \frac{d}{c} = a$
24. Wherefore by subtracting $\sqrt{\frac{dd-fc-bc}{cc}}$
from each part of the twenty-third step, it's
evident that $a \mp \frac{d}{c} = \sqrt{\frac{dd-fc-bc}{cc}}$
25. Again, by supposition in the ninth step, $\frac{2da+baa-b}{aa+1} \rightarrow g$
26. Whence by multiplying each part by $aa+1$, $2da+baa-b \rightarrow gaa+g$
27. And by subtracting baa from each part of
the twenty-sixth step, $2da-b \rightarrow gaa-baa+g$
28. And by adding b to each part of the twenty-
seventh step, $2da \rightarrow gaa-baa+g+b$
29. By supposition in the first and fourth steps,
(agreeable to the Preparation to the Resolu-
tion of this Question,) g is greater than b ,
suppose therefore, $n = g - b$
30. Then from the twenty-eighth and twenty-
ninth steps it follows, that $2da \rightarrow naa+g+b$
31. Whence, by arguing in like manner as before
from the fourteenth step to the twenty-first,
it will appear that $a \mp \frac{d}{n} + \sqrt{\frac{dd-gn-bn}{nn}} = a$
32. Again, by arguing in like manner as before
from the twenty-second step to the twenty-
fourth, it will be evident that $a \mp \frac{d}{n} - \sqrt{\frac{dd-gn-bn}{nn}} = a$
33. Then out of the twenty-fifth and thirty-first steps, after a, c and n are exchanged
for $s, f-b$ and $g-b$, for these are equal to those, as appears by the Positions
in the seventh, thirteenth and twenty-ninth steps, the following Canon 1. ariseth for
limiting the number s , when $r=1$; viz.

CANON 1.

$$s \rightarrow \frac{d + \sqrt{dd+bb-ff}}{f-b} \quad (12.562, \&c.)$$

$$s \rightarrow \frac{d + \sqrt{dd+bb-eg}}{g-b} \quad (8.439, \&c.)$$

Again, out of the twenty-fourth and thirty-second steps, after a, c and n are exchanged
for $s, f-b$ and $g-b$, (as before,) another Canon ariseth for limiting the number s ,
when $r=1$; viz.

CANON 2.

$$s \rightarrow \frac{d - \sqrt{dd+bb-ff}}{f-b} \quad (0.788, \&c.)$$

$$s \rightarrow \frac{d - \sqrt{dd+bb-eg}}{g-b} \quad (0.852, \&c.)$$

Therefore if 1 be taken for the value of r , and there be given, $2=b$, $3=d$,
 $\sqrt{6}=f$, and $\sqrt{7}=g$, (as before in the first, second, third and fourth steps,) then
by Canon 1. above-express'd, s may be any number between $12\frac{562}{1000}$ and $8\frac{439}{1000}$; and
consequently, if $r=1$ and $s=9$, (which value of s is within the limits last before-
mentioned, then the sides of the two Squares fought (being expounded according to the
two Quantities in the sixth step of the Resolution of this Quest. 4.) shall be $\frac{121}{100}$ and
 $\frac{121}{100}$, viz.

275d+

$$\frac{275d+15b-rrb}{ss+rr} = \frac{107}{41}, \text{ and } \frac{ssd-rrd+275b}{ss+rr} = \frac{102}{41}.$$

Therefore the two Squares fought are $\frac{1144}{1681}$ and $\frac{121}{1681}$, whose sum is $\frac{1265}{1681}$,
that is, 13 ; and the first of those Squares is greater than 6 , but less than 7 , as was
required.

Again, if $1=r$; $2=b$; $3=d$; $\sqrt{6}=f$; $\sqrt{7}=g$; (as before,) then by Canon 2.
 s may be any Fraction between $\frac{1265}{1681}$ and $\frac{1265}{1681}$; and consequently, if $1=r$ and $\frac{121}{100}=s$,
(which value of s is within the last mentioned limits,) then the sides of the two Squares
fought will be found $\frac{1144}{1681}$ and $\frac{121}{1681}$, viz.

$$\frac{275d+15b-rrb}{ss+rr} = \frac{1391}{533}, \text{ and } \frac{ssd-rrd+275b}{ss+rr} = \frac{1226}{533}.$$

Therefore the two Squares fought are $\frac{1144}{284089}$ and $\frac{121}{284089}$, whose sum is $\frac{1265}{284089}$,
viz. 13 ; and the first of those Squares is greater than 6 , but less than 7 , as was required.

Again, if $1=r$, and $s=\frac{1}{2}$, (which value of s is also within the limits discovered
by Canon 2.) then the sides of the two Squares fought being expounded as before, will be
found $\frac{121}{4}$ and $\frac{121}{4}$, which are the same with those before-found in the Example of Canon 1.

Note. If 1 be put equal to r , and the number s be taken by Canon 2. then because
in this case s is less than r , the Algebraical Rules of $+$ and $-$ in adding, subtracting, &c.
must be observed to resolve the aforesaid literal values of the sides of the Squares fought into
numbers, as in the two last Examples.

QUEST. 5. (Quest. 11. Lib. 2. Diophant.)

To find two square numbers whose difference shall be equal to a given number, suppose
60, (or d)

RESOLUTION.

- To the given difference 60, that is, d
- Let some number whose Square is less than the given dif- b
- ference be represented by a
- For the side of the lesser Square fought put a
- And for the side of the greater Square fought put $a+b$
- Therefore the lesser Square is aa
- And the greater Square is $aa+2ba+bb$
- And the difference of those Squares is $2ba+bb$
- But the said difference must be equal to the given difference d , $2ba+bb=d$
- therefore $n = \frac{d-bb}{2b}$
- Which Equation, after due Reduction, makes known the
value of the side of the lesser Square, viz. $a+b = \frac{d+bb}{2b}$
- And from the ninth and fourth steps, the value of the side
of the greater Square is also discovered, viz. $a+b = \frac{d+bb}{2b}$
- The two last steps give the following

CANON 1.

Take any square number less than the given difference, and subtract it from the said
difference; then divide the Remainder by the double of the side of the Square first taken;
and the Quotient shall be the side of the lesser of the two Squares fought; lastly, this side
added to the side of the Square first taken, gives the side of the other Square fought.

So if two Squares be desired whose difference shall be 60, I take a square number less
than 60, as 36, this subtracted from that leaves 24, which divided by 12 the double
of the Square Root of 36, gives the Quotient 2, which shall be the side of the lesser Square
fought; and then by adding 6 the Square Root of the said 36, to the side 2, the sum 8
is the side of the greater Square fought; lastly, the Squares of the said sides 2 and 8,
to wit, 4 and 64 will solve the Question, for their difference is 60; as was required.

Observations upon Quest. 5.

- It is evident by the two last steps of the preceding Reso- $\frac{d-bb}{2b}$ and $\frac{d+bb}{2b}$
lution, that the values of the sides of the two Squares fought are $\frac{d-bb}{2b}$ and $\frac{d+bb}{2b}$
Now

Now if we suppose $bc = d$, then those sides will be converted? $\frac{bc - 1 - bb}{2b}$ and $\frac{bc - bb}{2b}$ into these, viz.

Which last mentioned sides or Quotients, after the common? $\frac{1}{2}c + \frac{1}{2}b$ and $\frac{1}{2}c - \frac{1}{2}b$ Factor b is cast away, will be reduced to these, to wit,

Hence ariseth this elegant Canon, often used by *Diophantus* to find out two Squares in a given difference, viz.

CANON 2.

Take two such unequal numbers that the Product of their multiplication may be equal to the given difference; then half the sum and half the difference of those two numbers shall be the sides of the two Squares sought.

As, for example, if two Squares be desired whose difference shall be 60, I take two such numbers (10 and 6) which being mutually multiplied make 60; then half the sum of 10 and 6 and half their difference are 8 and 2 the sides of the two Squares sought, and consequently the Squares themselves are 64 and 4, whose difference is 60; as was required.

Again, instead of 10 and 6 taken as before, we may take 30 and 2, for the Product of these is equal to the given difference 60; then half the sum of 30 and 2, and half their difference, give 16 and 14, whose Squares 256 and 196 have 60 for their difference, as was required.

After the same manner, Fractions being admitted, innumerable pairs of Squares may be found out, such, that the difference of each pair shall be equal to one and the same number given: For if the given number be divided by a number taken at pleasure, half the sum, and half the difference of the Divisor and Quotient shall be the sides of two Squares whose difference is equal to the given number.

2. But for farther illustration of the truth of the preceding Canon 2. let c and b represent two unequal numbers, and suppose c to be the greater; then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb$

The Square of $\frac{1}{2}c - \frac{1}{2}b$ is $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb$

The difference of those Squares is cb

Whence it is manifest, That the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of half the sum of the said two numbers; and the lesser is the Square of half their difference. Wherefore the truth of the foregoing Canon 2. doth evidently appear.

3. *Vinea* useth the following Canon (which differs but little from the preceding Canon 2.) to find out two Squares in a given difference.

CANON 3.

Take two such unequal numbers, that the Product of their multiplication may be equal to a quarter of the given difference of two Squares sought; then the sum and difference of those two numbers first taken shall be the sides of the desired Squares.

As, for example, if it be desired to find out two Squares whose difference shall be 60; first, I take $\frac{1}{4}$ of the said 60, to wit, 15; then I chuse two such unequal numbers that the Product of their multiplication may make 15, as 5 and 3; lastly, the sum, and difference of 5 and 3, give 8 and 2 for the sides of two Squares whose difference is 60.

The truth of this Canon 3. may be demonstrated thus; let c and b represent two unequal numbers, and suppose c to be the greater; then

The Square of $c + b$ is $cc + 2cb + bb$

The Square of $c - b$ is $cc - 2cb + bb$

The difference of those Squares is $4cb$

Whence it is manifest, That the quadruple of the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of the sum of those numbers, and the lesser Square is the Square of the difference of the same two numbers. Wherefore the truth of Canon 3. is evident.

4. If a Square be equal to two Squares, then (by *prop. 47. Elem. 1. Euclid.*) the sides of those three Squares will constitute a right-angled Triangle, viz. the greatest side shall be the Hypotenuse, and the other two the sides about the right-angle; whence it follows, that the Square of one of the sides about the right-angle is equal to the difference of the Squares of the other two sides: And therefore if any Rational number be given for

for one of the sides about the right-angle of a right-angled Triangle, the other side about the right-angle and the Hypotenuse shall be given also in Rational numbers by the help of any of the three preceding Canons: As, for example, if 4 be given for the Side, the Square thereof is 16, then by any of the said Canons find out two Squares whose difference may be 16, such are 25 and 9, (and innumerable other pairs of Squares;) therefore their Square Roots or sides, viz. 5 and 3, shall be the desired Hypotenuse and Perpendicular. Whence it is evident, that by the like Operation innumerable right-angled Triangles may be found out in Rational numbers, which shall have one common Side (or Perpendicular) prefribed.

QUEST. 6.

To find two such numbers, that the Product of their multiplication may be equal to a given number, suppose d , and that the Square of half the sum of the said numbers may be greater than a number given, suppose b .

RESOLUTION.

- For one of the numbers sought put a .
- Then by dividing the given Product d by a , the Quotient $\frac{d}{a}$ shall be the other number sought, to wit,
- Therefore half the sum of those two numbers is $\frac{aa + d}{2a}$
- Now suppose it be desired that the Square of the said half sum may be greater than the given number b , then it necessarily follows, that the sum itself must be greater than the Square Root of b , viz.
- Therefore from the fourth step, by multiplying each part by $2a$, it follows, that $aa + d > 2a\sqrt{b}$
- That is, (because $a\sqrt{4b} = 2a\sqrt{b}$), $aa + d > a\sqrt{4b}$
- And by subtracting d from each part of the sixth step, it follows that $aa > a\sqrt{4b} - d$
- And by subtracting $a\sqrt{4b}$ from each part of the seventh step, $aa - a\sqrt{4b} < -d$
- And by adding a quarter of the Square of the known Coefficient $\sqrt{4b}$, to wit, b , to each part of the eighth step, it follows that $aa - a\sqrt{4b} + b < b - d$
- And by extracting the square Root out of each part of the ninth step, $a - \sqrt{b} < \sqrt{b - d}$
- Wherefore by adding \sqrt{b} to each part of the tenth step, $a < \sqrt{b} + \sqrt{b - d}$
- Again, because $-a + \sqrt{b}$ (as well as $a - \sqrt{b}$) may be the side of the Square $aa - a\sqrt{4b} + b$ in the first part of the ninth step, it thence follows that $-a + \sqrt{b} < \sqrt{b - d}$
- And by adding a to each part of the twelfth step, $\sqrt{b} < a + \sqrt{b - d}$
- And by subtracting $\sqrt{b} - d$ from each part of the thirteenth step, $\sqrt{b} - \sqrt{b - d} < a$
- Wherefore from the fourteenth step, by comparing the latter part to the former, $a > \sqrt{b} - \sqrt{b - d}$
- The eleventh and fifteenth steps give limits for the choice of (a) one of the two numbers sought by this sixth Question, when it requires that the Square of half the sum of the same numbers may be greater than a given number; and the premises afford this following

CANON 1.

For one of the numbers sought take any number greater than $\sqrt{b} + \sqrt{b - d}$; or less than $\sqrt{b} - \sqrt{b - d}$; then divide d the given Product of the multiplication of the two numbers sought, by the number first taken, so shall the Quotient be the other number sought.

An Example in Numbers.

Suppose $d = 128$, and $b = 192$;

Thence it follows, that $\sqrt{b} + \sqrt{b - d} = 21,856, &c.$

Also, $\sqrt{b} - \sqrt{b - d} = 5,856, &c.$

D

Therefore

Therefore according to the direction of the preceding Canon, I take for one of the two numbers sought some number greater than $21\frac{1}{2}$, or less than $57\frac{1}{2}$, as the number 2, by this I divide the given number $128 = d$, and the Quotient gives 64 for the other number sought; which two numbers, 2 and 64, will solve the Question, as will be evident by

The Proof.

The Product of the multiplication of 2 and 64 makes the given number 128 (or d), and the Square of half the sum of 2 and 64, viz. the Square of 33 is 1089, which is greater than 192, (or b), as was required. But to the end there may be a possibility of solving the Question proposed, the Canon above-express'd shew there is a necessity that the number d must not exceed the number b .

17. The preceding Resolution of *Quest. 6.* presupposeth it to be desired that the Square of half the sum of the two numbers sought may be greater than a number given; but if it were desired that the said Square might be less than a number given, then \rightarrow being used instead of \leftarrow in the said Resolution, there would at length arise this following Canon to solve the said Question in the latter Case.

CANON 2.

For one of the numbers sought take any number less than $\sqrt{b} + \sqrt{b-d}$: but greater than $\sqrt{b} - \sqrt{b-d}$: then divide d the given Product of the multiplication of the two numbers sought, by the number first taken, and the Quotient shall be the other number sought.

An Example of this Canon.

Suppose (as before) . . . $d = 128$, and $b = 192$,
Thence it follows, that . . . $\sqrt{b} + \sqrt{b-d} = 21.856$, &c.
Also, . . . $\sqrt{b} - \sqrt{b-d} = 5.856$, &c.

Therefore (according to the latter Canon) I take for one of the two numbers sought some number between $57\frac{1}{2}$ and $21\frac{1}{2}$, as 16; by this I divide the given number 128 (or d), and the Quotient gives 8 for the other number sought; which two numbers, 16 and 8, will solve the Question when it requires that the Square of half their sum may be less than the given number 192, as may easily be proved: For the Product of the said 16 and 8 makes the given number 128, and the Square of half the sum of 16 and 8, viz. the Square of 12, is 144, which is less than the given number 192; as was required.

QUEST. 7.

To find two square numbers in a given difference, and that one of those Squares may be greater or less than a given number.

1. Let it be required to find two such square numbers, that their difference may be equal to a given number, suppose d , and that the greater Square may exceed a given number, suppose b .

RESOLUTION.

It is manifest by Canon 2. of the foregoing *Quest. 5.* That if two numbers be taken, such, that the Product of their multiplication is equal to the given difference of two Squares sought, then half the sum and half the difference of the numbers so taken shall be the sides of those Squares: Therefore if two numbers be found out, such, that their Product is equal to the given difference d , and that the Square of half the sum of the same numbers is greater than the given number b , then that Square shall be the greater of the two Squares required, and the Square of half the difference of the said numbers shall be the lesser Square required: But two such numbers may be found out by the first Canon of the preceding last Question, and consequently this seventh Question may be solved by the following

CANON 1.

Take some number greater than $\sqrt{b} + \sqrt{b-d}$ or less than $\sqrt{b} - \sqrt{b-d}$: then divide d the given difference of the two Squares sought by the number first taken, and reserve the Quotient; lastly, half the sum and half the difference of the said Quotient and number first taken shall be the sides of the two Squares sought.

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An Example in Numbers.

Suppose . . . $d = 128$, and $b = 192$,
Thence it follows that . . . $\sqrt{b} + \sqrt{b-d} = 21.856$, &c.
Also, . . . $\sqrt{b} - \sqrt{b-d} = 5.856$, &c.

Therefore according to the direction of the Canon, I take some number greater than $57\frac{1}{2}$, or less than $21\frac{1}{2}$, as 2; then by this I divide 128, (to wit, d) and the Quotient is 64; lastly, half the sum of the said 2 and 64 is 33, and half their difference is 31, which 33 and 31 are the sides of two Squares that will solve the Question proposed, as will be evident by

The Proof.

The Squares of 33 and 31 are 1089 and 961; the difference of these is equal to the given difference 128, (to wit, d), and the greater Square 1089 is greater than 192, (or b), as was required.

2. In like manner, if it were required to find out two Squares whose difference shall be equal to a given number d , and the greater Square less than a given number b ; the sides of the said Squares may be found out by this following

CANON 2.

Take some number less than $\sqrt{b} + \sqrt{b-d}$, but greater than $\sqrt{b} - \sqrt{b-d}$: then divide d the given difference of the Squares sought, by the number so taken, and reserve the Quotient; lastly, half the sum and half the difference of the said Quotient and number first taken shall be the sides of the two Squares sought.

An Example in Numbers.

Suppose . . . $d = 128$, and $b = 192$,
Thence it follows, that . . . $\sqrt{b} + \sqrt{b-d} = 21.856$, &c.
Also, . . . $\sqrt{b} - \sqrt{b-d} = 5.856$, &c.

Therefore according to the direction of the last preceding Canon, I take some number between $57\frac{1}{2}$ and $21\frac{1}{2}$, as 16; by this I divide the given number 128, (to wit, d), and the Quotient is 8; lastly, half the sum of the said 16 and 8 is 12, but half their difference is 4; which 12 and 4 are the sides of two Squares that will solve the Question, as will be evident by

The Proof.

The Squares of 12 and 4 are 144 and 16; the difference of these is equal to the given difference 128, (or d), and the greater Square 144 is less than 192, (or b), as was required.

3. But if it were required to find out two Squares in a given difference d , and that the lesser Square might be greater than a given number g ; they may be discovered by the help of the preceding Canons of this seventh Question, in this manner, viz.

Let it be required to find two Squares whose difference shall be 24 (or d), and that the lesser Square may be greater than 12, (or g). Here the scope must be to find out two such Squares that their difference may be 24, and that the greater Square may exceed 36, that is, $24 + 12$, and then the lesser Square will consequently exceed 12. Therefore

Suppose . . . $d = 24$, and $g = 12$,
Suppose also . . . $b = 36 = d + g$,
Thence it follows, that . . . $\sqrt{b} + \sqrt{b-d} = 9.464$, &c.
Also, . . . $\sqrt{b} - \sqrt{b-d} = 2.533$, &c.

Then (according to the first Canon of this seventh Question) I take some number greater than $97\frac{1}{2}$, or less than $27\frac{1}{2}$, as 2; by this I divide 24, (to wit, d), and the Quotient is 12; then half the sum of 2 and 12 is 7, and half their difference is 5; which 7 and 5 are the sides of two Squares 49 and 25; whose difference is 24, (to wit, d); and the lesser Square 25 is greater than 12, (or g), as was required. But for the greater evidence, let ff be put for the lesser Square found out, and hh for the greater; then:

By Construction, . . . $hh = d + ff$,
Also by Construction, . . . $hh = d + g (= b)$,
Therefore . . . $ff = g$. Which was to be proved.
D 2 4. Lastly,

4. Lastly, if it were desired to find out two Squares in a given difference d , and that the lesser Square might be less than a given number g , let the sum of those two given numbers, (to wit, $d + g$) be called b (as before,) and then by the latter of the two preceding Canons of this seventh Question find out two Squares that their difference may be equal to the given difference d ; and that the greater Square may be less than the sum b , so shall the lesser Square be less than the given number g .

QUEST. 8.

[This is the twelfth of the second Book of Diophantus, and the seventh of the fourth Book of Vietæ's Zeteticæ.]

Two numbers being given, suppose 192 and 128, to find a third, which added to each of those given may make each sum to be a Square.

RESOLUTION 1.

For the number sought put a , no part of it. Then the Question requires that each of these two sums, $192 + a = \square$, may be a square number, viz. $128 + a = \square$.

3. Now that Duplicate Equality (for so Diophantus calls it) may be resolved thus, viz. First, subtract $128 + a$ from $192 + a$, and the Remainder 64 is the difference both of the given numbers and likewise of the two Squares sought; then (by the preceding seventh Question) find two such square numbers that their difference may be 64, and that the greater Square may exceed 192 the greater number given; such are the Squares 289 and 225, whose sides are 17 and 15.

4. Then equate $192 + a = 289$, thus, $192 + a = 289$

5. Or equate $128 + a = 225$, thus, $128 + a = 225$

6. Lastly, from either of those Equations in the fourth and fifth steps, the number a sought will be also made known, viz. $a = 97$

I say 97 will solve the Question; for if it be added to 192 and 128 severally, the sums 289 and 225 are Squares, as was required: And out of the premises well examined, respect also being had to the preceding seventh Question, there will arise this following Canon to find out innumerable Answers to the Question proposed.

CANON.

7. Take any number greater than the sum, or less than the difference of the square Roots of the two numbers given; divide the difference of the two numbers given, by the number first taken; and reserve the Quotient; then from the Square of half the sum of this Quotient and the number taken; subtract the greater of the two numbers given, or, from the Square of half the difference between the said Quotient and the number first taken, subtract the lesser number given; so shall either of the Remainders (for they are equal to one another) be the numbers sought.

An Example of the Canon.

8. Let there be two numbers given to find a third, according

9. Their difference is

10. Also

11. And

12. Let a number be taken, either greater than 12553, or

13. Divide 88 in the ninth step, by 4 in the twelfth, and the

14. Quotient is

15. The half of the sum of 4 and 22 in the twelfth and

16. Thirteenth steps is

17. The Square of the said 13 is

18. From that Square subtract the greater of the two numbers

19. given in the eighth step, to wit

20. So the Remainder is the number sought, to wit

21. The Proof.

22. $(\sqrt{96} + 73) = 169$, whose $\sqrt{}$ is 13.

23. $(\sqrt{88} + 73) = 81$, whose $\sqrt{}$ is 9.

In like manner, if it were desired to find out some number signified by (a) that $104 + 54$ may make a Square, also that $104 + 6$ may make a Square, the preceding Canon will give innumerable values of a , among which 1 will be found a true value, for if $a = 1$, then $104 + 54 = 64$, and $104 + 6 = 16$.

Observations upon Quest. 8.

1. Diophantus in resolving this Question makes an entrance into one of his peculiar subtilties; which he calls a Duplicate Equality, (an ingenious Invention variously used by him; as divers knotty Questions in this Book will make manifest;) the principal hinge whereof depends on the finding of two Squares whose difference shall be equal to the difference of two Algebraical Quantities, each of which is proposed to be found equal to some known square number: As, in the preceding Resolution of this Quest. 8. two Squares, to wit, 289 and 225 are found out; whose difference 64 is equal to the difference of the two Algebraical Quantities $192 + a$ and $128 + a$, each of which; according to the import of the Question, is to be found equal to some square number; and therefore the number a sought must be such as will cause that effect.

2. But that the reason of the Operation in resolving the Duplicate Equality in this eighth Question may clearly appear, two things are to be proved, viz.

First, That the greater of the two square numbers found out in the third step of the Resolution must necessarily exceed the greater of the two numbers given in the Question, and the lesser Square exceed the lesser number given. To prove this, first, it is intended that the desired number a should be affirmative, that is, greater than nothing; but if any square number not greater than 192 were set in the place of 289 in the Equation in the fourth step, viz. $192 + a = 289$, or any square number not greater than 128, in the place of 225 in the Equation in the fifth step, viz. $128 + a = 225$, the value of a would be less than nothing; and therefore the necessity of the before-mentioned qualification of the said Squares is apparent.

Secondly, That when two square numbers are found out, such, that their difference is equal to the difference of the two numbers given, and that the greater Square exceeds the greater of those given numbers, then consequently the lesser Square shall exceed the lesser number, and these two last mentioned excesses or differences shall be equal to one another. The truth of this consequence will be evidently by the following

THEOREM.

If two square numbers, suppose dd the greater and ff the lesser, have the same difference as two other numbers, suppose b the greater and c the lesser, and that the greater Square dd exceeds the greater number b , then the lesser Square ff shall exceed the lesser number c , and the former excess $dd - b$ shall be equal to the latter $ff - c$. For,

By supposition

Therefore by adding ff to each part, it follows, that $dd - ff = b - c$

But by supposition

Wherefore, by subtracting b from each part of the Equation in the last step, but one, that which the Theorem affirms

is manifest, viz. $dd - b = ff - c$

And consequently either of those equal excesses or differences (in the last Equation) shall be a number to solve the Question proposed; for it is manifest that if the former excess $dd - b$ be added to b , and the latter excess $ff - c$ to c , the sums will be Squares, to wit, dd and ff .

Another manner of resolving the foregoing Quest. 8. which is here repeated, viz.

To find a number which added severally to 128 and 192, may make the sums to be Squares.

RESOLUTION 2.

1. For the number sought put $aa - 128$ whereby part of the Question is satisfied; for if $aa - 128$ be added to 128, it makes a Square, to wit, aa ; let therefore the number sought be feigned to be

2. But

2. But the Question requires also, that if the number sought be added to 192, the sum may be a Square; add therefore $aa - 128$ to 192, so the sum $aa - 128 + 192$, that is, $aa + 64$ must be equal to a Square number, viz.
3. It remains therefore to equate the said $aa + 64$ to a Square, whose side (to the end the value of aa may be greater than 128, as the number $aa - 128$ assumed in the first step requires) may be feigned to be $a +$ any Absolute number greater than $2\sqrt{128}$, or $a +$ any Absolute number greater than $25\sqrt{2}$, (which limits are discovered by the following third way of resolving this Question;) let therefore the said side be feigned to be $a + 2$, and then the Square of this side being equated to $aa + 64$ as the second step requires, this Equation aritheth, viz.

$$aa + 4a + 4 = aa + 64.$$

4. Which Equation, after due Reduction, makes known the value of a , viz. $a = 15$

Therefore by the first and fourth steps the number sought will be found 97, which will solve the Question; for if 97 be added to 128 and 192 severally, the sums are Squares, to wit, 225 and 289; and the limits in the third step for feigning the side of one of the Squares sought do shew that the Question is capable of innumerable Answers.

A third manner of resolving the preceding Quest. 8. which is here repeated, viz.

To find a number, which added first to a given number (f) and then to a greater given number (b), may make the sums to be Squares.

RESOLUTION 3.

1. For the difference of the two given numbers b and f put d , viz. $d = b - f$
suppose
2. And for the number sought put $aa - f$, (a representing a number unknown,) whereby the first part of the Question is satisfied; for if $aa - f$ be added to f it makes a Square, to wit, aa ; let therefore the number sought be feigned to be $aa - f$
3. But the Question requires also, that if the number sought be added to b it may make a Square; add therefore $aa - f$ to b , and it makes $aa + b - f$, that is, $aa + d$, (for d was put equal to $b - f$), $aa + d = \square$
which must be equal to a Square, viz.
4. It remains then to equate $aa + d$ to some Square, whose side may be feigned to be either $a + e$, or $a + u$, (which numbers, a , e and u are all yet unknown;) First, let the side of the said Square be feigned to be $a + e$, so its Square being equated to $aa + d$ this Equation aritheth, viz.

$$aa + 2ae + ee = aa + d.$$

5. Which Equation, after it is duly reduced to find what a is equal to, gives $a = \frac{d - ee}{2e}$
6. But by the second step $a = \sqrt{f}$
7. Therefore by the fifth and sixth steps $\frac{d - ee}{2e} = \sqrt{f}$
8. And by multiplying each part of the seventh step by $2e$, it follows, that $d - ee = 2e\sqrt{f}$, or $e\sqrt{4f}$
9. And by adding ee to each part of the eighth step, it gives $d = ee + e\sqrt{4f}$
10. And by adding f , that is, $\frac{1}{4}$ of the Square of the known Coefficient $\sqrt{4f}$ in the ninth step, to each part thereof, it follows, that $d + f = ee + e\sqrt{4f} + f$
11. And by extracting the Square Root out of each part of the tenth step, $\sqrt{d + f} = e + \sqrt{f}$
12. And by subtracting \sqrt{f} from each part of the eleventh step, $\sqrt{d + f} - \sqrt{f} = e$
13. And by comparing the latter part of the twelfth step to the first part, it's manifest that $e = \sqrt{d + f} - \sqrt{f}$
14. But by the first step, $b = d + f$
15. And consequently $\sqrt{b} = \sqrt{d + f}$

16. Where.

16. Wherefore, by setting \sqrt{b} in the place of $\sqrt{d + f}$, $e = \sqrt{b} - \sqrt{f}$
in the thirteenth step, it's evident that $ee = d$
17. And because by the fifth step, $e = \sqrt{d}$
18. Therefore
19. Again, forasmuch as the side of the Square mentioned in the fourth step may be feigned to be $a + u$, let the Square of $a + u$ be equated to $aa + d$, as the third step requires, so this Equation aritheth, viz.

$$aa + d = aa + 2au + uu.$$

10. Which Equation, after due Reduction, to find out what a is equal to, gives $a = \frac{uu - d}{2u}$
11. But by the second step, $a = \sqrt{f}$
12. Therefore from the twentyeth and twenty-first steps, $\frac{uu - d}{2u} = \sqrt{f}$
13. And by arguing to find out limits for u , in like manner as before for e from the seventh to the sixteenth step inclusive, it will at length appear, that $u = \sqrt{b} + \sqrt{f}$
14. And because by the twentyeth step $uu = d$
15. Therefore $u = \sqrt{d}$
16. Now suppose $192 = b$
128 = f
 $64 = b - f = d$
17. And consequently
18. And from the sixteenth and twenty-sixth steps it follows that $e = 2\sqrt{128}$, &c. ($\sqrt{b} - \sqrt{f}$)
19. And from the twenty-third and twenty-sixth steps, $u = 25\sqrt{2}$, &c. ($\sqrt{b} + \sqrt{f}$)
20. Likewise from the eighteenth and twenty-sixth steps, $e = 8$ (\sqrt{d})
21. And from the twenty-fifth, and twenty-seventh steps $u = 8$

But of the limits found out in the four last preceding steps; the two former only are necessary for choosing the numbers e and u , for the two latter not being so strict as the two former are useless. Then after the number e or u is duly chosen according to the said limits in the twenty-eighth and twenty-ninth steps, the number a will be discovered either by the fifth, or by the twentieth step; and lastly, the number sought by the second and twenty-sixth steps. All which will be farther illustrated by the following Canon and Examples.

CANON.

32. Take any number less than the difference, or greater than the sum of the square Roots of the two numbers given in the Question: then divide the difference between the Square of the number first taken, and the difference of the two numbers given, by the double of the number taken; and from the Square of the Quotient subtract the lesser of the two numbers given, so the Remainder shall be the number sought.

Example 1.

Let there be two numbers given to find a third, according to Quest. 8. as, $b = 192$, and $f = 128$

Thence it follows, that $b - f = 64 = d$

Also, $\sqrt{b} - \sqrt{f} = 2.542$, &c.

And $\sqrt{b} + \sqrt{f} = 25.170$, &c.

Now according to the Canon, take some number less than $2\sqrt{128}$, and call it e , as $e = 1$

Thence it follows, that $\frac{d - ee}{2e} = 31\frac{1}{2} = u$

And lastly, $aa - f = 864\frac{1}{4}$ sought.

The Proof.

$192 + 864\frac{1}{4} = 1056\frac{1}{4}$, whose $\sqrt{\quad}$ extracted is $32\frac{1}{2}$.

$128 + 864\frac{1}{4} = 992\frac{1}{4}$, whose $\sqrt{\quad}$ extracted is $31\frac{1}{2}$.

Example 4.

Example 2.

Again, the same things being given as in Example 1. take $u = 32$
 some number greater than $257\frac{1}{2}$, and call it u , as $\frac{uu-d}{2u} = 15 = e$
 Thence it follows that $aa - f = 97$ sought.

The Proof.

$$192 + 97 = 289, \text{ whose } \sqrt{\text{is}} 17.$$

$$128 + 97 = 225, \text{ whose } \sqrt{\text{is}} 15.$$

Note. In divers of Diophantus's Questions, where Algebraick Quantities are to be equated to Squares, there is great use of finding out Limits, (after the manner delivered in the last preceding Resolution,) to direct how to feign the sides of the said Squares, so, as that their values in numbers may be greater than nothing; and therefore for the more ample Illustration of that Method I have framed the five Questions next following, in the Resolutions whereof, the industrious Learner will meet with no difficulty, if he be well exercis'd in the manner of resolving Quadratick Equations according to Sect. 5, 7, 9. Chap. 15. Book 1. as also in the Observations upon the first Question of this third Book.

QUEST. 9.

To find a number, call it a , that shall be less than 3, and cause $aa + 12$ to be a square number.

RESOLUTION.

- Put Letters for the given numbers, viz. $d = 12$
 $f = 3$
- Then the Question requires that $aa + d$ may be equal to a Square, but its Side (or Root) must be so feigned that the value of a may be less than f , and greater than nothing, to which end the said Side may be feigned to be either $a + e$, or $-a + n$; (which a , e and n do represent numbers yet unknown;) First therefore supposing the said Side to be $a + e$, the Square thereof is $aa + 2ae + ee$, which must be equated to $aa + d$ above-mentioned; hence this following Equation ariseth, viz.
 $aa + 2ae + ee = aa + d$
- Which Equation, after due Reduction to find out the value of a , gives $a = \frac{d - ee}{2e}$
- And since the Question requires that a may be less than 3, viz. $a < f$
- Therefore from the third and fourth steps $\frac{d - ee}{2e} < f$
- And by multiplying each part of the fifth step by $2e$, it follows that $d - ee < 2fe$
- And by adding ee to each part of the sixth step, $d < ee + 2fe$
- And by adding the Square of half the Coefficient $2f$ to each part of the seventh step, it gives $d + ff < ee + 4fe + ff$
- And by extracting the square Root out of each part of the eighth step, $\sqrt{d + ff} < e + f$
- And by subtracting f from each part of the ninth step, $\sqrt{d + ff} - f < e$
- Wherefore from the tenth step, by comparing the latter part to the first, $e < \sqrt{d + ff} - f$
- And since by the third step $ee < d$, therefore, $e < \sqrt{d}$
- Again, for as much as the Side of the Square mentioned in the second step may be feigned to be $-a + n$, let the Square of $-a + n$ be equated to $aa + d$ as the Question requires, so this Equation ariseth, viz.
 $aa + d = aa - 2an + nn$

- Which Equation, after due Reduction to find out the value of a , gives $a = \frac{nn - d}{2n}$

15. And

- And because the Question requires a to be less than 3, viz. $a < f$
 - Therefore from the fourteenth and fifteenth steps it follows that $\frac{nn - d}{2n} < f$
 - And by continuing the Process to find out Limits for n , in like manner as before for e from the fifth step to the eleventh inclusive, it will at length appear that $n < \sqrt{d + ff} + f$
 - And since by the sixteenth step, $nn < d$, therefore $n < \sqrt{d}$
- From the first, eleventh, twelfth, seventeenth, eighteenth, third, and fourteenth steps the following Canon is deduced, by the help whereof innumerable Answers may be found out to the Question propos'd.

CANON.

- Take any number (e) between $\sqrt{d + ff} - f$ and \sqrt{d} , that is, between $17\frac{1}{2}$ &c. and $37\frac{1}{2}$ &c. Or any number (n) between \sqrt{d} and $\sqrt{d + ff} + f$, that is, between $37\frac{1}{2}$ &c. and $77\frac{1}{2}$ &c. Then divide the difference between the Square of the number taken and d , or 12 , by the double of the number taken, so the Quotient shall be the number a sought.

Example 1.

Let there be two numbers given in such manner as before $d = 12$, and $f = 3$
 is supposed in this Quest. 9. viz. Then according to the first limits in the Canon take some number between $17\frac{1}{2}$ and $37\frac{1}{2}$ as 2, and call this e , viz. suppose $e = 2$
 And then by the latter part of the Canon, $\frac{d - ee}{2e} = 2 = a$ sought.
 Which number 2, to wit, a , will solve the Question, for it is less than 3; and $aa + 12$, that is, 16, is a Square, as was required.

Example 2.

Again, the same things being given as in Example 1. take some number between $17\frac{1}{2}$ and $37\frac{1}{2}$, (according to the latter limits in the Canon,) as 4, and call this n , viz. suppose $n = 4$
 And then you will find $\frac{nn - d}{2n} = \frac{16 - 12}{8} = \frac{4}{8} = \frac{1}{2} = a$
 Which Fraction $\frac{1}{2}$, to wit, a , will solve the Question, for it is less than 3; and $aa + 12$, that is, $12\frac{1}{4}$, is a Square as was required.

QUEST. 10.

To find out a number, call it a , that $aa - 60$ may be greater than 54, but less than 84.

RESOLUTION.

- Put Letters for the given numbers, as $b = 60$
 $c = 54$
 $d = 84$
- Then the Question requires that $aa - b$ may be greater than ca , yet less than da ; first then let us suppose $aa - b < ca$
- Thence it follows, by adding b to each part, that $aa < ca + b$
- And by subtracting ca from each part in the third step $aa - ca < b$
- And by adding the Square of half the known Coefficient c to each part of the fourth step, it follows that $aa - ca + \frac{1}{4}cc < b + \frac{1}{4}cc$
- And by extracting the square Root out of each part of the fifth step, it gives $a - \frac{1}{2}c < \sqrt{b + \frac{1}{4}cc}$
- Wherefore by adding $\frac{1}{2}c$ to each part of the sixth step, $a < \frac{1}{2}c + \sqrt{b + \frac{1}{4}cc}$
- Again, let us suppose, as the Question also requires, that $aa - b > da$
- Whence by arguing in like manner as before from the second step to the seventh inclusive; saving that instead of $<$ there, $>$ is to be used in this latter argumentation, it will at length appear that $a > \frac{1}{2}d + \sqrt{b + \frac{1}{4}dd}$

F

10. Thus,

10. Thus, (by the seventh and eighth steps) limits are discovered, within which any number may be taken for the value of a the number sought, viz.

$$a \sqsubset \frac{1}{2}e + \sqrt{b} + \frac{1}{2}ee : (10; \frac{6324102}{1000000}, \&c.)$$

$$a \sqsupset \frac{1}{2}d + \sqrt{b} + \frac{1}{2}dd : (12; \frac{7177979}{1000000}, \&c.)$$

As, for example, if $a = 12$, which is within the said Limits, then $aa - 60 = 84$; also $5a = 60$, and $8a = 96$: But 84 (that is, $aa - 60$) is greater than 60 , (that is, $5a$), and less than 96 , (that is, $8a$;) and therefore the number 12 , (that is, a ;) doth manifestly solve the Question proposed.

QUEST. 11.

To find out a number, call it a , that shall be greater than $10; \frac{6324102}{1000000}$, but less than $12; \frac{7177979}{1000000}$, and cause $aa - 60$ to be equal to some square number.

RESOLUTION.

- Put Letters for the given numbers, as, $\begin{cases} b = 60, \\ f = 10; \frac{6324102}{1000000}, \\ d = 12; \frac{7177979}{1000000}. \end{cases}$
- Then, (according to the import of the Question,) $aa - b$ must be equal to some Square, but the side thereof must be so feigned that the value of a may be greater than f , but less than d ; to which purpose, the said side may be feigned to be $a - e$, or $e - a$, (which a and e do represent numbers unknown,) and then the Square of the said $a - e$, or $e - a$ being equated to $aa - b$ above mentioned, gives this Equation, viz.

$$aa - b = aa - 2ae + ee.$$

- Which Equation, after due Reduction to find out the value of a , gives $a = \frac{ee + b}{2e}$.
- But according to the Question, $a \sqsubset \frac{f}{e}$.
- Therefore from the third and fourth steps, $\frac{ee + b}{2e} \sqsubset f$.
- And by multiplying each part of the fifth step by $2e$, it follows that $ee + b \sqsubset 2fe$.
- And by subtracting b from each part of the sixth step, $ee \sqsubset 2fe - b$.
- And by equal subtraction of $2fe$ from each part of the seventh step, $ee - 2fe \sqsubset -b$.
- And by adding ff , that is, the Square of half the known Coefficient $2f$ in the eighth step, to each part, it follows that $ee - 2fe + ff \sqsubset ff - b$.
- And by extracting the Square Root out of each part of the ninth step, $e - f \sqsubset \sqrt{ff - b}$.
- Wherefore by adding f to each part of the tenth step, it's evident that $e \sqsubset f + \sqrt{ff - b}$.
- Again, because $f - e$, (as well as $e - f$;) may be the side of the Square $ee - 2fe + ff$ in the first part of the ninth step, it thence follows, that $f - e \sqsubset \sqrt{ff - b}$.
- And by adding e to each part of the twelfth step, $f \sqsubset e + \sqrt{ff - b}$.
- And by subtracting $\sqrt{ff - b}$ from each part of the thirteenth step, $f - \sqrt{ff - b} \sqsubset e$.
- Wherefore from the fourteenth step, by comparing the latter part to the former, 'tis manifest that $e \sqsupset f - \sqrt{ff - b}$.
- Again, because the Question requires $a \sqsupset d$.
- It follows from the third and sixteenth steps, that $\frac{ee + b}{2e} \sqsupset d$.
- Whence by arguing in like manner as before from the fifth step to the fifteenth *inclusive*, saving that d is to be used here, instead of f there, and \sqsupset instead of \sqsubset , it will at length appear that $e \sqsupset d + \sqrt{dd - b}$.

From the eleventh, eighteenth, sixteenth, first and third steps the following Canon ariseth, which will find out innumerable Answers to the Question proposed.

CANON.

CANON.

19. Take any number (e) greater than $f + \sqrt{ff - b}$: but less than $d + \sqrt{dd - b}$: (that is, any number between $17; \frac{221}{100}$, &c. and $22; \frac{803}{100}$, &c.) or any number greater than $d - \sqrt{dd - b}$: but less than $f - \sqrt{ff - b}$: (that is, any number between $2; \frac{41}{100}$, &c. and $3; \frac{24}{100}$, &c.) Then $\frac{ee + b}{2e}$ shall be equal to (a) the number sought.

Examples.

First, for the number e take 22 which is within the former limits in the Canon; then $\frac{ee + b}{2e}$ gives $12; \frac{4}{10}$ for the number a sought by the Question: For if from the Square of $12; \frac{4}{10}$, to wit, $144; \frac{16}{100}$, you subtract the given number 60 , (or b ;) the Remainder $84; \frac{16}{100}$ is a Square whose side is $9; \frac{4}{10}$; and the said $12; \frac{4}{10}$ (that is, a ;) is greater than $10; \frac{6324102}{1000000}$, &c. but less than $12; \frac{7177979}{1000000}$, &c. as the Question requires.

Again, for the number e take 3 which is within the latter limits in the Canon; then $\frac{ee + b}{2e}$ gives $11; \frac{1}{2}$ (the number a ;) which will likewise solve the Question proposed: For if from the Square of $11; \frac{1}{2}$ you subtract 60 , there will remain a Square, to wit, $121; \frac{1}{4}$, whose side is $11; \frac{1}{2}$; and the said $11; \frac{1}{2}$ (that is, a) is greater than $10; \frac{6324102}{1000000}$, &c. but less than $12; \frac{7177979}{1000000}$, &c. as was required.

QUEST. 12.

To find out a number, call it a , that shall be greater than $2; \frac{1}{2}$, (a number given,) and cause $aa + 4a + 2$ to be equal to some square number.

RESOLUTION.

- Put letters for the given numbers, as, $\begin{cases} b = 4, \\ d = 2; \frac{1}{2}, \\ f = 2. \end{cases}$
- Then the Question requires that $aa + ba + f$ may make a square number, but its side must be so feigned that the value of a may be greater than d . Now to cause those effects, the said side may be feigned to be $a - e$, or $e - a$, (which e and a do represent numbers yet unknown,) and then the Square of $a - e$ or $e - a$, that is, $aa - 2ae + ee$, being equated to $aa + ba + f$, gives this Equation, viz. $aa + ba + f = aa - 2ae + ee$.
- Which Equation, after due Reduction to find out the value of a , gives $a = \frac{ee - f}{2e + b}$.
- And because the Question requires $a \sqsupset d$.
- It follows from the third and fourth steps, that $\frac{ee - f}{2e + b} \sqsupset d$.
- And by multiplying each part of the fifth step, by $2e + b$, $ee - f \sqsupset 2de + db$.
- And by adding f to each part of the sixth step, $ee \sqsupset 2de + db + f$.
- And by subtracting $2de$ from each part of the seventh step, $ee - 2de \sqsupset db + f$.
- And by adding the Square of half the known Coefficient $2d$ in the eighth step, to each part, it's manifest that $ee - 2de + dd \sqsupset db + f + dd$.
- And by extracting the Square Root out of each part of the ninth step, $e - d \sqsupset \sqrt{db + f + dd}$.
- Wherefore by adding d to each part of the tenth step, it's evident that $e \sqsupset \sqrt{db + f + dd} + d$.
- And consequently by resolving the latter part of the eleventh step into numbers, according to the Positions in the first, $e \sqsupset 6; \frac{33}{100}$, &c.
- The third step also shews that $ee \sqsupset f$, and consequently $e \sqsupset \sqrt{f}$, ($1; \frac{41}{100}$, &c.)

But this latter limit for the chusing of e is useless, for if e be greater than $6\frac{7}{10}$, &c. as appears by the twelfth step, it is evidently greater than $1\frac{1}{10}$, &c.

14. Lastly, from the eleventh, twelfth, third and first steps the following Canon ariseth, which will find innumerable Answers to the Question proposed.

CANON 1.

Take any number greater than $\sqrt{db+fd+dd} : + d$, (viz. greater than $6\frac{7}{10}$, &c.) and call the number taken e . Then $\frac{ee-f}{2e+b}$ shall be equal to the number a sought.

15. But if it were desired to find a number a that might be less than $2\frac{1}{2}$, and greater than nothing, and make $aa+4a+2$ to be a square number, then the same Positions and Process being made as before, saving that $-$ is to be used instead of $+$ from the fourth step to the twelfth inclusive, at length there would arise this following

CANON 2.

Take any number (e) greater than \sqrt{f} , but less than $\sqrt{db+fd+dd} : + d$ (viz. any number between $1\frac{1}{10}$, &c. and $6\frac{7}{10}$, &c.) Then $\frac{ee-f}{2e+b}$ will give the number a sought.

An Example of the first Canon.

For the number e take 8 which exceeds $6\frac{7}{10}$, &c. as the first Canon doth direct; Then $\frac{ee-f}{2e+b}$ gives $3\frac{1}{2}$ for the number a sought; for 'tis greater than $2\frac{1}{2}$ (or d), and $aa+4a+2$ makes a Square, to wit, $\frac{25}{4}$, whose side is $\frac{5}{2}$, as was required.

Note, That $a+u$ might be feigned to be the side of the Square mentioned in the second step, and thence limits would be discovered to chuse the number u , by which the number a would consequently be made known; but I leave the search of these latter limits as an exercise for the Learner.

QUEST. 13.

To find out a number, call it a ; that shall be greater than 1, but less than 4, and make $121+45a-9aa$ to be a square number.

RESOLUTION.

1. First put Consonants to represent the numbers given in the Question, as, $\left\{ \begin{array}{l} b = 1 \\ d = 4 \\ f = 11 \\ ff = 121 \\ g = 45 \\ h = 9 \end{array} \right.$
2. Then the Question requires that $ff+ga-haa$ may make a square number, whose side must be feigned that the value of a may be greater than b , but less than d : To which purpose the said side may be feigned to be $f+ea$, or $f-na$; (where a, e, u do represent numbers unknown:) First then let the said side be feigned $f+ea$, and let its Square $ff+2fea+eaa$ be equated to $ff+ga-haa$ above-mentioned, so this following Equation ariseth, viz.
3. Which Equation, after due Reduction to find out the value of a , gives $a = \frac{g-2fe}{b+ee}$
4. And because the Question requires $a < b$
5. It follows from the third and fourth steps, that $\frac{g-2fe}{b+ee} < b$
6. And by multiplying each part in the fifth step, by the Denominator $b+ee$, it follows, that $g-2fe < bb+bee$
7. And by subtracting bb from each part in the sixth step, $g-2fe-bb < bee$

S. And

8. And by adding $2fe$ to each part in the seventh step $g-bb < bee+2fe$
9. And by dividing every quantity in the eighth step by b , that bee may be freed from its Coefficient b , it follows that $\frac{g-bb}{b} < ee+\frac{2fe}{b}$
10. And by adding $\frac{ff}{bb}$, that is, the Square of half the Coefficient $\frac{2f}{b}$ to each part of the ninth step, it gives $\frac{ff+bg-bbb}{bb} < ee+\frac{f}{b}$
11. And by extracting the square Root out of each part of the tenth step, $\sqrt{\frac{ff+bg-bbb}{bb}} < e+\frac{f}{b}$
12. And by subtracting $\frac{f}{b}$ from each part of the eleventh step, $\sqrt{\frac{ff+bg-bbb}{bb}} - \frac{f}{b} < e$
13. Therefore from the twelfth step, by comparing the latter part to the first, it's manifest that $e < \sqrt{\frac{ff+bg-bbb}{bb}} - \frac{f}{b}$
14. Again, because the Question requires $a < d$
15. And by the third step, $a = \frac{g-2fe}{b+ee}$
16. It follows from the fourteenth and fifteenth steps, that $\frac{g-2fe}{b+ee} < d$
17. Whence by arguing in like manner as before from the fifth step to the thirteenth inclusive, it will at length appear that $e < \sqrt{\frac{ff+dg-dbb}{dd}} - \frac{f}{d}$
18. Again, let the side of the Square mentioned in the second step be feigned to be $f-na$, and then the Square of $f-na$ being equated to $ff+ga-haa$, (as the Question requires) this following Equation ariseth, viz.
19. Which Equation gives this value of a , viz. $a = \frac{g+2fu}{b+uu}$
20. But the value of a last mentioned must (as the Question requires) be greater than b ; and less than d ; and if the Process be continued from the last preceding step, to find our limits for u in like manner as before for e from the third step to the seventeenth inclusive, it will at length appear that

$$u > \frac{f}{b} + \sqrt{\frac{ff+bg-bbb}{bb}},$$

$$u < \frac{f}{d} + \sqrt{\frac{ff+dg-dbb}{dd}}.$$

Now after any number is taken for the value of e within the limits in the thirteenth and seventeenth steps, the number a required by the Question will be discovered by the third and first steps. Or, after any number is taken for the value of u within the limits in the preceding twentieth step, the number a sought will be made known by the nineteenth and first steps. All which will be made manifest by the following Canon and Examples.

CANON.

21. Take any number less than $\sqrt{\frac{ff+bg-bbb}{bb}} - \frac{f}{b}$, but greater than $\sqrt{\frac{ff+dg-dbb}{dd}} - \frac{f}{d}$, (that is, any number between $1\frac{1}{10}$, &c. and $3\frac{1}{10}$, &c.) and call the number taken e ; then $\frac{g-2fe}{b+ee}$ shall be the number a sought. Or take any number less than $\frac{f}{b} + \sqrt{\frac{ff+bg-bbb}{bb}}$, but greater than $\frac{f}{d} + \sqrt{\frac{ff+dg-dbb}{dd}}$, (that is, any number between $23\frac{1}{10}$, &c. and $5\frac{8}{10}$, &c.) and call the number taken u ; then $\frac{g+2fu}{b+uu}$ will give the number a sought.

Examples.

Examples.

Suppose $e = 1$, which is within the first limits in the Canon, then $\frac{g-2fe}{b+ee} = \frac{14}{10} = a$ the number sought: For $\frac{14}{10}$ is greater than 1, but less than 4; and if $a = \frac{14}{10}$, then $121 - 45a - 9aa$ makes a Square, (for its side is $\frac{11}{10}$) as the Question requires.

Again, suppose $e = \frac{1}{2}$, (which is likewise within the first limits,) then $\frac{g-2fe}{b+ee} = \frac{14}{13} = a$ the number sought: For $\frac{14}{13}$ is greater than 1, but less than 4; and if $a = \frac{14}{13}$, then $121 - 45a - 9aa$ makes a Square, to wit, $\frac{11}{13}$, whose side is $\frac{11}{13}$.

Again, suppose $e = 18$, which is within the latter limits in the Canon, then $\frac{g-2fe}{b+ee} = \frac{22}{27} = a$ the number sought: For if $a = \frac{22}{27}$, then $121 - 45a - 9aa$ makes a Square, to wit, $\frac{11}{27}$, whose side is $\frac{11}{27}$; and $\frac{22}{27}$ (or a) is greater than 1, but less than 4, as the Question requires.

QUEST. 14. (Quest. 13. Lib. 2. Diophant.)

To find a number, that if it be subtracted first from 192, and then from 64, each Remainder may be a Square.

RESOLUTION.

1. For the number sought put a .
2. Which number must be such, that each of these Quantities (or Remainders) may make a Square, viz. $192 - a = \square$
 $64 - a = \square$
3. Now to resolve that Duplicate Equality, first, (by Canon 2. Quest. 7. of this Book.) find out two such square numbers that their difference may be equal to 128, that is, the difference of the two given numbers 192 and 64, or the difference between the two Algebraick Quantities $192 - a$ and $64 - a$, and that the greater Square may be less than 192, (the greater of the two numbers given in the Question;) but two such Squares are 144 and 16.
4. Then from either of these Equations, $192 - a = 144$
 $64 - a = 16$
5. One and the same value of a , that is, the number sought will be discovered, viz. $a = 48$
7. I say 48 will solve the Question, as will be evident by

The Proof.

$$\begin{array}{l} 192 - 48 = 144, \\ 64 - 48 = 16, \end{array} \} \text{ which are Squares, as was required.}$$

The premises give this following

CANON.

8. First, (by the second Canon of the seventh Question of this third Book,) find out two square numbers in the same difference with the numbers given, and that the greater Square may be less than the greater number given; whence consequently, (as will appear by the following Theorem,) the lesser Square shall be less than the lesser number given: Then from the greater number given subtract the greater Square, or from the lesser number subtract the lesser Square, so shall either of those Remainders (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by this following

THEOREM.

9. If two Square numbers have the same difference as two other numbers, and that the greater number exceeds the greater Square, then the lesser number shall exceed the lesser Square, and the excess of the greater number above the greater Square shall be equal to the excess of the lesser number above the lesser Square. To make this manifest, let dd and gg represent two square numbers, whereof dd is the greater; also b the greater, and e the lesser of two other numbers; Then, (according to the import of the Theorem.)

10. Sup-

10. Suppose $b - c = dd - gg = \square$
11. And $b \sqsubset dd$
12. Then by adding c to each part of the Equation in the tenth step, it follows that $b = c + dd - gg$
13. Therefore by subtracting dd from each part of the last Equation, (which subtraction appears to be possible by the eleventh and tenth steps,) that which the Theorem affirms is manifest, viz. $b - dd = c - gg$
14. And consequently either of those two equal Excesses or Remainders which make the last Equation, shall be a number that will solve the Question proposed; for it is manifest, that if the first Excess $b - dd$ be subtracted from b , and the latter Excess $c - gg$ from c , the Remainders will be Squares, to wit, dd and gg .

To solve the foregoing 14th Question after another manner, viz.

To find a number, that if it be subtracted first from 9, and then from 21, each Remainder may be a Square.

RESOLUTION.

1. It is evident, that if $9 - aa$ be subtracted from 9, there will remain a Square, to wit, aa , therefore for the number sought put a
2. And then if $9 - aa$ be subtracted from 21, the Remainder must likewise make a Square, therefore $aa + 12 = \square$
3. It remains to equate $aa + 12$ to some Square, whose side must be so feigned that the value of a may be less than 3, (for by the first step $aa = 9$, and consequently $a = 3$;) But to cause that effect, the said side may be feigned to be either $a + 1$ any absolute number between $1\frac{1}{3}$ and $3\frac{1}{3}$, or else $a + 2$ any absolute number between $3\frac{1}{3}$ and $7\frac{1}{3}$, (which limits are found out by the ninth Question of this third Book;) let therefore the said side be feigned $a + 2$, and then by equating the Square of $a + 2$ to $aa + 12$ before-mentioned in the second step, this Equation arith, viz. $aa + 4a + 4 = aa + 12$

4. Which Equation gives $a = 2$.
- Therefore from the fourth and first steps the number sought is 5; for if it be subtracted from 9 and 21 severally, it leaves the Squares 4 and 16.
- From this latter Resolution of Quest. 14. (respect being had to the foregoing ninth Question,) a Canon may be deduced to find out innumerable Answers to the said fourteenth Question; but I leave it to the Learners exercise.

QUEST. 15. (Quest. 14. Lib. 2. Diophant.)

To find a number, from which if 27 and 15 (two numbers given) be severally subtracted, each Remainder may be a Square.

RESOLUTION.

1. For the number sought put a
 2. Which number must be such that each of these Quantities or Remainders may make a Square, viz. $a - 27 = \square$
 $a - 15 = \square$
 3. Now to resolve that Duplicate Equality, find out (by Canon 2. of the preceding Quest. 5.) two square numbers whose difference may be 12, that is, the difference of the two given numbers 27 and 15; but here is no need of limiting either of the said Squares: Suppose then the said Squares are found 16 and 4, $a - 27 = 4$
 $a - 15 = 16$
 4. Then from either of these Equations, $a = 31$
 5. The number a sought will be discovered, viz. $a = 31$
- I say 31 will solve the Question; for if from 31 you subtract 27 and 15 severally, the Remainders are Squares, to wit, 4 and 16.
- From the premises there arith this following

CANON.

CANON.

6. First, (by the preceding fifth Question,) find two square numbers that shall have the same difference as the two numbers given; then add the lesser Square to the greater number, or the greater Square to the lesser number; so shall either of those summs (for they are equal to one another) be the number sought.

The truth of which Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose dd the greater and gg the lesser, have the same difference as two other numbers, suppose b the greater and c the lesser; then the sum of the lesser Square and the greater number shall be equal to the sum of the greater Square and the lesser number: For,

By supposition, $b - c = dd - gg$
 And by adding gg to each part, $b - c + gg = dd$

Wherefore by adding c to each part of the last Equation, $b + gg = c + dd$
 that which the Theorem affirms is manifest, viz.

8. And consequently either of those two equal summs in the last Equation shall be a number to solve the Question proposed: For it is evident, that if b be subtracted from the first sum $b + gg$, and c from the latter sum $c + dd$, the Remainders are Squares, to wit, gg and dd .

To solve the foregoing Quest. 15. after another manner, viz.

To find a number, from which if 27 and 15 be severally subtracted, each Remainder may be a Square.

RESOLUTION 2.

1. It is evident that if 27 be subtracted from $aa + 27$, the Remainder will be a Square, to wit, aa ; therefore for the number sought put $aa + 27$
 2. But then 15 being subtracted from the said $aa + 27$, the Remainder must likewise be equal to a Square, therefore $aa + 12 = \square$
 3. It remains to equate $aa + 12$ to some Square, whose side may be feigned either $a +$ any absolute number less than $\sqrt{12}$, or $3\frac{3}{4}$, &c. or else $a +$ any absolute number greater than the said $3\frac{3}{4}$, &c. (which limits are found out in like manner as in the foregoing Quest. 9.) Let therefore the said side be feigned $a + 3$, and then by equating the Square of $a + 3$, that is, $aa + 6a + 9$ to $aa + 12$, the value of a will thence be found $\frac{3}{2}$, and consequently the number sought, (which in the first step was put $aa + 27$) shall be $27\frac{9}{4}$, which will solve the Question: For if from $27\frac{9}{4}$ the given numbers 27 and 15 be severally subtracted, the Remainders will be Squares, to wit, $\frac{9}{4}$ and $\frac{25}{4}$.

But if this second manner of resolving Quest. 15. be formed by Literal Algebra, (like to the third manner of resolving the preceding Quest. 8.) there will arise this

CANON.

Take any number greater than the sum, or less than the difference of the square Roots of the two numbers given; then divide the difference between the Square of the number taken and the difference of the given numbers by the double of the number taken; lastly, to the Square of that Quotient add the greater of the numbers given, so shall the sum be the number sought.

A third way of solving the preceding Quest. 15.

Let the Positions in the first and second steps of the preceding Resoln. 2. be resumed; then since $aa + 12$ must be equal to a Square, 'tis evident that 12 is the difference between that Square and aa ; therefore by the preceding fifth Question find two Squares whose difference may be 12; such are 16 and 4, the lesser of which shall be the value of aa ; therefore $aa + 27$ which was put for the number sought will be found 31, as before in the first Resolution of this Question.

QUEST. 16.

QUEST. 16.

To find a number, that if 12 be added to it, and 8 subtracted from the same, as well the Sum as the Remainder may be a Square.

RESOLUTION.

1. For the number sought put a
 2. Then each of these Quantities must make a Square, viz. $a + 12 = \square$
 $a - 8 = \square$
 3. Now to resolve that Duplicate Equality, first, subtract $a - 8$ from $a + 12$, and the Remainder is 20; this is equal as well to the sum of the given numbers as to the difference of the two Squares sought: Then (by the second Canon of the fifth Question of this third Book) find two Squares whose difference shall be 20; such are 36 and 16.
 4. Then from either of these Equations $a + 12 = 36$
 $a - 8 = 16$
 5. The number a sought will be made known, viz. $a = 24$
 Which number found out, to wit, 24, will solve the Question; for if it be increased with 12, and lessened by 8, the Sum and Remainder are Squares, to wit, 36 and 16.
 The substance of the Resolution is contain'd in this following

CANON.

6. First, (by the second Canon of the fifth Question foregoing) find out two square numbers whose difference shall be equal to the sum of the two numbers given, then subtract the number given to be added (whether it be the greater or the lesser of those given) from the greater Square, or add the number given to be subtracted to the lesser Square; so as well the Remainder as the Sum (for they are equal to one another) shall be the number sought.

The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by this following

THEOREM.

7. If there be two square numbers, suppose dd the greater, and gg the lesser, whose difference is equal to the sum of two other numbers, b and c ; then the excess of the greater Square above either of the two numbers, shall be equal to the sum of the lesser Square and the other number.
 8. For, by supposition $dd - gg = b + c$
 9. Whence by adding gg to each part, it follows that $dd = gg + b + c$
 10. Therefore by subtracting b from each part of the last Equation, $dd - b = gg + c$
 11. Likewise, by subtracting c from each part of the Equation in the ninth step, there remains $dd - c = gg + b$
 Wherefore from the two last Equations, the truth of the Theorem, and consequently of the Canon is manifest.

QUEST. 17.

To find a number, that if it be added to, and subtracted from a given square number, suppose 4, the Sum and Remainder may be Squares.

RESOLUTION.

1. For the number sought put a
 2. Which number must be such, that if it be added to and subtracted from 4, as well the Sum as the Remainder may make a Square, viz. $4 + a = \square$
 $4 - a = \square$
 3. To resolve that Duplicate Equality, first, (after the manner of Example 3. Canon 1. Resoln. 2. Quest. 2. of this Book,) divide 8 the double of the given Square 4 into two unequal Squares, $\frac{25}{4}$ and $\frac{9}{4}$,
 4. Then

4. Then from either of these Equations, $\begin{cases} 4 + a = \frac{13}{2} \\ 4 - a = \frac{1}{2} \end{cases}$
 5. The number sought will be discovered, viz. $\begin{cases} a = \frac{3}{2} \end{cases}$

Which number $\frac{3}{2}$ will solve the Question proposed; for if it be added to, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{25}{4}$ and $\frac{7}{4}$, whose sides are $\frac{5}{2}$ and $\frac{7}{4}$. By the like Operation (the substance whereof is contained in the following Canon,) you may find out innumerable Answers to this 17th Question.

CANON.

6. Divide the double of the given Square into two unequal Squares, (by the preceding *Quest.* 2.) then from the greater of the two Squares found out subtract the given Square, or from this subtract the lesser of those two, so shall either of the Remainders, (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in this 17th Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose cc the greater, and dd the lesser, be equal to the double of a square number, as $bb + bb$, or $2bb$; then the excess of the greater of those unequal Squares above one of the equal Squares shall be equal to the excess of the other of the equal Squares above the lesser of the unequal Squares.
 8. For by supposition $cc + dd = bb + bb$
 9. Also by supposition $cc - dd = bb - bb$
 10. Therefore $cc - dd = bb - bb$
 11. And $cc - dd = bb$
 12. And by subtraction of bb from each part of the Equation $cc - dd - bb = bb - bb$ in the eighth step,
 13. Wherefore by subtracting dd from each part of the last Equation, (which subtraction the 10th and 11th steps do shew to be possible,) that which the Theorem asserts is manifest, viz. $cc - bb = bb - dd$
 14. And consequently either of the Excesses (or Remainders) which make the last preceding Equation shall be the number a sought: For if the first Excess $cc - bb$ be added to bb , and the latter Excess $bb - dd$ subtracted from bb , the Summ and Remainder are Squares, to wit, cc and dd .

QUEST. 18.

To find a number, that if a given Square 9 be added to that number, and from another given Square 4 the same number sought be subtracted, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put a
 2. Then the Question requires that $\begin{cases} 9 + a = \square \\ 4 - a = \square \end{cases}$
 3. To resolve that Duplicate Equality, first (by the preceding *Quest.* 4.) divide 13 the summ of the given Squares 9 and 4 into two such other Squares that one of these found may exceed 9 the Square given to be added, but two such Squares are $\frac{16}{4}$ and $\frac{1}{4}$, whose summ is 13, and the greater of them exceeds 9.

4. Then from either of these Equations $\begin{cases} 9 + a = \frac{16}{4} \\ 4 - a = \frac{1}{4} \end{cases}$
 5. The number (a) sought is discovered, viz. $a = \frac{3}{4}$

Which number found out, to wit, $\frac{3}{4}$, will solve the Question; for if it be added to 9, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{16}{4}$ and $\frac{1}{4}$, whose sides are $\frac{4}{2}$ and $\frac{1}{2}$. By the like Operation (the substance whereof is express'd in the following Canon,) you may find out innumerable Answers to this 18th Question, because (by *Quest.* 4. of this Book) the summ of two Squares may be divided into as many pairs of Squares as you please, such, that one of each pair shall consist within given limits.

CANON.

CANON.

7. First, (by *Quest.* 4. of this Book,) divide the summ of the two Squares given into two such Squares, that the greater of these found out may exceed the Square given to be added; then from the greater of the two Squares found out subtract the Square given to be added; or, from the other Square given subtract the other Square found out; so shall either of the Remainders (for they are equal to one another) be the number sought. The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question proposed will be manifest by this following

THEOREM.

8. If the summ of two square numbers, suppose bb the greater, and cc the lesser, be found equal to the summ of two other unequal Squares, dd and ff , and that the greater of the two former exceeds either of the two latter, then the other of the two latter shall exceed the lesser of the two former, and one excess shall be equal to the other. For,
 9. By supposition $bb + cc = dd + ff$
 10. By supposition also $bb - cc = dd - ff$
 11. Therefore $bb - cc = dd - ff$
 12. And by subtracting dd from each part of the Equation in the ninth step, this ariseth, viz. $bb - cc - dd = ff - dd$
 13. Wherefore by subtracting cc from each part of the last Equation, (which subtraction the tenth and eleventh steps do shew to be possible) that which the Theorem asserts is manifest, viz. $bb - dd = ff - cc$
 14. And consequently the truth of the Canon and Resolution of the Duplicate Equality in this 18th Question is evident.

QUEST. 19.

To find a square number, that if it be increased or lessened by its side, may make a Square.

RESOLUTION.

1. For the side of the Square sought put a
 2. Therefore the Square it self is aa
 3. Then the Question requires $\begin{cases} aa + a = \square \\ aa - a = \square \end{cases}$
 4. Which Duplicate Equality differs but little from that in the foregoing seventeenth Question, and may be resolved thus: First, (after the manner of *Example* 3. *Canon* 1. *Resolut.* 2. of this Book, divide 2, which is compos'd of two Squares, 1 and 1, suppos'd to be prefix'd to aa and aa in the Duplicate Equality in the third step. Into two unequal Squares, $\frac{4}{4}$ and $\frac{1}{4}$; then multiply each of these by aa , and equate the greater Product $\frac{4}{4}aa$ to $aa + a$, or the lesser Product $\frac{1}{4}aa$ to $aa - a$, so from either of those Equations one and the same value of a will be discovered, viz. $a = \frac{1}{2}$, which is the side of the Square sought; for if $\frac{1}{2}$ be added to and subtracted from its Square $\frac{1}{4}$, the Summ and Remainder are Squares, to wit, $\frac{9}{4}$ and $\frac{1}{4}$, whose sides are $\frac{3}{2}$ and $\frac{1}{2}$. It is also easie to be perceived that this Question is capable of innumerable Answers.

QUEST. 20.

To find two numbers in a given Reason, suppose the greater to the lesser as 2 to 1; and that the Square of the summ of the two numbers being added to each of them may make square numbers.

RESOLUTION.

1. For the lesser number sought put a
 2. Then the greater, to the end it may be to the lesser as 2 to 1, shall be $2a$
 3. Therefore the summ of the two numbers sought is $3a$
 4. And the Square of their summ is $9a^2$

5. To

5. To which Square if the two numbers $2a$ and a be severally added, each sum must be a Square, therefore $\left. \begin{array}{l} 9aa - 12a = \square \\ 9aa - a = \square \end{array} \right\}$
6. Which Duplicate Equality, (according to *Diophantus's* Method, before explain'd and demonstrated in the Observations upon the first manner of solving the eighth Question of this Book,) may be resolv'd thus; viz. First, subtract $9aa - 12a$ from $9aa - a$, and the Remainder a is the difference of the two Squares that are to be equated to those Algebraick Quantities; then find two Squares whose difference may be equal to the said difference a , but with this Caution, that in each of those Squares there may be found $9aa$, to the end that when the greater Square is equated to $9aa - 12a$, or the lesser to $9aa - a$, the said $9aa$, after due Reduction, may vanish, and an Equation remain between some number of a and some known Rational number, whence the value of a will be expressible by some known number either Affirmative or Negative. Now to find out two such Squares, let two numbers be taken, such, that (according to *Canon 2. Quest. 5.* of this Book,) the Product of their Multiplication may make a , and that the half of their sum may consist of $3a +$ some absolute number, and the half of their difference of $3a -$ some absolute number, (for then it will follow that as well in the Square of the said half sum, as in the Square of the said half difference there will be found $9aa$;) whence we may infer, that the sum of the said two numbers must consist of $a +$ some absolute number, and their difference of $6a -$ some absolute number, to which purpose, divide 1 by 6 , (1 because the difference is $1a$, and 6 because it is the double of the square Root of 9 which is prefix to aa ,) so the Quotient is $\frac{1}{6}$, and consequently $6a$ multiplied by $\frac{1}{6}$ makes a ; whence 'tis evident that the only two numbers fit for the purpose aforesaid are $6a$ and $\frac{1}{6}$, whose half sum is $3a + \frac{1}{12}$, and half difference $3a - \frac{1}{12}$; (or $3a + \frac{1}{12}$.) therefore the Squares of the said half sum and half difference are $9aa + \frac{1}{4}a + \frac{1}{144}$ and $9aa - \frac{1}{4}a + \frac{1}{144}$; now let the greater of those Squares be equated to $9aa - 12a$, and the lesser to $9aa - a$, so these two following Equations will arise,

$$\text{viz. } \left\{ \begin{array}{l} 9aa - 12a = 9aa + \frac{1}{4}a + \frac{1}{144} \\ 9aa - a = 9aa - \frac{1}{4}a + \frac{1}{144} \end{array} \right.$$

7. Then from either of those Equations, after due Reduction, the value of a will be found to be $\frac{1}{12}$, and consequently, (from the first and second steps,) the numbers sought are $\frac{1}{12}$ and $\frac{1}{6}$, which will solve the Question proposed: For first, the greater hath such proportion to the lesser as 2 to 1 ; and if the Square of the sum of $\frac{1}{12}$ and $\frac{1}{6}$ be added to them severally, the two sums will be Squares, to wit $\frac{1}{4}$ and $\frac{1}{36}$, whose sides are $\frac{1}{2}$ and $\frac{1}{6}$.

8. In like manner, to resolve this Duplicate Equality, viz.

$$\text{If } \left\{ \begin{array}{l} 4aa - 3a - 1 = \square \\ 4aa - a - 1 = \square \end{array} \right\} \text{ what is } a = ?$$

First, I find the difference of those two Algebraick Quantities to be $2a$; then I search out two Quantities that being mutually multiplied may make $4a$, and that as well in half their sum as in half their difference there may be found $2a$, (that is, the square Root of $4aa$;) so by working as before is directed, I find $4a$ and 1 to be the only two Quantities agreeing with those conditions: Then the Square of half the sum of $4a$ and 1 , viz. the Square of $2a + \frac{1}{2}$ being equated to $4aa - 3a - 1$ will give $a = \frac{1}{2}$; or, the Square of half the difference of $4a$ and 1 , viz. the Square of $2a - \frac{1}{2}$ being equated to $4aa - a - 1$, gives $a = \frac{1}{2}$, as before.

QUEST. 21.

To find two numbers in a given Reason, suppose the greater to the less as 3 to 2 , and that the sum of the numbers being added to each of their Squares, may make Squares.

RESOLUTION.

1. For the lesser number put $\dots \dots \dots 2a$
 2. Then the greater (to the end that both numbers may be in the Reason prescribed,) shall be $\dots \dots \dots 3a$
 3. Therefore their sum is $\dots \dots \dots 5a$

4. Which

4. Which sum added to the Square of each of the said two numbers $2a$ and $3a$, must (as the Question requires) make a Square, therefore $\left\{ \begin{array}{l} 4aa - 12a = \square \\ 9aa - 9a = \square \end{array} \right\}$
5. Now in order to resolve that Duplicate equality, it must first be reduced to another, wherein there may be equal square numbers prefix to aa , which may be done thus, Divide 9 the greater of the two Squares that are prefix to aa by the lesser 4 , and then by the Quotient $\frac{9}{4}$ multiply that Algebraick Quantity where the said Divisor 4 is prefix to aa , viz. $4aa - 12a$ by $\frac{9}{4}$, and it produceth $9aa - \frac{27}{2}a$ to be equated to a Square; so now instead of the Duplicate equality in the fourth step, this ariseth,

$$\text{viz. } \left\{ \begin{array}{l} 9aa - \frac{27}{2}a = \square \\ 9aa - 9a = \square \end{array} \right.$$

6. In which Duplicate equality last above express'd, there are equal Squares, to wit, 9 and 9 prefix to aa , and therefore it may be resolv'd after the manner before shewn in the 20th Question, and when the value of a is discovered in this latter duplicate equality, it will necessarily constitute the former duplicate equality in the fourth step; for as a Square multiplied by a Square produceth a Square, so conversely, a Square divided by a Square gives the Quotient a Square. In order then to resolve the said duplicate equality in the fifth step, subtract $9aa - 9a$ from $9aa - \frac{27}{2}a$, and the Remainder is $\frac{9}{2}a$, this must be esteem'd the Product made by the mutual multiplication of two quantities to be taken with such Caution, that as well in half their sum as in half their difference there may be found $3a$, (because the square Root of $9aa$ in each of the two quantities to be equated to a Square is $3a$;) so by considering well that Caution, and what hath been said to the like purpose in the sixth step of the Resolution of the foregoing 20th Question, you will find that $6a$ and $\frac{3}{2}a$ are the only two numbers, that being mutually multiplied make $\frac{9}{2}a$, and have $3a$ as well in half their sum as in half their difference: Therefore let the Square of half the sum of the said $6a$ and $\frac{3}{2}a$, viz. the Square $9aa + \frac{9}{4}a + \frac{9}{16}$ be equated to $9aa - \frac{27}{2}a$, or, let the Square of half the difference of the said $6a$ and $\frac{3}{2}a$, viz. the Square $9aa - \frac{9}{4}a + \frac{9}{16}$ be equated to $9aa - 9a$; so from either of those Equations, one and the same value of a will be discovered, viz. $a = \frac{1}{2}$, and consequently $2a$ and $3a$, which in the first and second steps were put for the numbers sought, will be discovered to be $\frac{1}{6}$ and $\frac{1}{2}$, which will solve the Question: For, first, the greater is in proportion to the lesser as 3 to 2 ; and if their sum be added to their Squares severally, the two sums made by such addition will be Squares, to wit, $\frac{1}{4}$ and $\frac{1}{36}$, whose sides are $\frac{1}{2}$ and $\frac{1}{6}$.
7. But the Duplicate equality in the fourth step may be reduced to another wherein there shall be equal square numbers prefix to aa by this following Operation, which differs from that in the sixth step, viz. because 9 times 4 makes the same Product as 4 times 9 , and because a Square multiplied by a Square produceth a Square, let $4aa - 12a$ (in the fourth step) be multiplied by 9 , and $9aa - 9a$ by 4 ; so there will necessarily be found $9aa$ in each Product, and this following duplicate equality comes now to be resolv'd instead of that in the fourth step,

$$\text{viz. } \left\{ \begin{array}{l} 36aa - 108a = \square \\ 36aa - 36a = \square \end{array} \right.$$

8. Lastly, this Duplicate equality having equal square numbers prefix to aa , may be resolv'd like that in the preceding fifth step, and at length the value of a will be found $\frac{1}{2}$, as before.

QUEST. 22.

To find two such square numbers, that if to the Product of their multiplication a given number (d) be added, the sum may be a Square.

RESOLUTION.

1. For one of the Squares sought take any known square number which may be represented by bb
 2. And for the other Square sought put aa
 3. Then the Product of their multiplication is $bbaa$
 4. To which Product the given number d being added, the sum is $bbaa + d$
 5. Which

5. Which summ must be equal to a Square, the side whereof may be feigned to be $ba -$ any known number greater than \sqrt{d} , suppose $ba - c$; then the Square of $ba - c$, that is, $bbaa - 2bca + cc$ being equated to $bbaa - d$, this Equation ariseth, viz.
- $$bbaa - d = bbaa - 2bca + cc.$$

6. Whence, after due Reduction, $\dots \dots \dots \rightarrow a = \frac{cc - d}{2bc}$

From the premises ariseth this following

CANON.

For one of the Squares sought take any square number; then from any square number subtract the given number, and divide the Remainder by the double of the Product made by the multiplication of the sides of those two Squares; so the Quotient shall be the side of the other Square sought.

An Example in Numbers.

Let the number given be $\dots \dots \dots \rightarrow 12 = d$
 For one of the Squares sought take any square number, as $\dots \dots \dots \rightarrow 4 = bb$
 Take also some other square number greater than 12, (or d), as $\dots \dots \dots \rightarrow 36 = cc$

Then (by the Canon) the side of the other Square sought shall be $\dots \dots \dots \rightarrow 1 = \frac{cc - d}{2bc}$

I say, 4 and 1 are two Squares, which will solve the Question when the number given is 12; for if to 4, the Product of 4 and 1, you add 12, the summ makes a Square, to wit, 16. By the like Operation you may find out innumerable Answers to the Question without varying the given number; and 'tis ealie also to find out other Canons to solve the same.

QUEST. 23. (Quest. 15. Lib. 2. Diophant.)

To divide a given number (b) into two parts, and to find a square number, which if it be increased with each of those parts, may make a Square.

RESOLUTION.

1. Take two such numbers, that the summ of their Squares $\dots \dots \dots \rightarrow c$ and d may be less than the given number b ; suppose these, $\dots \dots \dots \rightarrow$
2. Then for the side of the Square sought put $\dots \dots \dots \rightarrow a$
3. The Square thereof is $\dots \dots \dots \rightarrow aa$
4. To the side a add severally c and d , and assume the summs $\dots \dots \dots \rightarrow a + c$ to be the sides of two Squares, so the first side will be $\dots \dots \dots \rightarrow$
5. And the other side $\dots \dots \dots \rightarrow a + d$
6. The Square of $a + c$ (in the fourth step) is $\dots \dots \dots \rightarrow aa + 2ca + cc$
7. The Square of $a + d$ (in the fifth step) is $\dots \dots \dots \rightarrow aa + 2da + dd$
8. Then for one of the desired parts of (b) put $2ca + cc$, (for it's evident, that if the Square aa in the third step be increased with $2ca + cc$, it makes the Square $aa + 2ca + cc$ $\dots \dots \dots \rightarrow 2ca + cc$ $- cc$ in the sixth step.)
9. And for the other part of b put $2da + dd$, for this added $\dots \dots \dots \rightarrow 2da + dd$ to aa makes a Square, to wit, that in the seventh step $\dots \dots \dots \rightarrow$
10. But the summ of the parts in the eighth and ninth steps $\dots \dots \dots \rightarrow 2ca + cc + 2da + dd = b$ must be equal to b , therefore $\dots \dots \dots \rightarrow$
11. Which Equation, after due Reduction, gives $\dots \dots \dots \rightarrow a = \frac{b - cc - dd}{2c + 2d}$

The premises well examined, afford this following

CANON.

12. Take two numbers, with this Caution, that the summ of their Squares may be less than the number given to be divided; then subtract the summ of those Squares from the given number, and divide the Remainder by the double summ of the numbers taken, so the Quotient shall be the side of the Square sought; then multiply the double of the said side severally by the numbers first taken, and to the Products add severally the respective Squares of the numbers taken; so the summs made by those additions shall be the desired parts of the number given.

Am

An Example in Numbers.

Let the number given to be divided be $\dots \dots \dots \rightarrow 33 = b$
 Let two numbers be taken, such, that the summ of their Squares $\dots \dots \dots \rightarrow 2 = c$
 may be less than 33, as $\dots \dots \dots \rightarrow 3 = d$
 Then (by the Canon) the side of the Square sought, (which) $\dots \dots \dots \rightarrow 2 = \frac{b - cc - dd}{2c + 2d}$
 side is represented by a in the Resolution, shall be $\dots \dots \dots \rightarrow$
 Also, one of the desired parts of 33 is $\dots \dots \dots \rightarrow 12 = 2ca + cc$
 And the other is $\dots \dots \dots \rightarrow 21 = 2da + dd$

The Proof.

$\left\{ \begin{array}{l} 4 = aa \text{ the Square sought;} \\ 12 + 21 = 33 \text{ the number given;} \\ 4 + 12 = 16 \\ 4 + 21 = 25 \end{array} \right\}$ which are Squares; as was required.

QUEST. 24.

To find two such numbers, that their summ may make a Square: Also, that each number being added to the Square of the other number, may make a Square.

RESOLUTION.

1. For the summ of the two numbers sought assume some Square, as $\dots \dots \dots \rightarrow ee$
 2. And for the first of the two desired numbers put $\dots \dots \dots \rightarrow a$
 3. Therefore the other shall be $\dots \dots \dots \rightarrow ee - a$
 4. Which added to the Square of the first number a , makes the summ $\dots \dots \dots \rightarrow aa - a + ee$
 5. Which summ last exprest, the Question requires to be a Square, and such it will be, if we suppose $e = \frac{1}{2}$; for then the said $aa - a + ee$ (in the fourth step) will be equal to $aa - a + \frac{1}{4}$, which is the Square of $a - \frac{1}{2}$, or $\frac{1}{2} - a$; Now therefore let the Resolution be renewed thus,
 6. For the summ of the two numbers sought, (instead of ee) put $\dots \dots \dots \rightarrow \frac{1}{4}$
 7. And for the first number put (as before) $\dots \dots \dots \rightarrow a$
 8. Therefore the other shall be $\dots \dots \dots \rightarrow \frac{1}{4} - a$
 9. Which latter number added to the Square of the first, makes the summ $\dots \dots \dots \rightarrow aa - a + \frac{1}{4}$
- But the summ last exprest is manifestly a Square, whose side is $a - \frac{1}{2}$, or $\frac{1}{2} - a$; therefore two of the conditions in the Question are satisfied.
10. Again, if to the Square of the second number $\frac{1}{4} - a$, viz. to $aa - \frac{1}{2}a + \frac{1}{16}$, the first number a be added, the summ is $aa + \frac{1}{2}a + \frac{1}{16}$, which the Question likewise requires to be a Square, and so it is, for 'tis the Square of $a + \frac{1}{4}$; but if the last mentioned summ had not happened to have been a Square, then a Square might have been feigned equal to it, according to the method in divers preceding Questions of this Book 3.

The premises discover this following

THEOREM.

11. If the Fraction $\frac{1}{4}$ be divided into any two parts, each part increased with the Square of the other part shall make a Square.
- By the help therefore of this Theorem, innumerable Answers to the Question proposed may be found out.

An Example.

Let two Fractions be taken whose summ makes $\frac{1}{4}$, as $\frac{1}{8}$ and $\frac{1}{8}$; I say these will solve the Question: For first, their summ is a Square; secondly, the first Fraction $\frac{1}{8}$ increased with $\frac{1}{64}$, (the Square of the latter Fraction $\frac{1}{8}$) makes the Square $\frac{9}{64}$; likewise the latter Fraction $\frac{1}{8}$ increased with $\frac{1}{64}$, the Square of the first Fraction $\frac{1}{8}$ makes a Square, to wit, $\frac{9}{64}$.

Moreover, by the help of the said Theorem, this following Question may be solved, viz.

12. To find two numbers in a given Reason, suppose the greater to the lesser as 3 to 2; and that each number being added to the Square of the other number, may make a Square.

Divide the Fraction $\frac{1}{4}$ into two such parts, that the greater may be to the lesser as 3 to 2, so you will find $\frac{9}{16}$ and $\frac{1}{16}$, which will solve the Question: For first,

by

by Construction they are in the Reason prescribed; secondly, $\frac{3}{2}a$ with $\frac{1}{2}a$ (the Square of $\frac{1}{2}a$) makes the Square $\frac{1}{4}a^2$; and lastly, $\frac{3}{2}a$ with $\frac{1}{2}a$ (the Square of $\frac{1}{2}a$) makes the Square $\frac{1}{4}a^2$.

13. Another manner of solving the Question last proposed may be this, viz.

For the two numbers sought in the given Reason of 3 to 2, put $3a$ and $2a$. Then, since the Question requires, that each number being added to the Square of the other number may make a Square, this Duplicate equality aritheth, viz.

$$\begin{array}{r} 9aa + 2a = \square \\ 4aa + 3a = \square \end{array}$$

Which Duplicate equality may be resolved like that in the foregoing twenty-first Question; so the value of a will be found $\frac{1}{10}a$, and consequently $3a$ and $2a$ give $\frac{3}{10}a$ and $\frac{2}{10}a$ for the numbers sought: For first, they are in proportion as 3 to 2; secondly, if to the Square of the first you add the second number, it makes the Square $\frac{1}{10}a^2$, whose side is $\frac{1}{10}a$; lastly, if to the Square of the second number you add the first, it makes the Square $\frac{1}{10}a^2$, whose side is $\frac{1}{10}a$.

QUEST. 25. (Quest. 29. Lib. 2. Diophant.)

To find two such square numbers, that each being increased with the Product of their multiplication may make a Square.

RESOLUTION.

1. First, (by the preceding Quest. 5.) find out two Squares that may differ by unity, such are $\frac{1}{4}a^2$ and $\frac{1}{16}a^2$, and take the lesser for one of the Squares sought, as
2. Then for the other Square sought assume
3. Therefore the Product of their multiplication is

Which Product $\frac{1}{16}a^2$ being added to the second Square aa doth manifestly make a Square, to wit, $\frac{1}{16}a^2 + aa$; but if the said Product $\frac{1}{16}a^2$ be added to the first Square $\frac{1}{4}a^2$ it must also make a Square, therefore $\frac{1}{16}a^2 + \frac{1}{4}a^2$ not being a Square, must be equated to a Square, the side whereof may be variously feigned, let it be $\frac{1}{2}a - 1$; then the Square of $\frac{1}{2}a - 1$ being equated to $\frac{1}{16}a^2 + \frac{1}{4}a^2$ gives $a = \frac{1}{2}a$, and consequently $aa = \frac{1}{4}a^2$ is the second Square sought. I say, $\frac{1}{4}a^2$ and $\frac{1}{16}a^2$ are two Squares, which will solve the Question, as will appear by

The Proof.

The two Squares found out are $\frac{1}{4}a^2$ and $\frac{1}{16}a^2$
 The Product of their multiplication is $\frac{1}{16}a^2$
 Which Product added severally to $\frac{1}{4}a^2$ and $\frac{1}{16}a^2$, makes these $\frac{1}{4}a^2$ and $\frac{1}{16}a^2$
 Squares, the sides of which last mentioned Squares are $\frac{1}{2}a$ and $\frac{1}{4}a$

After the same manner you may easily find out two such Squares, that if the Product of their multiplication be subtracted from them severally, the Remainders may be Squares.

QUEST. 26. (Quest. 35. Lib. 2. Diophant.)

To find three such numbers, that the Square of every one of them being added to the sum of the three numbers, may make a Square.

RESOLUTION.

1. If to the Product of the multiplication of any two unequal numbers the Square of half their difference be added, the sum shall be a Square, to wit, the Square of half the sum of the two numbers multiplied: Therefore by the help of this Theorem numbers proper for the Resolution of the Question proposed may be taken, viz. Take some number at pleasure, as 12, and divide it thrice into two such numbers that the Product of the two numbers of each pair may make 12, such are these three pairs of numbers, viz. 1, 12; 2, 6; 3, 4; then take the half-difference of the two numbers of each pair, so you will find the three half-differences to be these, $\frac{11}{2}$, 2 and $\frac{1}{2}$. Now by the Theorem above-mentioned, if to the Product 12 the Squares of the said three half-differences be severally added, every one of the sums will be a Square; therefore

2. For

2. For the sum of the three numbers sought put $12aa$
3. And for the first number $\frac{1}{2}a$
4. And for the second number $2a$
5. And for the third number $\frac{1}{4}a$
6. Therefore the sum of those three numbers is $8a = 12aa$
7. Which sum must be equal to $12aa$ in the second step, therefore $a = \frac{1}{2}a$
8. Which Equation, after due Reduction, gives $a = \frac{1}{2}a$
9. Therefore from the eighth, third, fourth and fifth steps the three numbers sought are these, viz. $\frac{1}{2}a$, $2a$, $\frac{1}{4}a$
10. Which three numbers will solve the Question, for if their Squares be severally added to their sum, the three sums will be Squares, to wit, $\frac{1}{4}a^2$, $4a^2$ and $\frac{1}{16}a^2$. It is also evident, that the Question may be extended to four, five, or as many numbers as shall be desired, by this following

CANON.

11. Take some number at pleasure, and divide it into two numbers as many times as there be numbers desired by the Question, but so, as that the Product of the two numbers of each pair may make the number first taken; then divide the sum of the half-differences of the two numbers of each pair by the number taken, and multiply the Quotient by the said half-differences severally; so the Products shall be the numbers sought.

The certainty of this Canon depends upon the Theorem assumed in the Resolution, which Theorem may be demonstrated thus;

Let two numbers be assumed, as, a the greater, e the lesser.
 The Product of their multiplication is ae
 The half-difference of the said two numbers is $\frac{1}{2}a - \frac{1}{2}e$
 The Square of the said half-difference is $\frac{1}{4}aa - \frac{1}{2}ae + \frac{1}{4}ee$
 To which Square if ae the Product above-mentioned be added, $\frac{1}{4}aa - \frac{1}{2}ae + \frac{1}{4}ee + ae$ the sum is $\frac{1}{4}aa + \frac{1}{4}ee$

Which sum is the Square of $\frac{1}{2}a + \frac{1}{2}e$, to wit, half the sum of the two numbers a and e first taken, as the Theorem affirmed.

QUEST. 27. (pars Quest. 9. Lib. 3. Diophant.)

To find three Squares in Arithmetical proportion, and such, that the half of their sum may exceed the greatest of the three Squares.

RESOLUTION.

1. For the least of the three Squares sought put aa
2. And to the end the mean Square may exceed the least, let the side of the mean Square be $a + 1$, therefore the mean Square is $aa + 2a + 1$
3. Therefore the excess of the mean Square above the least is $2a + 1$
4. And by adding the said excess to the mean Square, the sum must be equal to the greatest, which sum is $aa + 4a + 2$
5. Therefore $aa + 4a + 2$ must be equated to some Square, with this condition, that $\frac{1}{2}aa + 3a + \frac{1}{2}$, which is the half-sum of the three Squares in the first, second and fourth steps, must (according to the Question) exceed the greatest of the said three Squares; supposing therefore $\frac{1}{2}aa + 3a + \frac{1}{2} = aa + 4a + 2$, it will thence follow (by arguing after the manner of searching out limits in divers of the foregoing Questions) that $a \leq 1 - \frac{1}{2} \sqrt{2}$, that is, $a \leq 2.414$, &c.

Therefore $aa + 4a + 2$ (in the fourth step) must be so equated to a Square, that the value of a may be greater than 2.414 , &c. Now to cause that effect, innumerable values of a may be found out by the first Canon of the twelfth Question of this Book. Suppose therefore a be found equal to $\frac{1}{2}a$, (as in the Example of that Canon) then from the first, second and fourth steps, these three Squares will be discovered; to wit, $\frac{1}{4}a^2$, $4a^2$ and $\frac{1}{16}a^2$, which will solve the Question.

And because a Square multiplied by a Square produces a Square, by multiplying severally the said three Squares found out by the common Denominator 100, the Products

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961;

961, 1681 and 2401 will be Squares, whose sides are 31, 41 and 49; also the difference of the first and second is equal to the difference of the second and third, (each difference being 720;) therefore the said three Squares are in Arithmetical proportion, and the half of their sum is manifestly greater than every one of them: Therefore all the conditions in the Question are satisfied.

QUEST. 28. (Quest. 9. Lib. 3. Diophant.)

To find three numbers in Arithmetical proportion, and such, that the sum of every two of them may make a Square.

RESOLUTION.

1. By the preceding twenty-seventh Question find three Squares in Arithmetical proportion, and such, that the half of their sum may exceed every one of them, (the reason of which condition will be evident by the following eighth, ninth and tenth steps,) such are these three Squares, whose sides are 31, 41 and 49, 961, 1681, 2401
2. For the three numbers sought put a, b, c
3. Then equate the sum of the first and second numbers to the least of the three Squares before found, viz. suppose $a + b = 961$
4. Likewise equate the sum of the first and third numbers to the mean Square, and it makes $a + c = 1681$
5. Equate also the sum of the second and third numbers to the greatest Square, and it gives $b + c = 2401$
6. The sum of the three last Equations is $2a + 2b + 2c = 5043$
7. The half of the said sum is $a + b + c = 2521\frac{1}{2}$
8. Then by subtracting the third Equation from the seventh, there remains $a = 120\frac{1}{2}$
9. And by subtracting the fourth Equation from the seventh, there remains $b = 840\frac{1}{2}$
10. And by subtracting the fifth Equation from the seventh, there remains $c = 1560\frac{1}{2}$

I say these three numbers, $120\frac{1}{2}$, $840\frac{1}{2}$, $1560\frac{1}{2}$ will solve the Question; for the difference between the first and second, to wit, 720, is equal to the difference between the second and third; therefore they are in Arithmetical proportion, and the sum of every two of them makes a Square,

$$\text{viz. } \begin{cases} 120\frac{1}{2} + 840\frac{1}{2} = 961, & \text{whose } \sqrt{\text{is}} \text{ } 31; \\ 120\frac{1}{2} + 1560\frac{1}{2} = 1681, & \text{whose } \sqrt{\text{is}} \text{ } 41; \\ 840\frac{1}{2} + 1560\frac{1}{2} = 2401, & \text{whose } \sqrt{\text{is}} \text{ } 49. \end{cases}$$

QUEST. 29. (Quest. 12. Lib. 3. Diophant.)

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three sums shall be Squares.

This Question may be resolved divers ways; I shall here shew three of my own, and if the curious Reader desire to see more variety, he may consult Vietæ's Zetæ. 7. lib. 5. also Bachet's Comment, and the Solution of Simplicius in pag. 177. of his Methodum, printed in 1668.

RESOLUTION 1.

1. First, by Quest. 22. of this Book, I seek two such Squares, that if to the Product of their multiplication the given number 12 be added, the sum shall be a Square, such are these Squares 1 and 4, which may be taken for two of the three numbers sought, 1 and 4
2. Then for the third number sought I put a
3. Now (according to the Question) the Product of the multiplication of the first and third numbers being increased with 12 must make a Square; also, the Product of the

the second and third numbers together with 12 must make a Square; it remains therefore to resolve this Duplicate equality,

$$\text{viz. } \begin{cases} 1a + 12 = \square \\ 4a + 12 = \square \end{cases}$$

4. But before the said Duplicate equality can be resolved, it must be reduced to another that shall have equal numbers of a , to which purpose I multiply the first of the two quantities to be equated, to wit, $1a + 12$, by 4, (which is prefix to the latter of those two quantities) and it produceth $4a + 48$ to be equated to a Square, so this Duplicate equality ariseth, (instead of the former,)

$$\text{viz. } \begin{cases} 4a + 48 = \square \\ 4a + 12 = \square \end{cases}$$

5. Now to resolve this latter Duplicate equality, (and consequently the former,) I proceed according to the first manner of solving the preceding Quest. 8. viz. First, the difference between $4a + 48$ and $4a + 12$ is 36, then I seek two such Squares that their difference may be 36, and that the greater of them may exceed 48, but two such Squares are 100 and 64, (found out by Canon 1. of the preceding Quest. 7.) $\begin{cases} 4a + 48 = 100 \\ 4a + 12 = 64 \end{cases}$

6. Then from either of these Equations, $a = 13$
 7. The third number sought is discovered, viz. 13
- I say, the numbers 1, 4 and 13 will solve the Question proposed; for if 12 be added to the Product of the first and second, likewise to the Product of the first and third; and lastly to the Product of the second and third, the three sums will be Squares, to wit; 16, 25 and 64. The premises shew how to solve the Question by innumerable Answers.

Another way of resolving Quest. 29. which is here repeated, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three sums shall be Squares.

RESOLUTION 2.

1. For the given number 12 put b
2. For the three numbers sought put a, c, d
3. Then supposing b to be the difference of two Squares, search out those Squares (by the fifth Question of this Book) and let cc represent the greater, and xx the lesser, therefore $cc - b = xx$
4. Now the Question requires, that if b be added to the Product of the multiplication of a and c (the first and second numbers sought) the sum must be a Square, therefore by supposing that Square to be cc above found, this Equation ariseth, viz. $ac + b = cc$
5. Therefore from the last Equation by equal subtraction of b , $ac = cc - b$
6. And because by Construction in the third step $xx = cc - b$
7. Therefore from the fifth and sixth steps $ac = xx$
8. Therefore by dividing each part of the last Equation by a , it gives $c = \frac{xx}{a}$
9. Again, (by the fifth Question) find two other Squares whose difference shall be equal to b , suppose dd the greater Square, and zz the lesser, therefore $dd - b = zz$
10. Then the Product of the multiplication of the first and third numbers sought, with b added, makes $ad + b$, which must be equal to some Square, let it be dd before found, therefore $ad + b = dd$
11. Therefore by equal subtraction of b from the last Equation, $ad = dd - b$
12. And because by Construction in the ninth step $zz = dd - b$
13. Therefore from the two last Equations $ad = zz$
14. Therefore by dividing each part of the last Equation by a , it gives $d = \frac{zz}{a}$
15. Now since by the second, eighth, sixth, fourteenth and eleventh steps, the positions for the three numbers sought are $a, \frac{xx}{a}, \frac{zz}{a}$, (or $\frac{cc-b}{a}, \frac{xx}{a}, \frac{dd-b}{a}$) it is evident,

evident, that if b be added to the Product of the multiplication of the first and second numbers, the sum is a Square, to wit, cc . Likewise, if b be added to the Product of the first and third numbers the sum is a Square, to wit, dd ; but if b be added to the Product of the second and third numbers the sum must also be a Square; therefore $\frac{xxz}{aa} + b$ must be equal to a Square whose side we may feign to be $\frac{xz}{a} + z$,

or $\frac{xz}{a} - z$, and consequently a will be found equal to $\frac{2xz}{b \pm z}$; But x, z, b and z are known numbers, therefore a the first number sought is known also; and from the eighth and fourteenth steps the second and third numbers will be discovered.

From this second Resolution of *Quest. 29*, it will not be difficult to deduce the following

CANON.

First, supposing the given number (b) to be the difference of two Squares, find out (by the second Canon of the foregoing fifth Question) two pair of Squares in that difference, and let the side of the lesser Square of the one pair be called (x), and the side of the lesser Square of the other pair, (z); then take some square number whose side may be called (t), and let the difference between (t and b) be called (g); then divide the double of the solid Product of the three sides x, z, t , viz. $2xzt$, by (g), and the Quotient shall be one of the three numbers sought; lastly, multiply severally the sides x and z by g , and divide the first Product by $2xt$, and the latter by $2xt$, so the Quotients shall be the two other numbers sought. Compare this Canon with the two following Examples.

$b = 12$	12	$\frac{1}{2}$	given in the Question.
$x = 2$	2		
$z = \frac{1}{2}$	$\frac{1}{2}$		
$t = \frac{1}{2}$	$\frac{1}{2}$		
$g = \frac{1}{2}$	$\frac{1}{2}$		
$\frac{2xzt}{g} = \frac{1}{2}$	$\frac{1}{2}$		
$\frac{gx}{2xt} = 5$	5		
$\frac{gz}{2xt} = \frac{1}{2}$	$\frac{1}{2}$		
		$\frac{1}{2}$	found out by the Canon to solve the Question.

I say the number given being 12 , the Question may be solved by these three numbers $\frac{1}{2}, 5, \frac{1}{2}$; likewise by these, $\frac{1}{2}, \frac{2}{3}, \frac{1}{3}$, as may easily be proved.

A third way of resolving *Quest. 29*, which is here repeated, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12 , be added, the three sums shall be Squares.

RESOLUTION 3.

- For the given number 12 put b .
- For the three numbers sought put a, e, n .
- Then, according to the Question, $ae + b$ must be equal to a Square, let it be some known Square cc , therefore $ae + b = cc$.
- Therefore from the last Equation by equal subtraction of b , $ae = cc - b$.
- And by dividing each part of the last Equation by a , it gives $e = \frac{cc - b}{a}$.
- Thus, the first of the three numbers sought being a , and the second $\frac{cc - b}{a}$, it is evident, that if b be added to the Product of their multiplication, the sum is a Square, to wit, cc .
- Again, according to the conditions in the Question, $an + b$ must be equal to a Square, let it be some known Square dd , therefore $an + b = dd$.
- Therefore by equal subtraction of b , $an = dd - b$.

g. And

9. And by dividing each part of the last Equation by a , $\dots \therefore n = \frac{dd - b}{a}$.
10. Thus, the first of the three numbers sought being a , and the third $\frac{dd - b}{a}$, it is evident, that if b be added to the Product of their multiplication the sum is a Square; to wit, dd . But the Question requires also, that if to the Product of the second and third numbers the given number b be added, the sum must be a Square, therefore let the second and third numbers, to wit, $\frac{cc - b}{a}$ and $\frac{dd - b}{a}$ be mutually multiplied, and to the Product add b , so the sum will be $\frac{baa + ddc + bb - bdd - bcc}{aa}$, which

must be equal to a Square, the side whereof may be feigned to be either $\frac{dc + b}{a}$ or $\frac{dc - b}{a}$; first let the side be $\frac{dc + b}{a}$, then its Square being equated to the sum above-mentioned, after due Reduction of that Equation it will appear that $a = c + d$.

11. Therefore, the Equation last express'd, to wit, $a = c + d$ being compared with the Quantities in the second, fifth and ninth steps, the three numbers sought, to wit, a, e, n will be found equal to these known Quantities, viz.

$$c + d, \quad \frac{cc - b}{c + d}, \quad \frac{dd - b}{c + d}.$$

12. Again, for as much as the side of the Square to be equated to $\frac{baa + ddc + bb - bdd - bcc}{aa}$ (the sum above-mentioned in the tenth step) may be feigned to be $\frac{dc + b}{a}$, (as well as $\frac{dc - b}{a}$), let the Square of $\frac{dc + b}{a}$ be equated to the said sum, then by proceeding as before, three other numbers capable of solving the Question will be found equal to these, viz.

$$c - d, \quad \frac{cc - b}{c - d}, \quad \frac{dd - b}{c - d}.$$

From the premises two excellent Canons are deducible to solve the foregoing *Quest. 29*:

CANON 1.

13. Subtract the given number from two Squares severally, then divide each of the Remainders by the sum of the sides of the same Squares, so shall the two Quotients and the said sum of the sides be three numbers which will solve the Question.

For example, let the given number be 12 , subtract it from the Squares 36 and 64 , the Remainders are 24 and 52 , which being severally divided by 14 , (the sum of 6 and 8 , which are the sides of the said Squares 36 and 64), the Quotients $\frac{1}{2}$ and $3\frac{1}{2}$, with the said 14 , are three numbers to solve the Question, as will be evident by

The Proof.

$$\left. \begin{array}{l} 14 \times \frac{1}{2} + 12 = 36 \\ 14 \times 3\frac{1}{2} + 12 = 64 \\ 1\frac{1}{2} \times 3\frac{1}{2} + 12 = \frac{25}{2} \end{array} \right\} \text{Which are Squares, as was required.}$$

CANON 2.

14. Subtract the given number from two Squares severally, then divide each of the Remainders by the difference of the sides of the same Squares, so shall the two Quotients and the said difference be three numbers, which will solve the Question.

For example, let the given number be 12 , subtract it from the Squares 36 and 64 , and divide each of the Remainders 24 and 52 , by 2 , (which is the difference of 6 and 8 , the sides of the said Squares 36 and 64), so the Quotients 12 and 26 , with the said 2 , are three numbers that will solve the Question, as will appear by

The Proof.

$$\left. \begin{array}{l} 2 \times 12 + 12 = 36 \\ 2 \times 26 + 12 = 64 \\ 12 \times 26 + 12 = 324 \end{array} \right\} \text{Which are Squares, as was required.}$$

Bachet

Bachet in his Comment upon *Quest. 12. Lib. 3. Dioph.* (which is the same with the preceding 29th Question) delivers two Canons, one of which is the same with *Canon 2.* above express'd, and the other is this following

CANON 3.

15. Subtract the given number from two Squares severally, divide the Remainders severally by the difference of the sides of the same Squares; then shall the two Quotients and their double sum lessened by the aforesaid difference be three numbers which will solve the Question propounded.

For example, let the given number be 12, subtract it from two Squares, suppose 36 and 64, the Remainders 24 and 52 being divided severally by 2, (the difference of the sides of the said Squares 36 and 64) give the Quotients 12 and 26, which are two of the three numbers sought; then from 76 (the double sum of the said Quotients) subtracting 2 (the before-mentioned difference,) the Remainder 74 shall be the third number sought. By this Operation it is evident that the two first numbers are the same with those found out by *Canon 2.* but the third numbers are different: I say the three numbers 12, 26 and 74 will solve the Question, as may easily be proved.

But to manifest the certainty of *Canon 3.* both its Operation and Demonstration may be symbolically express'd in this manner, viz.

Operation.

16. Take two unequal numbers, as $\left\{ \begin{array}{l} d \text{ the greater;} \\ c \text{ the lesser.} \end{array} \right.$
17. Their difference is $d - c$
18. Take any number less than the Square of c , as, b
19. Subtract the number b from the Squares of c and d severally, $\left\{ \begin{array}{l} cc - b \text{ and } dd - b \\ \text{so the Remainders are} \end{array} \right.$
20. Divide each of those Remainders by $d - c$, and the Quo- $\left\{ \begin{array}{l} \frac{cc - b}{d - c} \text{ and } \frac{dd - b}{d - c} \\ \text{tients are} \end{array} \right.$
21. The double sum of those Quotients is $\frac{2cc + dd - 4b}{d - c}$
22. From that double sum subtract $d - c$, and the Remain- $\frac{cc + dd + 2cd - 4b}{d - c}$
der is
23. Thus, the three numbers found out by *Canon 3.* last afore-going to solve *Quest. 29.* are equal to these, viz.

$$\frac{cc - b}{d - c}; \quad \frac{dd - b}{d - c}; \quad \frac{cc + dd + 2cd - 4b}{d - c}.$$

Now I say, if every two of those three numbers be mutually multiplied, and to the Products severally the number b be added, the three sums shall be Squares.

Demonstration.

24. By *Canon 1. Resolnt. 3. Quest. 29.* if the Product of the multiplication of the two first numbers, to wit, of $\frac{cc - b}{d - c}$ and $\frac{dd - b}{d - c}$ be increased with b , the sum will be a Square; it remains to prove, that if the Product of the multiplication of the first and third numbers be increased with b , the sum shall also be a Square; likewise that the Product of the multiplication of the second and third numbers increased with the number b , makes a Square. But if the number b be added to the Product of the multiplication of the first and third numbers, the sum is $\frac{ccc - ccdd + 2ccd - 4ccb - 4cdb + 4bb}{dd + cc - 2cd}$,

which is a Square, whose side is $\frac{cc + cd - 2b}{d - c}$; And if b be added to the Product of the multiplication of the second and third numbers, the sum is $\frac{ddd + ddc + 2ddc - 4ddb - 4ddb + 4bb}{dd + cc - 2cd}$, which is a Square whose side is $\frac{dd + cd - 2b}{d - c}$. Therefore the truth of *Canon 3.* is evident.

By the help of the second and third Canons last before express'd, *Bachet* extends the preceding *Quest. 29.* to four numbers, as I shall shew in the next Question.

QUEST. 30.

QUEST. 30. (*Bachet* in *Quest. 13. Lib. 3. Diophant.*)

To find four such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 3, be added, the four sums shall be Squares.

RESOLUTION.

1. The number given in the Question is 3
2. Let two Squares be feigned from $a + 1$ two such known numbers that their difference may be a Square, and that each of them may exceed the given number 3. To which end let the side of one of those Squares be feigned $a + 2$, and let the other side be $a + 6$, for the difference of 2 and 6 is 4, (a square number;) then will the Squares of the said sides $a + 2$ and $a + 6$ be these,
viz. $\left\{ \begin{array}{l} aa + 4a + 4, \text{ (the Square of } a + 2.) \\ aa + 12a + 36, \text{ (the Square of } a + 6.) \end{array} \right.$
3. From each of those Squares subtract the given number 3, so the Remainders will be these;
viz. $\left\{ \begin{array}{l} aa + 4a + 1 \\ aa + 12a + 33 \end{array} \right.$
4. Divide the said Remainders severally by 4, (the difference of the aforesaid sides $a + 2$ and $a + 6$) and let the Quotients be assumed for two of the four numbers sought,
viz. $\left\{ \begin{array}{l} \frac{1}{4}aa + a + \frac{1}{4}, \text{ (the first number.)} \\ \frac{1}{4}aa + 3a + 8\frac{1}{4}, \text{ (the second number.)} \end{array} \right.$
5. The double sum of the said Quotients (assumed in the last step for the first and second numbers sought) is $aa + 8a + 17$, from which subtract 4, (the difference before-mentioned) and let the Remainder be assumed for the third number,
viz. $\frac{1}{4}aa + 8a + 13$, (the third number.)
6. If the Construction hitherto be compared with the third Canon in the third Resolution of the foregoing 29th Question, it will thence be evident, that if to the Product made by the multiplication of every two of those three numbers before assumed in the fourth and fifth steps, there be added the given number 3, the three sums will be Squares.
7. For the fourth number sought assume the difference $\left\{ \begin{array}{l} 4, \text{ (the fourth number.)} \\ \text{of the sides } a + 2 \text{ and } a + 6 \text{ before-mentioned, to wit,} \end{array} \right.$
8. Then by comparing the Construction in the second, third, fourth and seventh steps with *Canon 2.* in the third Resolution of the foregoing twenty-ninth Question, it will be manifest, that if to the Product made by the multiplication of every two of these three numbers, to wit, the first, second and fourth numbers before assumed in the fourth and seventh steps, there be added the given number 3, the three sums will be Squares.
9. It remains to make the Product of the multiplication of the third and fourth numbers, the given number 3 being added, equal to a Square; but the third number $aa + 8a + 13$, multiplied by the fourth number 4, gives the Product $4aa + 32a + 52$, to which adding 3, the sum is $4aa + 32a + 55$, which must be equal to a Square, the side whereof may be feigned to be $2a - 10$, then the Square thereof being equated 55; therefore let the said side be $2a - 10$, then the Square thereof being equated 55; the value of a will thence be found equal to $\frac{11}{2}$, by the help to $4aa + 32a + 55$, the value of a will thence be found equal to $\frac{11}{2}$, by the help whereof, recourse being had to the fourth, fifth and seventh steps, the four numbers sought will be found these, viz. $\frac{11}{2}, \frac{11}{2}, \frac{11}{2}, \frac{11}{2}$ and 4, which will solve the Question. For if the first be multiplied by every one of the other three, and the Products be severally increased with the given number 3, the three sums will be these Squares, to wit, $\frac{11}{2}, \frac{11}{2}, \frac{11}{2}, \frac{11}{2}$, whose sides are $\frac{11}{2}, \frac{11}{2}, \frac{11}{2}$ and $\frac{11}{2}$. Also, if the second be multiplied by the third and fourth numbers severally, and each Product be increased with 3, these two Squares will arise, to wit, $\frac{11}{2}, \frac{11}{2}$, whose sides are $\frac{11}{2}$ and $\frac{11}{2}$; lastly, the Product of the third number multiplied by the fourth being added to 3 makes the Square $\frac{11}{2}$, whose side is $\frac{11}{2}$.

QUEST. 31. (*Quest. 15. Lib. 3. Diophant.*)

To find three such numbers, that if the Product of the multiplication of every two of them be lessened by the third, the three Remainders shall be Squares.

RESOLUTION.

1. For the first number put a
2. And

2. And for the second number put $a + 1$ some known Square number, as $a + 4$
3. Then the Product of the multiplication of the first and second numbers is $aa + 4a$
4. From which Product if $4a$ be subtracted, it is evident the Remainder will be a Square, to wit, aa , therefore for the third number we may put $4a$, (and so one of the conditions in the Question will be satisfied) $4a$
5. Then (from the second and fourth steps,) the Product of the multiplication of the second and third number is $4aa + 16a$, from which the first number a being subtracted, there remains $4aa + 15a$, which (according to the Question) must be equal to a Square, viz. $4aa + 15a = \square$
6. Also (from the first and fourth steps,) the Product of the multiplication of the first and third numbers is $4aa$, from which subtracting the second number $a + 4$, there remains $4aa - a - 4$, which (according to the Question) must be equal to a Square, viz. $4aa - a - 4 = \square$
7. So in the two last preceding steps we are faln upon a Duplicate equality, which (upon the grounds before demonstrated) may be resolved thus, viz. First, the difference of the two Algebraical quantities to be equated to two Squares, (by subtracting the lesser from the greater) is manifestly $16a + 4$; then two Squares must be found, such, that their difference may be $16a + 4$, and that $4aa$ may be in each of those Squares: Therefore (agreeable to Canon 2. of Quest. 5. of this Book,) two numbers are to be taken, that being mutually multiplied will produce $16a + 4$; moreover, that $2a$ may be found as well in the half-sum as in the half-difference of the numbers taken; but the only two numbers that will agree with those conditions are $4a + 1$ and 4 , whose half-sum is $2a + \frac{1}{2}$, and their half-difference is $2a - \frac{1}{2}$.
8. Then by equating the Square of the said $2a + \frac{1}{2}$ to $4aa + 15a$ (in the fifth step,) or the Square of $2a - \frac{1}{2}$ to $4aa - a - 4$ (in the sixth step,) from either of those Equations the value of a will be found $\frac{1}{2}$, which is the first of the three numbers sought, and consequently from the second and fourth steps, the second and third numbers are $2\frac{1}{2}$ and 5 , which three numbers will solve the Question; for if from the Product of every two of them the other be subtracted, the three Remainders are Squares, to wit, $\frac{1}{4}$, 25 and 1 .

QUEST. 32.

To find three numbers, such, that if to the Square of every one of them the sum of the other two be added, the three summs may be Squares.

RESOLUTION.

1. Take any number of $a + 1$ some known number, as $a + 1$, for the side of a Square, then the Square of $a + 1$ is $aa + 2a + 1$; now if for the first number sought we put a , for the second $2a$, and for the third 1 , then it is evident that the Square of the first number, together with the sum of the second and third, makes a Square, to wit, $aa + 2a + 1$, whereby one of the conditions in the Question is satisfied; therefore,
- For the three numbers sought put $a, 2a, 1$
2. Then (according to the import of the Question,) the Square of the second number, together with the sum of the first and third must make a Square, therefore $4aa + a + 1 = \square$
3. Likewise the Square of the third number, together with the sum of the first and second must make a Square, therefore $a + 3a + 1 = \square$
4. So we are faln upon a Duplicate equality, which differs from any of the preceding Forms, but (upon the same foundation by which those have been resolved) this may be resolved thus; First, supposing the former of the two quantities to be equated to a Square to exceed the latter, (for here we may indifferently take either of them for the greater,) their difference, by subtracting $3a + 1$ from $4aa + a + 1$, is manifestly $4aa -$

$4aa - 2a$; this difference, (as in most of the Duplicate equalities hitherto) must be esteemed the Product made by the mutual multiplication of two quantities, or factors; but here these two factors must be such, that as well in half their sum as in half their difference there may be found 1 , that is, the side of the known Square, (or absolute number) in each of the two quantities to be equated, to the end that when the Square of the said half-sum is equated to $4aa + a + 1$, or the Square of the said half-difference to $3a + 1$, the Square number 1 , by due Reduction of either of those Equations, may vanish. Now to find out two factors qualified as aforesaid, first, take 2 , the double of the side of the known Square 1 in each of the two quantities to be equated, with — prefix, (because — is prefix to $2a$ in the difference $4aa - 2a$ above-mentioned,) for part of the first of the two desired factors; then divide $2a$, (which is part of the said difference,) by 2 the double of the side of the said known Square 1 , and take the Quotient a for the latter factor; then divide $4aa$, (the other part of the said difference $4aa - 2a$) by the said latter factor a , and the Quotient $4a$ connected with — 2 first taken, shall be the compleat first factor: So two factors or quantities to agree with the conditions above mentioned are found to be $4a - 2$ and a ; for the Product of their multiplication is $4aa - 2a$, and 1 is found both in half their sum and half their difference: Then by equating the Square of half the sum of the said factors $4a - 2$ and a , viz. the Square of $\frac{5a - 1}{2}$, to $4aa + a + 1$; or by equating the Square of half the difference of the same factors, viz. the Square of $\frac{3a - 1}{2}$, to $3a + 1$, the value of a will be discovered, viz.

5. From either of these Equations, after due Reduction, $\frac{5a - 1}{2}$ $\frac{3a - 1}{2}$ $4aa + a + 1 = 4aa + a + 1$ $3a + 1$

6. The same value of a will be discovered, viz. $a = \frac{1}{2}$

Therefore by the sixth and first steps, these three numbers are found out, to wit, $\frac{1}{2}$, 1 , and 1 , which will solve the Question proposed: For $\frac{1}{4}$ the Square of the first number, together with $\frac{1}{2}$ the sum of the second and third, makes a Square, to wit, $\frac{1}{4}$; also $\frac{1}{4}$ the Square of the second, together with $\frac{1}{2}$ the sum of the first and third, makes a Square, to wit, $\frac{1}{4}$; and lastly, 1 the Square of the third, with $\frac{1}{2}$ the sum of the first and second makes a Square, to wit, 9 . By what hath been said in the first step of the Resolution this Question is capable of innumerable Answers.

QUEST. 33.

To find a number less than $2a$ number given, and such, that if it be multiplied by two given numbers severally, suppose by 8 and 6 , and if to each of the Products a given square number, suppose 4 , be added, the summs may be square numbers.

RESOLUTION.

1. For the number sought put a
2. Then if that position be prosecuted according to the conditions in $8a + 4 = \square$ the Question, this Duplicate equality will arise, to wit, $6a + 4 = \square$
3. Which kind of Duplicate equality Diophantus useth in divers Questions, and because the Resolution thereof is a very subtil invention, I have framed this Question purposely to explain it.

First, observe well these three numbers, $8a + 4$, $6a + 4$ and 4 .

Then seek what proportion the excess of the greatest of those three numbers above the mean hath to the excess of the mean above the least; so you will find that the former excess is to the latter as 1 to 3 . For the excess of $8a + 4$ above $6a + 4$ is $2a$, and the excess of $6a + 4$ above 4 is $6a$; but $2a$ is to $6a$ as 1 to 3 . Therefore the former excess is to the latter as 1 to 3 , and consequently the former excess is one third part of the latter.

4. Now the principal scope in resolving the said Duplicate equality is to find out two square numbers with this condition, that the excess of the greater above the less may have such proportion to the excess of the lesser above 4 (the Square given in the Question) as 1 to 3 ; to wit, as the difference of the numbers 8 and 6 , which are prefix to a in the Duplicate equality, is to 6 the lesser number prefix: For when two such Squares are found out, then if the greater be equated to $8a + 4$, or the lesser to $6a + 4$,

one and the same value of a will come forth. But to find out the said two Squares I proceed thus:

5. The least of the three Squares above mentioned, to wit, that given in the Question, by the help whereof the other two are to be found out, is 4
6. And to the end the mean Square may exceed the least, let the side of the mean Square be feigned $e+2$, (2 being the side of the given Square 4;) therefore the mean Square it self is $ee+4e+4$
7. Therefore the excess of the mean Square above the least is manifestly $ee+4e$
8. But by what hath been said before, the excess of the greatest Square above the mean must be $\frac{1}{2}$ part of the excess of the mean Square above the least; therefore (from the last step) the excess of the greatest Square above the mean shall be $\frac{1}{2}ee+\frac{1}{2}e$
9. Therefore by adding the last mentioned excess, to wit, $\frac{1}{2}ee+\frac{1}{2}e$ to the mean Square in the sixth step, the sum will be the greatest of the said three Squares, to wit, $\frac{3}{2}ee+\frac{1}{2}e+4$
10. Which $\frac{3}{2}ee+\frac{1}{2}e+4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out, viz. Forasmuch as the two greatest of the three Squares above mentioned must be such, that when the greatest is equated to $8a+4$, or the mean to $6a+4$, the value of a may be less than 2, (according to the conditions in the Question;) therefore such a square number must be found out equal to the said $\frac{3}{2}ee+\frac{1}{2}e+4$, that when 4 is subtracted from the said square number, $\frac{1}{2}$ part of the Remainder may be less than 2. Therefore from $\frac{3}{2}ee+\frac{1}{2}e+4$ subtract 4, and the Remainder is $\frac{3}{2}ee+\frac{1}{2}e$, whereof $\frac{1}{2}$ is $\frac{1}{4}ee+\frac{1}{4}e$, which must be less than 2; therefore
11. Suppose $\frac{1}{4}ee+\frac{1}{4}e=2$
12. Thence, by multiplying all by 4, it follows, that $ee+e=8$
13. And by adding the Square of half the Coefficient 4 to each part, there ariseth $ee+4e+4=16$
14. And by extracting the Square Root out of each part of the last step, $e+2=4$
15. Therefore by equal subtraction of 2, it is manifest that $e=2$
16. Thus we have found that $\frac{3}{2}ee+\frac{1}{2}e+4$ must be equated to a Square; with this condition, that the value of e may be less than 2. Now to cause that effect, the side of the said Square may be feigned $2+\frac{1}{2}$ any number of e greater than $3\frac{1}{2}$, therefore let the said side be feigned $3\frac{1}{2}e-2$, then the Square of $3\frac{1}{2}e-2$ being equated to the said $\frac{3}{2}ee+\frac{1}{2}e+4$, the value of e will thence be found $\frac{11}{12}$.
17. Now if $e=\frac{11}{12}$
18. Then consequently the Square of $3\frac{1}{2}e-2$, that is, the greater of the two Squares sought, will be $\frac{11}{12}$
19. And the Square of $e+2$ (which in the sixth step was put for the side of the lesser of the two Squares sought,) will be $\frac{11}{12}$
20. Which two Squares, to wit, $\frac{11}{12}$ and $\frac{11}{12}$, (whose sides are $\frac{11}{12}$ and $\frac{11}{12}$) together with 4, (the square number given in the Question,) are such, that the excess of the greatest above the mean is $\frac{1}{2}$ part of the excess of the mean above the least, (according to the scope designed in the fourth step.) Now if the greater Square $\frac{11}{12}$ be equated to $8a+4$, or the lesser Square $\frac{11}{12}$ to $6a+4$, from either of those Equations the value of a , to wit, the number sought by the Question will be found $\frac{11}{12}$: For first, it is less than 2, also eight times that number, together with 4, makes the Square $\frac{11}{12}$; and six times the same number $\frac{11}{12}$, together with 4, makes the Square $\frac{11}{12}$. It is also evident by the sixteenth step, that as many numbers as one will may be found out to solve the Question proposed.
21. But for the greater evidence of the infallibility of the method of resolving this Duplicate equality, I shall demonstrate the same in manner following, viz.

Suppose

Suppose $\left\{ \begin{array}{l} r=8 \\ s=6 \\ e=4 \\ d=\frac{11}{12} \\ f=\frac{11}{12} \end{array} \right\}$ two Multipliers given in the Question;
a square number given in the Question;
two square numbers found out according to the direction in the fourth step of the Resolution,
the number sought.

22. Suppose also, according to the Construction in the Resolution, that the excess of d above f hath such proportion to the excess of f above e , as the excess of r above s hath to s , viz. $as, d-f : f-e :: r-s : s$.
23. Then according to the Construction in the twentieth step of the Resolution, let these two Equations be instituted, viz. $ra+c=d$
 $sa+c=f$
24. Now since the Conclusion of the Resolution (in the said twentieth step) takes it for granted, that one and the same value of a , (to wit, the number sought) will be given by either of those two Equations, we must prove that these two Quotients are equal to one another, viz. $\frac{d-c}{r} = \frac{f-c}{s}$

Demonstration.

25. Forasmuch as by Construction in the 22^d step, $d-f : f-e :: r-s : s$
 26. Therefore by Composition of Reason, $d-c : f-c :: r-s : s$
 27. Therefore alternately $d-c : r :: f-c : s$
 28. But if four numbers be Proportionals, the Reason of the first to the second is equal to the Reason of the third to the fourth, therefore $\frac{d-c}{r} = \frac{f-c}{s}$
- Which was to be demonstrated.

Observat. 1. upon Quest. 33.

In the Duplicate-equality used in the preceding Quest. 33. both the numbers of a are affirmative, but if they were both negative, or one of them affirmative and the other negative, the Resolution would differ very little from the former, as will appear by the two following Questions.

QUEST. 1.

1. Let it be required to find out the number signified by a in this $4-2a=0$
Duplicate equality, viz. $4-3a=0$

RESOLUTION.

2. First, these three numbers are to be considered $4, 4-2a, 4-3a$
Then because the excess of 4 above $4-2a$, hath such proportion to the excess of $4-2a$ above $4-3a$, as 2 to 1, let 4 be considered as the greatest of three Squares, and find the other two, with this condition, that the excess of 4 above the mean may be the double of the excess of the mean above the least, to which end,
3. Let the greatest of the said three Squares be 4
4. And to the end the mean Square may be less than the greatest, let the side of the mean Square be $2-e$, therefore the mean Square shall be $ee-4e+4$
5. Therefore the excess of the greatest Square above the mean is $4-ee$
6. Therefore, according to the condition prescribed in the second step, the half of the excess in the fifth step shall be the excess of the mean Square above the least, to wit, $2e-\frac{1}{2}ee$
7. Which last excess, to wit, $2e-\frac{1}{2}ee$ being subtracted from the mean Square in the fourth step, the Remainder shall be equal $\frac{1}{2}ee-6e+4$ to the least Square, to wit,
8. Therefore $\frac{1}{2}ee-6e+4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out, viz. Forasmuch as the least of the three Squares above mentioned must be such, that when it is equated to $4-3a$ (in the first step) the value of a may be greater than nothing, it is evident the said least Square must necessarily be less than 4; Supposing therefore $\frac{1}{2}ee-6e+4=4$ from this supposition, $H 2$

3. Let it be required to find out the number signified by a in this } $3a + 13 = \square$
 Duplicate equality, viz. } $a + 7 = \square$
 4. Because $2a + 6$, the difference of the two Algebraick Quantities being divided by 2 gives the Quotient $a + 3$, which subtracted from $a + 7$ leaves a Square, to wit, 4; the Duplicate equality propos'd is explicable by the method used in the preceding *Quest.* 33. as I shall here make manifest.
 First, let these three numbers be considered, to wit, . . . $3a + 13$, $a + 7$, 4
 Then, by continuing the search in such manner as hath been shewn in solving the foregoing *Quest.* 33. you may find out (among innumerable values of a) $a = 29$, which will solve the Duplicate equality propos'd; for if 29 be multiplied by 3, and 13 be added to that Product, the sum makes a Square, to wit, 100; also, if 7 be added to 29, it makes the Square 36.

5. The second Case is, when the difference of the two Algebraick Quantities which are to be equated to two Squares is such, that if it be multiplied or divided by some known number, and the Product or Quotient be subtracted from the lesser of those two Algebraick Quantities, there remains a negative known number, which taken affirmatively hath such proportion to the said Multiplier or Divisor, as a square number to a square number. For example,

6. Let it be required to find a number, suppose it to be a , that } $6a + 25 = \square$
 will make } $2a + 3 = \square$

7. Because $4a + 22$, the difference of those two Algebraick Quantities, being divided by 2 gives the Quotient $2a + 11$, which subtracted from the lesser of the said two Quantities, leaves -8 ; and the number 8 to the Divisor 2, is as 4 to 1, that is, as a square number to a square number; the Duplicate equality propos'd may be resolved thus, viz.

8. Let these three numbers be considered, to wit, . . . $6a + 25$, $2a + 3$, and -8

Then forasmuch as $4a + 22$, to wit, the difference of the two greater of those three numbers, is the double of $2a + 11$ which is the difference of the two lesser of the said three numbers, seek two Squares, that the excess of the greater above the lesser may be the double of the excess of the lesser above -8 ; to which end you may proceed thus, viz.

9. For the lesser of the two Squares put . . . aa

10. Then the excess of aa above -8 is . . . $aa + 8$

11. The double whereof is . . . $2aa + 16$

12. Which added to the lesser Square makes a sum equal to the } $3aa + 16$
 greater, to wit, . . .

Now the said $3aa + 16$ must be equated to a Square, with this caution, That the square number found out must exceed 25, to the end that when the said Square is equated to $6a + 25$, the value of a may be greater than nothing: But to cause that effect, innumerable Squares may be found out, (by the method before-delivered in divers Questions of this Book,) such are 64 and 16; for whether you equate 64 to $6a + 25$, or 16 to $2a + 3$, one and the same value of a , to wit, $6\frac{1}{2}$, will be discovered to solve the Duplicate equality propos'd in the sixth step.

QUEST. 34. (Quest. 17. Lib. 3. Diophant.)

To find three such numbers, that the Product of the multiplication of every two of them, with the sum of the same two numbers, may make a Square.

RESOLUTION.

1. Forasmuch as the Product of the multiplication of two Squares whose sides differ by unity, being added to the sum of the said Squares, will make a Square, (by the Theorem at the end of the Resolution of this Question,) let there be put for the first and second numbers . . . 4 and 9

So is one of the conditions in the Question satisfied; for 4 multiplied by 9 produceth 36, to which if 13 (the sum of 4 and 9) be added, the sum is a Square, to wit, 49.

2. Let the third number be . . . a

3. Then

3. Then the Product of the multiplication of the second and third numbers is $9a$, to which adding their sum $a + 9$, it makes $10a + 9$, which (according to the Question) must be equal to a Square, viz. } $10a + 9 = \square$
 4. Likewise the Product of the multiplication of the first and third numbers is $4a$, which with their sum $a + 4$ makes $5a + 4$, which (according to the Question) must be equal to a Square, viz. } $5a + 4 = \square$
 5. So in the two last steps we are fallen into a Duplicate equality, which may be solved by innumerable values of a , as hath been shewn in the second Observation upon the foregoing *Quest.* 33. of this *Book*. For example, take that value of a there found, to wit, 28 ($= a$) for the third number sought by this Question; I say 4, 9 and 28 will solve this 34th Question, as will be evident by

The Proof.

The three numbers found out are . . . 4, 9, 28

Now according to the Question,

I. $4 \times 9, + 9 + 4 = 49$
 II. $9 \times 28, + 9 + 28 = 289$
 III. $4 \times 28, + 4 + 28 = 144$ } Which are Squares, as was required.

The first step of the Resolution of this 34th Question is grounded upon this

THEOREM.

6. If two numbers differ by unity, the Product made by the multiplication of their Squares, together with the sum of their Squares shall be a Square.
 The truth of this Theorem may be demonstrated thus,

7. Let there be two numbers which differ by 1 (or unity,) as . . . a , and $a + 1$

8. Then their Squares are . . . aa
 $aa + 2a + 1$

9. The sum of those Squares is . . . $2aa + 2a + 1$

10. The Product of the multiplication of the said Squares is . . . $aaaa + 2aaa + aa$

11. The sum of the said Sum and Product in the ninth and tenth steps is . . . $aaaa + 2aaa + 3aa + 2a + 1$

12. Which Aggregate is a Square whose side is . . . $aa + a + 1$
 As will easily appear by multiplying the said side by it self. Therefore the truth of the Theorem is manifest.

QUEST. 35. (Quest. 18. Lib. 3. Diophant.)

This Question is the same with the foregoing 34th, which is here repeated, and solved after another manner.

To find three such numbers, that the Product of the multiplication of every two of them, being added to the sum of the same two numbers, may make a Square.

RESOLUTION.

1. Let the first number be . . . 4
 2. And let the second number be . . . 9
 3. Then the Product of their multiplication added to their sum, makes } $4a + 3$
 4. Which $4a + 3$ must (according to the Question) be equal to } $4a + 3 = 25$
 some Square, let it be 25, therefore . . . $a = 5\frac{1}{2}$
 5. Therefore from that Equation, . . .
 6. So we have found two numbers, to wit, $5\frac{1}{2}$ and 3, which will satisfy one of the conditions in the Question, for the Product of their multiplication with their sum makes 25, which is a Square. It remains to find a third number, which must be such, that the Product of the second and third numbers being added to their sum may make a Square; also, that the Product of the first and third numbers being added to their sum, may make a Square: Now to find out the said third number, Diophantus begins again, thus,
 7. Let the first number be (as before it was found) . . . $5\frac{1}{2}$ 8. And

8. And let the second number be (as before it was assumed,) . . . 3
 9. Then for the third number put . . . a
 10. And since (according to the Question) the Product of the multiplication of the second and third numbers, with their sum, must make a Square; therefore, from the eighth and ninth steps, $4a + 3 = \square$
 11. Also the Product of the first and third numbers with their sum, must be a Square; therefore, from the seventh and ninth steps, $6\frac{1}{2}a + 5\frac{1}{2} = \square$
 12. So in the two last steps we are fallen into a Duplicate equality, but 'tis not resolvable by any of the preceding Rules of *Diophantus*; he frames therefore the Positions a-new, wherein his scope is to find such numbers of a in the two Algebraick Quantities to be equated to two Squares, that shall be in proportion one to the other as a square number to a square number, and then he shews how to resolve this new kind of Duplicate equality, which hath not hitherto happened. First, if we examine whence 4 and $6\frac{1}{2}$ (to wit, the numbers prefixt before a in the tenth and eleventh steps) do proceed, we shall find that they arise from the addition of unity to each of the numbers 3 and $5\frac{1}{2}$ first found; (for by multiplying a into those numbers severally, and by adding a to each Product, there ariseth $4a$ and $6\frac{1}{2}a$ above exprest.) Therefore the next search must be to find two such numbers, that being severally increased with unity, the one sum may be to the other as a square number to a square number: And because (by the Theorem in the following first Observation upon this Question,) if we add unity to each of two numbers whereof the greater exceeds the quadruple of the lesser by 3, the two sums will be in the Reason of a Square to a Square; therefore,

13. Let the first of the three numbers sought be . . . a
 14. Then (by the said Theorem) the second number shall be . . . $4a + 3$
 15. Now according to the Question, the Product of the first and second numbers together with their sum must be a Square, therefore from the two last preceding steps, $4aa + 8a + 3 = \square$
 16. The side of which Square may be feigned $2a - 3$, any known number whose Square is greater than 3, let therefore the said side be $2a - 3$, then its Square being equated to $4aa + 8a + 3$, the value of a will be found $\frac{1}{5}$ for the first number, and consequently $\frac{23}{5}$ (as $4a + 3$) shall be the second number.

So we have found two numbers which will answer the first part of the Question, and moreover they are fit to raise a Duplicate equality that will be explicable by a Rational number: Therefore now an effectual Resolution may be formed thus;

17. Let the first number be . . . $\frac{1}{5}$
 18. And the second number . . . $\frac{23}{5}$
 19. And let the third number be . . . a
 20. Then according to the Question, the Product of the multiplication of the second and third numbers, together with their sum, must be equal to a Square, therefore from the 18th and 19th steps, $\frac{23}{5}a + \frac{1}{5} = \square$
 21. Also according to the Question, the Product of the first and third numbers, with their sum, must be a Square; therefore, from the 17th and 19th steps, $\frac{1}{5}a + \frac{1}{5} = \square$
 22. Now because the numbers drawn into a , in the Duplicate equality exprest in the two last preceding steps, are (by Construction) in the Reason of a Square to a Square, for $\frac{23}{5} : \frac{1}{5} :: 4 : 1$, and consequently (by Prop. 19. Elem. 7. Euclid.) $\frac{23}{5} \times 1 = \frac{1}{5} \times 4$. Therefore by multiplying the Algebraick quantity in the twentieth step by 1, and that in the twenty-first step by 4, the numbers of a in the Products will be equal to one another, for the first Product will be $\frac{23}{5}a + \frac{1}{5}$, and the latter $\frac{1}{5}a + \frac{4}{5}$; hence a new Duplicate equality is formed,

$$\text{viz. } \begin{cases} \frac{23}{5}a + \frac{1}{5} = \square \\ \frac{1}{5}a + \frac{4}{5} = \square \end{cases}$$

23. Which Duplicate equality being of the same kind with that explained in the preceding eighth Question, may be solved by innumerable Answers.

But I shall exhibit only one Answer for an Example. First then, because the difference of the two Algebraick quantities in the Duplicate equality last before exprest is 3, let two Squares be found out (by Canon 1. Quest. 7. of this Book,) whose difference shall be

be 3, and that the greater Square may exceed $\frac{1}{5}$; such are the Squares $\frac{1}{5}$ and $\frac{16}{5}$, whose sides are $\frac{1}{5}$ and $\frac{4}{5}$: Then,

$$24. \text{ From either of these Equations, } \begin{cases} \frac{1}{5}a + \frac{1}{5} = \frac{1}{5} \\ \frac{1}{5}a + \frac{1}{5} = \frac{16}{5} \end{cases}$$

25. The same value of a will be discovered for the third number sought, viz. . . . $a = \frac{1}{5}$
 Thus three numbers are found out, to wit, $\frac{1}{5}$, $\frac{23}{5}$ and $\frac{1}{5}$, which will solve the Question, as will be evident by

The Proof.

$$\begin{cases} \frac{1}{5} \times \frac{23}{5} + \frac{1}{5} = \frac{16}{5} \\ \frac{1}{5} \times \frac{1}{5} + \frac{1}{5} = \frac{1}{5} \end{cases} \text{ Which are Squares; as the Question requires.}$$

Observations upon Quest. 35.

1. If the Resolution of this Question be well examined, it will appear, that the forming of the Duplicate equality in the twentieth and twenty-first steps, where the numbers prefixt to a have such Reason to one another as a square number to a square number; agreeable to the Scope before-mentioned in the twelfth step, doth depend upon this following

THEOREM.

If there be two such numbers, that the greater exceeds the quadruple of the lesser by 3, and if unity be added to each number, the sums shall have such Reason between themselves as a Square to a Square, viz. the greater sum shall be to the lesser as 4 to 1. Which Theorem may be easily demonstrated, thus,

Suppose $\begin{cases} a \\ 4a + 3 \end{cases}$ Two numbers, whereof the greater exceeds the quadruple of the lesser by 3.
 $\begin{cases} a + 1 \\ 4a + 4 \end{cases}$ The first number increased with unity.
 $\begin{cases} a + 1 \\ 4a + 4 \end{cases}$ The second number increased with unity.
 I say $4a + 4$ hath such proportion to $a + 1$, as a Square to a Square, for,

In like manner, if there be two numbers whereof the greater exceeds nine times the lesser by 8, as 17 and 1, then if you add 1 to each number, the sums shall be to one another as a square number to a square number, viz. the greater sum shall be to the lesser as 9 to 1; the like is to be understood of other Squares.

2. After the two numbers prefixt before a in the Duplicate equality formed in the twentieth and twenty-first steps of this Question, are found such, that they have such Reason one to the other as a Square to a Square, then may any two square numbers in that Reason be used as is directed in the twenty-second step: So instead of 4 and 1 there taken, we may take 100 and 25, which have the same Reason between themselves as $\frac{1}{5}$ and $\frac{1}{5}$; For, $\frac{23}{5} : \frac{1}{5} :: 100 : 25$.

$$\text{Therefore, } \frac{23}{5} \times 25 = \frac{1}{5} \times 100.$$

Then by multiplying the Algebraick quantity in the twentieth step of the Resolution by 25, and that in the twenty-first step by 100, the following Duplicate equality (being that which *Diophantus* useth in solving this Question) will arise,

$$\text{viz. } \begin{cases} 130a + 105 = \square \\ 130a + 30 = \square \end{cases}$$

Hence, by the Canon in the seventh step of *Resolut. 1. Quest. 8.* (among innumerable values of a that might be found out,) you may find $a = \frac{1}{5}$ (as before) for the third number sought.

QUEST. 36. (Quest. 20. Lib. 3. *Diophant.*)

To find two numbers, that the Product of their multiplication increased severally with each of them, and also with their sum, may make three Squares.

RESOLUTION.

1. For one of the numbers put . . . a
 2. And for the other, . . . $4a - 1$
 3. Then their Product is . . . $4aa - a$ Whence

Whence it is evident, that if the first number a be added to the said Product, the sum is a Square, to wit, $4aa$.

4. It remains, that the second number and the sum of both being severally added to the said Product may make a Square; but the second number added to the Product makes $4aa + 3a - 1 = \square$, and the sum of both numbers, together with their Product, makes $4aa + 4a - 1$; therefore
5. Which Duplicate equality may be resolved by the method before explained in the preceding twentieth and twenty-first Questions. For the difference of those two Algebraic quantities which are to be equated to Squares is a , which is to be divided into two such quantities that the Product of their multiplication may make a , and that both in the half-sum and in the half-difference of those two quantities there may be found $2a$, but such are the quantities $4a$ and $\frac{1}{4}$, whose Product is a ; also the half of their sum is $2a + \frac{1}{8}$, and the half-difference is $2a - \frac{1}{8}$; then by equating the Square of $2a + \frac{1}{8}$ to $4aa + 4a - 1$, or the Square of $2a - \frac{1}{8}$ to $4aa + 3a - 1$, from either of those Equations the value of a will be found $\frac{1}{24}$. Therefore the first number shall be $\frac{1}{24}$, and the second $\frac{5}{24}$; which numbers will solve the Question, as may easily be proved.

QUEST. 37. (Quest. 22. Lib. 3. Diophant.)

To find four such numbers, that every one of them being added to, and subtracted from the Square of the sum of them all, as well the four sums as the four remainders shall be Squares.

RESOLUTION.

1. In every right-angled Triangle; if the Square of the Hypotenusal be increased or lessened by the quadruple of the Area, that is, by the double Product of the multiplication of the two sides about the right-angle, it makes a Square, (which Theorem is made manifest at the end of the Resolution.) Therefore the chief scope is to find four right-angled Triangles in numbers having equal Hypotenuses: But those may be found out thus.

First, (by the Canon in *Observat.* 8. *Resolut.* 2. of *Quest.* 1. of this *Book*.) find out two unlike right-angled Triangles in numbers, such are these,

$$\begin{array}{ccc} 5 & 4 & 3 \\ 13 & 12 & 5 \end{array}$$

2. Then multiply the three sides of the first Triangle by the Hypotenusal of the second, also multiply the three sides of the latter Triangle by the Hypotenusal of the first; so the Products will give these two right-angled Triangles having equal Hypotenuses,

$$\begin{array}{ccc} 65 & 52 & 39 \\ 65 & 60 & 25 \end{array}$$

3. By the help of the two unlike right-angled Triangles first found, to wit, 5, 4, 3 and 13, 12, 5, the Canon in *Observat.* 4. upon *Resolut.* 2. and 3. of *Quest.* 2. of this *Book* will give two other right-angled Triangles unlike to those in the second step, but having the same Hypotenusal 65, to wit, these;

$$\begin{array}{ccc} 65 & 56 & 33 \\ 65 & 16 & 63 \end{array}$$

4. Then assume a to represent a number unknown, and let it be multiplied by every one of the sides of those four Triangles having 65 for a common Hypotenusal, so the Products will be these,

$$\begin{array}{ccc} 65a & 52a & 39a \\ 65a & 60a & 25a \\ 65a & 56a & 33a \\ 65a & 16a & 63a \end{array}$$

5. Now for the sum of the four numbers sought by the Question put $\frac{1}{24}$ 65a
6. Therefore the Square of the said sum is $\frac{1}{24}$ 4225aa
7. Then for the first number sought, take the quadruple of the Area of the first of the four Triangles in the fourth step, viz. multiplying 52a by 39a, take the double of that Product for the first number, to wit,

8. In

8. In like manner for the second number take the double Product of 60a by 25a, that is, 3000aa
9. And for the third number take the double Product of 56a by 33a, that is, 3696aa
10. And for the fourth number take the double Product of 63a by 16a, that is, 2016aa
11. The sum of the four numbers express'd in the seventh, eighth, ninth and tenth steps is 12768aa
12. Which sum must be equal to 65a, which in the fifth step was assumed for the sum of the four numbers sought, hence this Equation, 12768aa = 65a
13. Which Equation reduced, gives $a = \frac{1}{12768}$
14. Therefore from the thirteenth, seventh, eighth, ninth and tenth steps the four numbers required will be found these, viz. $\frac{1}{12768}$ 52a, $\frac{1}{12768}$ 60a, $\frac{1}{12768}$ 56a, $\frac{1}{12768}$ 63a; which four numbers will solve the Question, as will be evident to him that will take the pains of forming the Proof.
But because the Resolution of this Question is chiefly grounded upon a Theorem taken for granted in the first step, I shall here demonstrate the same

THEOREM.

15. If the Square of the Hypotenusal of a right-angled Triangle be increased or lessened by the quadruple of the Area, (that is, the double Product of the multiplication of the sides about the right-angle,) the sum, as also the remainder shall be a Square. For,

If a and e = the sides about the right-angle of a right-angled Triangle,
Then $2ae$ = the double Product of those sides,
Also aa and ee = the Squares of those sides,
And $aa + ee$ = the Square of the Hypotenusal, (per 47. Prop. 1. Elem. Euclid.)

Hence that which the Theorem asserts is manifest,

$$\begin{array}{l} \text{viz. } \left\{ \begin{array}{l} aa + ee + 2ae = \square, \text{ whose side is } a + e; \\ aa + ee - 2ae = \square, \text{ whose side is } a - e. \end{array} \right. \end{array}$$

It is also evident from the premises, that this 37th Question may be extended to five, six, or as many numbers as shall be desired; but first of all, so many numbers as are required, so many right-angled Triangles in numbers must be found out having equal Hypotenuses; which Triangles in whole numbers may be readily discovered by the method delivered in *Observat.* 13. upon *Resolut.* 2. of *Quest.* 1. of this *Book*.

QUEST. 38.

[This is Quest. 20. of the fourth Book of Vieta's *Zetetics*, and the same with Quest. 3. in Bachet's Comment upon the fourth Book of Diophantus.] *Quest. 2 of Book 3 of 4*

Two cube-numbers being given, such, that the double of the lesser exceeds the greater, to find two other cube-numbers whose difference shall be equal to the difference of the given Cubes. (But how to perform this when the double of the lesser Cube is less than the greater, I shall hereafter shew in *Quest.* 42.)

RESOLUTION.

1. Let the sides of the given Cubes be $\left\{ \begin{array}{l} d \text{ the greater;} \\ b \text{ the lesser.} \end{array} \right.$
2. Then the Cubes of those sides are d^3 and b^3
3. And the difference of the said Cubes is $d^3 - b^3$
4. For the side of the lesser Cube sought put $a - d$
5. And for the side of the greater Cube sought put $\frac{dd}{bb} a - b$
6. Therefore the greater Cube sought is $\frac{d^6}{b^6} aaa - \frac{3d^4}{b^3} aa + 3dda - b^3$
7. And the lesser Cube sought is $aaa - 2daa + 3dda - d^3$
8. Therefore the difference of the two Cubes sought is $\frac{d^6 a^3 - b^6 a^3}{b^6} - \frac{3d^4}{b^3} aa + 3dda + d^3 - b^3$
9. Which

9. Which difference must be equal to the difference of the given Cubes, therefore;
 $\frac{d^3 a^3 - b^3 a^3}{b^3} - \frac{3d^2 aa + 3daa + d^3 - b^3}{b^3} = d^3 - b^3.$

10. From that Equation, after due Reduction, this ariseth, } $a = \frac{3d^2 b^3 - 3db^4}{d^3 - b^3}.$
 viz. }
 11. And by reducing the Fraction in the latter part of the last preceding Equation into its least Terms, by the common Divisor for $d^3 - b^3$, it gives } $a = \frac{3db^3}{d^3 + b^3}.$
 12. Therefore from the first, fourth, fifth and eleventh steps, the sides of the Cubes sought will be found equal to these known quantities,

$$\text{viz. } \begin{cases} \frac{2db^3 - d^4}{d^3 + b^3} = \text{the lesser side;} \\ \frac{2db^3 - b^4}{d^3 + b^3} = \text{the greater side.} \end{cases}$$

13. The same sides will be produced, if you put $a - b$ for the side of the greater of the Cubes sought, and $\frac{bb}{da}a - d$ for the lesser side, (instead of the Positions in the fifth and fourth steps,) and it's evident that each of the sides found out in the twelfth step will be greater than nothing, if $2b^3$ exceeds d^3 , (that is, if the double of the lesser of the two Cubes given exceeds the greater, as the Question presupposeth.)

The twelfth step affords this following

CANON.

14. Multiply the excess of the double of the lesser of the two Cubes given above the greater, by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser, by the side of the lesser: then divide each of those Products by the sum of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

Examples in Numbers.

15. Let two Cubes be given, such, that the double of the } $125 = d^3$, and $64 = b^3$
 lesser exceeds the greater, as, }
 16. The sides of which Cubes are } $5 = d$, and $4 = b$

17. Then by the Canon, } $\frac{2db^3 - d^4}{d^3 + b^3} = \frac{2}{3}$ } the sides of the Cubes sought:
 } $\frac{2db^3 - b^4}{d^3 + b^3} = \frac{4}{3}$

18. The Cubes of those sides $\frac{2}{3}$ and $\frac{4}{3}$ are $\frac{8}{27}$ and $\frac{64}{27}$; whose difference is 61 , which is equal to the difference of the two given Cubes 125 and 64 .

19. In like manner, if these two Cubes be given, to wit, 1728 and 1000 , whose difference is 728 , the foregoing Canon will give $\frac{4}{3}$ and $\frac{2}{3}$, the sides of two Cubes whose difference is 728 .

Observations upon Quest. 38.

First, the chief scope in the Resolution of this Question is, to raise an Equation between some number of aaa and some number of aa , that a may be found equal to a Rational number; to which purpose, the side of one of the Cubes sought may be feigned to be a — one of the sides of the given Cubes, and the other side sought some number of a — the other side given; but this latter number of a must be such as will cause equal numbers of a to arise in the Cubes of those feigned sides, that when the lesser of the feigned Cubes is subtracted from the greater, the numbers of a may vanish, and then the Remainder being equated to the difference of the given Cubes, this difference will likewise vanish, (because 'tis also found in the difference of the feigned Cubes,) and an Equation remain between some number of aaa and some number of aa . Now to cause that effect, supposing (as before in the Resolution) d to represent the side of the greater Cube given, and b the side of the lesser, we may put $a - d$ for the side of the lesser of the Cubes sought, and then the greater side must necessarily be $\frac{db}{bb}a - b$; or $a - b$ may be put for the greater

side

side sought; and then the lesser must be $\frac{bb}{da}a - d$; from either of which ways of framing the Positions there will arise, after due Reduction, an Equation between aaa and aa ; whence a will be found equal to a Rational number. All which will be manifest to him that diligently examines the preceding Resolution.

Secondly, if two pairs of Cubes which shall have equal differences be desired in whole numbers, they may easily be found out by the help of the foregoing Canon, in this manner, viz. let d and b represent the sides of two such Cubes, that the double of the Cube of the lesser side b exceeds the Cube of the greater side d , then the said Canon gives this Equation,

$$\begin{cases} + \text{Cube of } \frac{2db^3 - b^4}{d^3 + b^3} \\ - \text{Cube of } \frac{2db^3 - d^4}{d^3 + b^3} \end{cases} = \text{Cube of } d - \text{Cube of } b.$$

Now to contrait that Equation, suppose f , h and g to be equal to the Numerators and common Denominator, so that Equation will be converted into this, viz.

$$\frac{f^3}{g^3} - \frac{h^3}{g^3} = d^3 - b^3.$$

Whence, by multiplying every Term by the Denominator g^3 , this Equation is produced, viz.

$$f^3 - h^3 = g^3 d^3 - g^3 b^3.$$

That is, } $+ \text{Cube of } f$ } = Cube of gd — Cube of gb .
 } $- \text{Cube of } h$

In which last Equation, if instead of f , h and g , you take $2ba^3 - b^4$; $2db^3 - d^4$, and $d^3 + b^3$, which were before supposed equal to f , h and g respectively, this following Equation will arise, viz.

$$\begin{cases} + \text{Cube of } \frac{2ba^3 - b^4}{2db^3 - d^4} \\ - \text{Cube of } \frac{2db^3 - d^4}{2db^3 - d^4} \end{cases} = \begin{cases} + \text{Cube of } \frac{d^4 + db^3}{2db^3 - d^4} \\ - \text{Cube of } \frac{b^4 + db^3}{2db^3 - d^4} \end{cases}$$

Which last Equation gives this following

CANON.

First, take two such Cubes in whole numbers that the double of the lesser may exceed the greater, and multiply the excess of the double of the greater above the less by the side of the lesser Cube; secondly, multiply the excess of the double of the lesser Cube above the greater by the side of the greater Cube; thirdly, multiply the sum of the same Cubes by the side of the greater; fourthly, multiply the sum of the said Cubes by the side of the lesser: then the difference of the Cubes of the first and second Products shall be equal to the difference of the Cubes of the third and fourth Products.

An Example in Numbers.

Let two such Cubes be taken, that the double of the lesser } $125 = ddd$
 exceeds the greater, as, } $64 = bbb$
 The sides of which Cubes are } $5 = d$
 } $4 = b$

Then by working according to the directions of the last preceding Canon, the four Products, that is, the sides of the } 248 , 5 , 315 , 252
 four Cubes sought, in their least terms will be found these, to wit.

Which four numbers will satisfy the Proposition; for the difference between the Cubes of 248 and 5 is equal to the difference between the Cubes of 315 and 252 , as may easily be proved.

Hence it is easie to find four Cubes in whole numbers, such, that the sum of two of them shall be equal to the sum of the other two, for if two pairs of Cubes be found out by the last preceding Canon, such, that the first pair hath the same difference as the latter, then the sum of the greater Cube of the first pair and the lesser of the latter, shall be equal to the sum of the lesser Cube of the first pair and the greater of the latter.

Thirdly, *Albert Girard* (in his Comment in *Simon Stevin's* Arithmetick, upon the 19th of the fifth Book of *Diophantus*), observes, but doth not demonstrate, that the Cubes found out by the preceding Quest. 38. are always less than the Cubes given; which Property, since it will be useful in the following Quest. 39. I shall here demonstrate.

Suppose

Suppose $\left. \begin{array}{l} d = 5 \\ b = 4 \end{array} \right\}$ the sides of two Cubes, such, that the double of the lesser exceeds the greater.
 $\left. \begin{array}{l} d^3 = 125 \\ b^3 = 64 \end{array} \right\}$ the Cubes of those sides.
 $d^3 - b^3 = 61$ the difference of the same Cubes.

Then by the Canon in *Sett.* 14. *Quest.* 38. the sides of two Cubes whose difference is equal to the difference of the given Cubes, whose sides are $d (= 5)$ and $b (= 4)$ will be found these that follow, to wit,

$$\frac{2db^3 - d^4}{d^3 + b^3} = \frac{5}{63}; \text{ and } \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63}.$$

Now because the Cubes given and found out have equal differences, if it be proved that the greater Cube found out is less than the greater Cube given, then consequently the lesser Cube found out shall be less than the lesser Cube given: But that the side of the greater Cube found out, is less than the side of the greater Cube given, (and by consequence the greater Cube found out less than the greater Cube given,) I prove thus,

The greater side found out (as before) is $\frac{2bd^3 - b^4}{d^3 + b^3}$.
 Therefore we must demonstrate that $\frac{2bd^3 - b^4}{d^3 + b^3} < d$

Demonstration.

By supposition, $d = b$
 Therefore by multiplying d and b severally by bb , it follows, that $dbb = b^3$
 And by adding b^3 to each part, $b^3 + dbb = 2b^3$
 By supposition $d^3 = 2b^3$
 Therefore from the two last preceding steps, $d^3 = b^3 + dbb$
 And by multiplying each part in the last step by b , $bd^3 = b^4 + db^3$
 But by multiplying each number in the first step of this Demonstration by d , $bd^3 = d^4$
 Therefore by comparing the sum of the numbers in the first parts of the two last preceding steps, to the sum of those in the latter parts, $2bd^3 = d^4 + b^4 + db^3$
 And by subtracting b^4 from each part of the last preceding step, $2bd^3 - b^4 = d^4 + db^3$
 Wherefore by dividing each part of the last step by $d^3 + b^3$, it's manifest that $\frac{2bd^3 - b^4}{d^3 + b^3} < d$
 Which was to be demonstrated.

Having proved that the greater of the two sides found out by the Canon before discovered for resolving *Quest.* 38. is less than the greater of the two sides given, it follows, that the Cube of that side found out is less than the Cube of the greater side given, and that the lesser Cube found out is less than the lesser Cube given, (because by Construction the two Cubes found out have the same difference as the Cubes given.) Therefore the truth of the property before affirmed is manifest.

Fourthly, if two pairs of numbers have equal differences, the lesser number of the lesser pair shall have lesser Reason (or Proportion) to the greater number of the same pair, than the lesser number of the greater pair hath to the greater number of this pair. To make this manifest,

Suppose $\left. \begin{array}{l} b = 8 \\ c = 6 \end{array} \right\}$ two unequal numbers taken at pleasure.
 $\left. \begin{array}{l} b - c = 2 \end{array} \right\}$ the difference of those numbers.
 $d = 5$ a number less than $c (= 6)$ the lesser of the two numbers first taken.
 $\left. \begin{array}{l} b - d = 3 \\ c - d = 1 \end{array} \right\}$ two numbers whose difference is equal to the difference of the numbers first taken.

Now I say that the Reason (or Proportion) of $c - d$ to $b - d$ is less than that of c to b ; therefore,

The Proposition to be demonstrated, is, that $\frac{c - d}{b - d} < \frac{c}{b}$
Demon.

Demonstration.

By supposition, $c = b$
 Therefore by multiplying c and b severally by d , it follows $dc = bd$
 And by adding bc to each part in the last step, $bc + dc = bc + bd$
 And by subtracting bd from each part, $bc + dc - bd = bc$
 And by subtracting dc from each part in the last preceding step, $bc - bd = bc - dc$
 And by dividing each part of the last step by $b - d$, $\frac{bc - bd}{b - d} = c$
 Wherefore by dividing each part of the last preceding step by b , $\frac{c - d}{b - d} < \frac{c}{b}$
 Which was to be demonstrated.

Fifthly and lastly, from the preceding 3^d and 4th Observations we may deduce this
COROLLARY.

If two cube-numbers be given, such, that the double of the lesser exceeds the greater, then (by the help of the preceding Canon in *Sett.* 14. *Quest.* 38.) two cube-numbers may be found out, whose difference shall be equal to the difference of the given Cubes, and the double of the lesser of the Cubes found out shall be less than the greater of them.

For if two given Cubes, (which I shall call the first pair) be such that the double of the lesser exceeds the greater, we may by the said Canon find out a second pair of Cubes, whose difference shall be equal to the difference of the first pair, (viz. the greater Cube of the second pair shall be less than the greater Cube of the first pair, and the lesser Cube of the second pair shall be less than the lesser Cube of the first pair, and the lesser Cube less than the lesser;) and the lesser Cube of the second pair shall have less Reason or proportion to the greater Cube of the same pair, than the lesser Cube of the first pair hath to the greater of the same pair, (by *Observat.* 4.) But if the double of the lesser Cube of the second pair doth yet happen to exceed the greater of the same pair, then by the help of the second pair of Cubes and the said Canon, we may find a third pair of Cubes, whose difference shall be equal to the common difference of the first and second pairs; and by proceeding in like manner, the double of the lesser of the two Cubes found out; and by proceeding in like manner, the double of the lesser of the two Cubes found out; will at length necessarily be less than the greater, because (as before hath been proved,) the lesser Cube of each pair found out hath less proportion to the greater of the same pair, than the lesser of the next precedent pair (by which the latter were found out) hath to the greater.

QUEST. 39.

Two cube-numbers being given, such, that the double of the lesser is either greater or less than the greater, to divide the difference of the given Cubes into two Rational cube-numbers.

Preparation.

1. When the double of the lesser of the given Cubes exceeds the greater, two others must be found out, (according to the directions following the Corollary in *Observat.* 5. *Quest.* 38.) such, that the difference of these Cubes may be equal to the difference of those given, and that the double of the lesser of the Cubes found out may be less than the greater. Then two cube-numbers being given or found out, such, that the double of the lesser is less than the greater, their difference may be divided into two Rational cube-numbers by the following Resolution, (which is the same in substance with that of the 1st of the 4th Book of *Vietâ's Zeteticus*, and of the first Question of *Bacher* in his Comment upon the fourth Book of *Diophantus*.)

RESOLUTION.

2. Let the sides of the given Cubes (qualified as above) be d the greater, and b the lesser; is supposed) be d^3 and b^3
 3. Therefore the Cubes of those sides are d^3 and b^3
 4. And the difference of the said Cubes is $d^3 - b^3$
 5. For the side of one of the Cubes sought put $d - a$

6. And

6. And for the side of the other Cube fought } $\frac{dd}{bb} a - b$
 put }
 7. Therefore the first Cube is } $-a^3 + 3daa - 3dda + d^3$
 8. And the latter Cube is } $\frac{d^3}{b^3}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$
 9. Therefore the sum of those Cubes is } $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3dda + d^3 - b^3$
 10. Which sum must be equal to the difference of the given Cubes; therefore;
 $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3dda + d^3 = d^3 - b^3.$

11. Which Equation, after due Reduction, gives } $a = \frac{3db^3}{d^3 + b^3}$
 12. Therefore from the first, fifth, sixth and eleventh steps, the sides of the two Cubes fought will be found equal to these known Quantities,

$$Viz. \begin{cases} \frac{d^3 - 2db^3}{d^3 + b^3} = \text{the first side,} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \text{the other side.} \end{cases}$$

13. The same sides will be produced, if instead of the Positions in the fifth and sixth steps, there be put $a - b$ and $d - \frac{bb}{da}a$ for the sides of the Cubes fought. And 'tis evident that each of the sides found out in the twelfth step will be greater than nothing if $2b^3$ be less than d^3 , that is, if the double of the lesser of the two Cubes given be less than the greater, as is supposed in the Preparation to the Resolution of this Question. The sides in the twelfth step, being express'd by words, will give this

CANON.

14. Multiply the excess of the greater of the two given Cubes above the double of the lesser by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser by the side of the lesser; then divide each of those Products by the sum of the said Cubes, and the Quotients shall be the sides of the Cubes fought.

Example 1. in Numbers.

15. Let two such Cubes be given, that the double of the } $8 = d^3$ and $1 = b^3$
 lesser is less than the greater, as, }
 16. The sides of those Cubes are } $2 = d$ and $1 = b$
 17. Then by the Canon, $\begin{cases} \frac{d^3 - 2db^3}{d^3 + b^3} = \frac{a}{3} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{b}{3} \end{cases}$ the sides of the Cubes fought:

The Cubes of those sides $\frac{a}{3}$ and $\frac{b}{3}$, are $\frac{a^3}{27}$ and $\frac{b^3}{27}$, whose sum is 7, which is equal to the difference of the two given Cubes 8 and 1, as was required by the Question.

Example 2.

18. Let it be required to divide 61, which is the difference of these Cubes 125 and 64, into two Rational cube-numbers.

Here, because the double of the lesser Cube exceeds the greater, the Canon above express'd in Sect. 14. is of no force; therefore by the help of the given Cubes, (according to the directions following the Corollary in Observat. 5. Quest. 38.) two other Cubes must be found out, such, that their difference may be equal to 61, to wit, the difference of the given Cubes 125 and 64, and that the double of the lesser of the two Cubes found out may be less than the greater of them: But two such Cubes are these, viz. $\frac{125}{27}$ and $\frac{64}{27}$, whose sides are $\frac{5}{3}$ and $\frac{4}{3}$; then using these Cubes as the Canon in the preceding Sect. 14. of this Quest. 39. doth direct, you will find $\frac{125}{27}$ and $\frac{64}{27}$ for the sides of the Cubes required; for the sum of the Cubes of the said sides is 61, which is equal to the difference of the given Cubes, 125 and 64.

19. Hence

19. Hence it is easie to find four cube numbers, the greatest of which shall be equal to the sum of the other three; for when the difference of two given Cubes is divided into two rational Cubes, these two, together with the lesser of the two given Cubes shall make the greater Cube given. But if four such Cubes be desired in whole numbers, they may be readily found out by the following Canon, which is rais'd by the like manner of arguing as was before used in the second Observation upon the preceding Quest. 38.

CANON.

20. First, take two Cubes in whole numbers, with this caution, That the double of the lesser may be less than the greater, and multiply the excess of the greater Cube above the double of the lesser by the side of the greater; secondly, multiply the excess of the double of the greater Cube above the less by the side of the lesser; thirdly, multiply the sum of the said Cubes by the side of the greater; fourthly, multiply the sum of the same Cubes by the side of the lesser: Then the sum of the Cubes of the first, second and fourth Products shall be equal to the Cube of the third Product.

An example in Numbers.

21. Let two such Cubes be taken, that the double of the lesser } $8 = ddd$
 is less than the greater, as, } $1 = bbb$
 22. The sides of which Cubes are } $2 = d$
 } $1 = b$
 23. Then by the last preceding Canon, the sides of the four }
 Cubes fought, in their least terms will be found these, viz. } $4, 5, 6, 3$
 24. I say the sum of the Cubes of 4, 5 and 3 is equal } $64 + 125 + 27 = 216$
 to the Cube of 6, viz. }

QUEST. 40.

To divide any cube-number, suppose 8, into three Rational cube-numbers.

RESOLUTION.

Take any cube-number less than the given Cube 8, as 1; then (by the preceding Quest. 39.) divide 7 the difference of those Cubes into two Cubes, suppose into these, $\frac{8}{27}$ and $\frac{1}{27}$, whose sum 7 is equal to the difference of the given Cubes 8 and 1.

$$\text{Therefore } \begin{cases} \frac{8}{27} + \frac{1}{27} = 8 - 1, \\ \frac{8}{27} + \frac{1}{27} + 1 = 8. \end{cases}$$

Whereby 'tis manifest that three Cubes, to wit, $\frac{8}{27}$, $\frac{1}{27}$ and 1 are found out, whose sum is equal to the given Cube 8, as was required.

Hence you may easily perceive a way to divide a given Cube into any odd number of Cubes; as to divide a Cube into five Cubes, first divide the given Cube into three Cubes, and then divide one of those three into three Cubes, so the other two, with the three Cubes last found out are five Cubes, whose sum is equal to the Cube first given, in like manner you may divide a Cube into 7, 9, 11, 13, &c. Cubes. But there is not any Rational cube-number that can be divided into two Rational Cube numbers; which negative Proposition the Learned Dr Wallis hath demonstrated.

QUEST. 41.

[This is the 15th of the 4th Book of Vieta's Zeteticis, and the same with Quest. 2. of Bachet in his Comment upon Quest. 2. of the 4th Book of Diophantus.]

Two cube-numbers being given, to find two other cube-numbers whose difference shall be equal to the sum of the given Cubes.

RESOLUTION.

1. Let the sides of the given Cubes be } d the greater,
 } b the lesser.
 2. Therefore the Cubes of those sides are } d^3 and b^3
 3. For the side of the greater Cube fought put } $a + d$
 4. And for the side of the lesser Cube fought put } $\frac{dd}{bb}a - b$

K

5. There;

5. Therefore the greater Cube sought is . . . $a^3 + 3daa + 3dda + d^3$
 6. And the lesser Cube sought is . . . $\frac{d^5}{b^5}aa - \frac{3d^4}{b^5}aa + 3dda - b^3$
 7. Therefore the difference of the Cubes sought is $\frac{b^5a^3 - d^5a^3}{b^5} + \frac{3d^4}{b^5}aa + 3daa + d^3 + b^3$
 8. Which difference must be equal to the summ of the given Cubes, therefore;
 $\frac{b^5a^3 - d^5a^3}{b^5} + \frac{3d^4}{b^5}aa + 3daa + d^3 + b^3 = d^3 + b^3.$

9. Which Equation, after due Reduction, gives $a = \frac{3db^3}{d^3 - b^3}$
 10. Therefore from the first, third, fourth and tenth steps, the sides of the two Cubes sought will be found equal to these known quantities,

$$\text{viz. } \begin{cases} \frac{2db^3 + d^4}{d^3 - b^3} = \text{the greater side;} \\ \frac{2bd^3 + b^4}{d^3 - b^3} = \text{the lesser side.} \end{cases}$$

The same sides will be produced, if instead of the Positions in the third and fourth steps there be put $d + \frac{bb}{da}a$ and $a - b$, and those sides above-express'd by Letters give this

CANON.

11. Add the greater of the two Cubes given to the double of the lesser, and multiply the sum by the side of the greater Cube; add also the lesser Cube to the double of the greater, and multiply this sum by the side of the lesser Cube; lastly, divide each of these Products by the difference of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

An Example in Numbers.

Let two Cubes be given, as, . . . $8 = ddd$, and $1 = bbb$
 The sides whereof are . . . $2 = d$, and $1 = b$

Then by the Canon, $\begin{cases} \frac{2db^3 + d^4}{d^3 - b^3} = \frac{22}{7} \\ \frac{2bd^3 + b^4}{d^3 - b^3} = \frac{12}{7} \end{cases}$ the sides of the Cubes sought;

The Cubes of those sides $\frac{22}{7}$ and $\frac{12}{7}$ are $\frac{10648}{343}$ and $\frac{1728}{343}$, whose difference 9 is equal to the summ of the given Cubes 8 and 1.

12. Hence 'tis easie to find out four cube-numbers, the greatest of which shall be equal to the summ of the other three: For when by this Question two Cubes are found out, having their difference equal to the summ of two given Cubes, the lesser of the two Cubes found out, together with the two given Cubes shall be equal to the greater of the Cubes found out. But if four such Cubes be desired in whole numbers, they may be readily found out by the following Canon, which is rais'd by the like manner of arguing as was before us'd in *Observat. 2. Quest. 38.*

CANON.

13. First take any two Cubes in whole numbers, add the greater to the double of the lesser and multiply the sum by the side of the greater; secondly, add the lesser Cube to the double of the greater, and multiply this sum by the side of the lesser; thirdly, multiply the difference of those Cubes by the side of the greater; fourthly, multiply the said difference by the side of the lesser Cube: Then the summ of the Cubes of the three latter Products shall be equal to the Cube of the first Product.

An Example in Numbers.

Let two Cubes in whole numbers be taken at pleasure, as $8 = ddd$, and $1 = bbb$
 The sides of those Cubes are . . . $2 = d$, and $1 = b$
 Then by the Canon last afore-going, the sides of the 20 , 17 , 14 , 7
 four Cubes sought will be found these, to wit, . . .
 I say the Cube of 20 is equal to the summ of the Cubes of 17 , 14 and 7 , viz. . . . $8000 = 4913 + 2744 + 343$

QUEST. 42.

QUEST. 42.

Two cube-numbers being given, such, that the double of the lesser is less than the greater, to find out two other Cubes whose difference shall be equal to the difference of the given Cubes. (But how this is to be done when the double of the lesser Cube exceeds the greater, hath already been shewn in *Quest. 38.*)

RESOLUTION.

1. Let there be two Cubes given, to wit, . . . $\begin{cases} ddd = 8 \\ bbb = 1 \end{cases}$
 2. By the Canon in *Secl. 14.* of the foregoing *Quest. 39.* find out two Cubes whose summ shall be equal to the difference of the given Cubes, such are these, . . . $\begin{cases} ccc = \frac{64}{27} \\ ggg = \frac{128}{27} \end{cases}$
 Therefore by that Construction, . . . $ddd - bbb = ccc + ggg = 8 - 1 = 7$.
 3. By the Canon in *Secl. 11.* of the foregoing *Quest. 41.* find two Cubes whose difference shall be equal to the summ of the Cubes ccc and ggg , (found out in the preceding second step,) such are these,
 $\begin{cases} kkk = \frac{201284656}{6128487}, \text{ whose side is } \frac{1266}{183} \\ lll = \frac{198185216}{6128487}, \text{ whose side is } \frac{1266}{183} \end{cases}$

Therefore by this Construction, . . . $kkk - lll = ccc + ggg = 7$

4. Therefore from the second and third steps, (per 1. *Axiom. 1. Elem. Euclid.*)
 $kkk - lll = ddd - bbb = 7$.

I say kkk and lll , that is, $\frac{201284656}{6128487}$ and $\frac{198185216}{6128487}$, (whose sides are $\frac{1266}{183}$ and $\frac{1266}{183}$), will solve the Question proposed; for their difference 7 is equal to the difference of the given Cubes 8 and 1.

Note. Although by the preceding Resolutions of this and *Quest. 38.* innumerable pairs of cube-numbers may be found out, such, that the difference of each pair shall be equal to the difference of two Cubes given, yet neither of those Resolutions will find out all the pairs of Cubes that have the same difference with two given Cubes; for example, if the Cubes 1728 and 1000 be given, whose difference is 728 , the Canon in the *14th* step of the foregoing *Quest. 38.* will not find out the Cubes 729 and 1 , whose difference is 728 ; although that Canon, with the help of the Resolution of this *Quest. 42.* will find out innumerable pairs of Cubes, such, that the difference of each pair shall be 728 .

QUEST. 43.

To divide a given number 28 compos'd of two cube numbers 27 and 1, into two other Rational cube-numbers.

[This Question was propounded in 1657, by Mons^r. Fermat, (as appears by an Epistolical Commerce printed at Oxford in 1658.) but his way of solving it came not to light, till it was publish'd (after his death) among other his Analytical Inventions, by way of Supplement to Mons^r. Bachet's Comment upon Diophantus, printed at Tholose in 1670. yet the very same way of solving this Question was found out long before by our Learned Dr John Wallis, (though, it seems, not timely enough to have been infer'd in the little Book above-mentioned,) and likewise by my self, before I had seen or heard of any Solution to the said Question, in such manner as here follows.]

RESOLUTION.

1. Let the Cubes 27 and 1, whose summ makes the given number 28, be represented by ddd and bbb , viz. $\begin{cases} ddd = 27 \\ bbb = 1 \end{cases}$
 2. By the Canon in *Secl. 11.* of the foregoing *Quest. 41.* find two cube-numbers whose difference may be equal to 28 the summ of the given Cubes 27 and 1, that is, ddd and bbb , such are these Cubes,

$$\text{viz. } \begin{cases} ggg = \frac{658603}{17576}, \text{ whose side is } \frac{87}{26} \\ ccc = \frac{166275}{17576}, \text{ whose side is } \frac{55}{26} \end{cases}$$

Therefore, . . . $ggg - ccc = ddd + bbb = 27 + 1 = 28$.
 K 2 3. By

3. By the foregoing *Quest.* 39. find out two Cubes whose sum shall be equal to 28 the difference of the two Cubes *ggg* and *ccc*, such are these,

$$kkk = \frac{25345235274412980702625}{9864820937041015055552}$$

$$lll = \frac{2276260963735440852831}{9864820937041015055552}$$

The sides of which Cubes are these, to wit, $\left\{ \begin{array}{l} k = \frac{612884101}{21446628} \\ l = \frac{28142}{21446628} \end{array} \right.$

Therefore, $ggg - ccc = kkk - lll$.

4. But by Construction in the 2^d step, $ggg - ccc = ddd + bbb = 27 + 1 = 28$.
5. Therefore from the two last Equations, (per 1. Axioms. 1. Elem. Euclid.) $kkk - lll = ddd + bbb = 27 + 1 = 28$.

Whence it is manifest that the two Cubes found out, to wit, *kkk* and *lll*, (which with their sides are before severally express'd by numbers in the third step,) will solve the Question, for their sum makes 28, which is the sum of the given Cubes 27 and 1. And because by the help of the known Cubes *ggg* and *ccc* in the second step, divers pairs of Cubes having the same difference with the said *ggg* and *ccc* may be found out, (by the 38th or 42^d Question foregoing:) Therefore by the help of any of the pairs of Cubes so found out, their difference may be divided into two Cubes whose sum shall be equal to the sum of the given Cubes 27 and 1.

Another Example.

Let it be required to divide 9, which is compos'd of the Cubes 8 and 1, into two other Cubes.

RESOLUTION.

1. Let the Cubes 8 and 1, whose sum makes the given number 9, be $ddd = 8$ represented by *ddd* and *bbb*, viz. $\left\{ \begin{array}{l} ddd = 8 \\ bbb = 1 \end{array} \right.$
2. By the Canon in Sect. 11. *Quest.* 41. of this Book, find out two Cubes whose difference may be equal to 9, the sum of the given Cubes 8 and 1, such are these Cubes,

$$\text{Viz. } \left\{ \begin{array}{l} ggg = \frac{8000}{343}, \text{ whose side is } \frac{20}{7} \\ ccc = \frac{4913}{343}, \text{ whose side is } \frac{17}{7} \end{array} \right.$$

Therefore, $ggg - ccc = ddd + bbb = 8 + 1 = 9$.

3. Then by the preceding 39th Question divide the difference of the Cubes *ggg* and *ccc* into two rational Cubes, viz. divide 9 the difference of the Cubes $\frac{8000}{343}$ and $\frac{4913}{343}$ into two Cubes: But here because the double of the lesser Cube $\frac{4913}{343}$ exceeds the greater $\frac{8000}{343}$, two Cubes must first be found out, (by the help of the foregoing *Quest.* 38.) that the difference of these may be equal to the difference of those, and that the double of the lesser of the Cubes found out may be less than the greater, such are these Cubes,

$$\text{Viz. } \left\{ \begin{array}{l} mmm = \frac{6695590844626239}{738542637646471}, \text{ whose side is } \frac{188479}{90391} \\ nnn = \frac{48707103808000}{738542637646471}, \text{ whose side is } \frac{36220}{90391} \end{array} \right.$$

4. Now so far as the double of the lesser of the two Cubes last found out is less than the greater, we may by the help of the preceding *Quest.* 39. divide 9 the difference of those Cubes *mmm* and *nnn*, (and likewise of *ggg* and *ccc*) into two rational Cubes, whose sides will be found these,

$$\text{Viz. } \left\{ \begin{array}{l} k = \frac{1243617733900094826481}{669623835676137297449} \\ l = \frac{4872615171435336560}{669623835676137297449} \end{array} \right.$$

5. Therefore, by Construction in the two last preceding steps, $ggg - ccc = kkk - lll$.
6. But by Construction in the second step, $ggg - ccc = ddd + bbb = 8 + 1 = 9$.
7. Therefore from the two last Equations, $kkk - lll = ddd + bbb = 8 + 1 = 9$.

Thus two Cubes (whose sides *k* and *l* are above-express'd in numbers) are found out, which added together make 9, the sum of the given Cubes 8 and 1, as was required.

QUEST. 44.

QUEST. 44.

To divide the double of any given cube-number into four cube-numbers. For example, let it be required to divide 54 the double of the Cube 27, into four cube-numbers.

RESOLUTION.

1. For the given Cube 27 put *ddd*, viz. suppose $\left\{ \begin{array}{l} ddd = 27 \\ 2ddd = 54 \end{array} \right.$
2. Therefore the double of that Cube is $2ddd = 54$.
3. Take any cube-number less than the given Cube 27, as 1, for $bbb = 1$ which put *bbb*, viz. suppose $\left\{ \begin{array}{l} bbb = 1 \\ ddd + bbb = 28 \end{array} \right.$
4. By the foregoing *Quest.* 43. find two Cubes whose sum may be equal to $ddd + bbb$, (to wit, $27 + 1$), suppose those which solved the said *Quest.* 43. in Example 1.

$$\text{Viz. } \left\{ \begin{array}{l} kkk = \frac{25345235274412980702625}{9864820937041015055552} \\ lll = \frac{2276260963735440852831}{9864820937041015055552} \end{array} \right.$$

The sides of which Cubes are these, to wit, $\left\{ \begin{array}{l} k = \frac{612884101}{21446628} \\ l = \frac{28142}{21446628} \end{array} \right.$

Therefore by that Construction, $ddd + bbb = kkk + lll = 28$.

5. By *Quest.* 39. of this Book divide 26 the difference of the Cubes 27 and 1, to wit, $ddd - bbb$ into two Cubes, suppose into these,

$$\text{Viz. } \left\{ \begin{array}{l} rrr = \frac{421871}{21952}, \text{ whose side is } \frac{75}{28} \\ sss = \frac{148877}{21952}, \text{ whose side is } \frac{53}{28} \end{array} \right.$$

Therefore by this Construction, $ddd - bbb = rrr + sss = 26$.

6. Therefore by adding together the Equations $2ddd = kkk + lll + rrr + sss = 54$ in the fourth and fifth steps, this will arise, viz. $2ddd = kkk + lll + rrr + sss = 54$

Therefore four Cubes are found out, to wit, *kkk*, *lll*, *rrr* and *sss*, which with their sides are before express'd in numbers in the fourth and fifth steps, the sum of whose Cubes makes 54, which is equal to the double of the Cube 27 given, as was required by the Question.

QUEST. 45. (Quest. 17. Lib. 4. Diophant.)

To find out three numbers whose sum may make a Square, and that the second number added to the Square of the first may make a Square; also, that the third number added to the Square of the second may make a Square; and lastly, that the first number added to the Square of the third may make a Square.

RESOLUTION.

1. For the first number sought put a — any known number, as, $\left\{ \begin{array}{l} a - 1 \\ aa - 2a + 1 \end{array} \right.$
2. The Square thereof is $aa - 2a + 1$.
3. To which Square if $+ 4a$ be added, (to wit, the double of $- 2a$, but with the contrary sign $+$), it makes a Square, to wit, $aa + 2a + 1$.
4. Therefore for the second number put $4a$.
Whereby one of the conditions in the Question is satisfied, for the second number $4a$ added to the Square of the first number $a - 1$ makes the Square $aa + 2a + 1$, whose Root is $a + 1$.
5. Then form a Square from $4a + 1$, (which is the sum of the second number $4a$ and the known number 1 in the first assumed number $a - 1$, but with the contrary sign $+$), the known number 1 will be $16aa + 8a + 1$; from which subtract the Square of the second number $4a$, to wit $16aa$, and put the Remainder $8a + 1$ for the third number: Whence it is evident, that if this third number be added to the Square of the second, the sum is a Square, whereby another of the conditions in the Question is satisfied.
6. From the first, fourth and fifth steps the sum of the three numbers sought is $13a$, which according to the Question must be a Square, let it therefore be equated to some Square, viz. suppose $13a = 169aa$, whence $a = 13aa$; now according to this value

value of a , the first number which was put $a - 1$ will be $13aa - 1$, the second number which was put $4a$ will be $52aa$; and lastly, the third number which was assumed $8a + 1$ will be $104aa + 1$. It remains that the Square of the third number $104aa + 1$, to wit, $10816aaa + 208aa + 1$, added to the first number $13aa - 1$ may make a Square; but it makes $10816aaa + 221aa$, this therefore must be equated to a Square, or the same divided by aa gives $10816aa + 221$ to be equated to a Square, whose side, to the end that a may be greater than $\sqrt{11}$, and consequently $13aa$ greater than 1 , may be feigned to be $104aa + 1$ any known number less than 475 , or $104a$ — any known number greater than 6075 ; let therefore the side of the said Square be feigned $104a + 1$, whence the Square it self is $10816aa + 208a + 1$, which being equated to the aforesaid $10816aa + 221$, will give $a = \frac{11}{2}$. Therefore the positions being resolved, the first number will be $\frac{11}{2} \times \frac{11}{2} = \frac{121}{4}$, the second $\frac{11}{2} \times 4 = \frac{22}{1}$, the third $\frac{11}{2} \times 8 + 1 = \frac{55}{1}$; which three numbers will solve the Question, for their sum is $\frac{121}{4} + \frac{22}{1} + \frac{55}{1} = \frac{121 + 88 + 220}{4} = \frac{429}{4}$, the Square of the side $\frac{21}{2}$; also the Square of the first number, to wit, $\frac{121}{4}$, added to the second makes the Square $\frac{429}{4}$ from the side $\frac{21}{2}$; moreover the Square of the second, to wit, 22 , added to the third makes the Square 77 from the side $\frac{11}{2}$; lastly, the Square of the third, to wit, 3025 , added to the first makes the Square 3136 from the side 56 .

QUEST. 46.

To find three numbers, that as well the sum of every two, as of all three, may make a Square.

RESOLUTION.

- Let b represent any known number, and a some number unknown, then from $a + 1$ some even number of b , (for avoiding Fractions) as from $a + 2b$ form a Square, which will be $aa + 4ba + 4bb$.
- Then for the first number sought put the two first terms of the said Square, as $aa + 4ba$.
- Then take the half of the said $4ba$, to wit, $2ba$, and prefixing the sign $-$ to it, it makes $-2ba$, to which add bb the Square of half the Coefficient $2b$, and take the sum for the second number sought, to wit, $-2ba + bb$.
- Subtract bb in the said second number from $4bb$ part of the Square first formed, and add the Remainder $3bb$ to $-2ba$, to wit, the same multitude of ba as is in the second number, but with a contrary sign, and put this sum for the third number sought, to wit, $-2ba + 3bb$.
- Then from the premises it necessarily follows, that the sum of the first and second numbers (in the second, and third steps) makes a Square, to wit, $aa + 2ba + bb$.
- And the sum of the second and third numbers (in the third and fourth steps) is a Square, to wit, $-2ba + 3bb + -2ba + bb = -4ba + 4bb$.
- Also the sum of all the three numbers is a Square, to wit, $aa + 4ba + 4bb - 2ba + bb = aa + 2ba + 5bb$.
- It remains that the sum of the first and third numbers make a Square, but it makes $aa + 6ba + 3bb$, which must be equated to a Square, yet so as the value of a may be less than $\frac{3b}{2}$, to the end that the second number $-2ba + bb$ may be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $-a +$ any number between $\frac{3b}{2}$ and $3b$, (as may be collected from the Canon in *Quest. 15. Quest. 12. of this Book.*) Let therefore the said side be $-a + 2b$, and then its Square $aa - 4ba + 4bb$ being equated to $aa + 6ba + 3bb$ (the sum of the first and third numbers) this Equation arithmetically, to wit, $aa - 4ba + 4bb = aa + 6ba + 3bb$.
- Whence after due Reduction, the value of a is made known, viz. $a = \frac{7b}{5}$.
- Therefore, the positions in the second, third and fourth steps being resolved according to that value of a , the three numbers sought are discovered, to wit, $\frac{49}{25}bb$, $\frac{3}{5}bb$ and $\frac{24}{25}bb$.

Hence this

CANON.

CANON.

Take any Square number, then $\frac{1}{25}$ of that Square, also $\frac{3}{5}$ of the same Square, and $\frac{1}{5}$ thereof, will give three numbers to solve the Question.

As, for example, if 10 be taken for the side of a Square, then these three numbers will be found out by the Canon, to wit, 41, 80 and 320, which will solve the Question: For the sum of 41 and 80 makes the Square 121, whose side is 11; also the sum of 80 and 320 makes the Square 400, whose side is 20; and the sum of 320 and 41 makes the Square 361, whose side is 19; lastly, the sum of all the three numbers 41, 80 and 320 makes the Square 441, whose side is 21. In like manner you may find out as many Answers in whole numbers as you please, by taking 20, 30, 40 or 50, &c. for the side of a Square, and then taking such parts thereof as the Canon directs.

QUEST. 47. (Quest. 23. Lib. 4. Diophant.)

To find three numbers, that if they be severally added to the Solid produced by their continual multiplication, every one of the three sums may be a Square.
(I shall wave Diophantus's Resolution, and use that of Fermat in his Observation upon this Question, which is much easier.)

RESOLUTION.

- Let a Square be formed from a — any known number, as from $aa - 2a + 1$, whose Square is $aa - 2a$.
- Then for the Solid of the three numbers sought put the two first terms of that Square, to wit, $aa - 2a$.
- And for the first number sought put the last term of the said Square, to wit, 1 .
Whence one of the conditions is satisfied; for if the said first number 1 be added to $aa - 2a$, (that is, the Solid of all the three numbers,) the sum is a Square, to wit, that first formed.
- For the second number put $\frac{1}{2}a - 1$, (the third number.)
This added to the said Solid $aa - 2a$ makes the Square aa , whereby another of the conditions in the Question is satisfied.
- Then divide $aa - 2a$, (the Solid of all the three numbers) by $2a$ the Product of the first and second, so the Quotient is $\frac{1}{2}a - 1$.
- Which third number added to the Solid of all the three must also make a Square, but it makes $aa - \frac{1}{2}a - 1$.
- Therefore $aa - \frac{1}{2}a - 1$ must be equated to a Square, yet so, as the value of a may be greater than 2, to the end that the third number $\frac{1}{2}a - 1$ may be greater than nothing: But to cause that effect, the side of the said Square may be feigned a — any number less than 2, but greater than $\frac{1}{2}$; or a — any number greater than 2; let then the said side be feigned $a - 3$, whose Square equated to $aa - \frac{1}{2}a - 1$, will give $a = \frac{10}{3}$. According to which value, the Positions being resolved, the first number sought is 1, the second $\frac{4}{3}$, and the third $\frac{5}{3}$, which will solve the Question: For the Solid Product of their multiplication one into another, to wit, $\frac{20}{27}$, taking to it severally the said three numbers, makes the Squares $\frac{49}{9}$, $\frac{16}{9}$ and $\frac{25}{9}$.

QUEST. 48. (Quest. 31. Lib. 4. Diophant.)

To find four square numbers, whose sum added to the sum of their sides may make a number given, suppose 12.

RESOLUTION.

Forasmuch as (by the first Proposition in the following Observation upon this Quest.) every Square increased with his side and $\frac{1}{4}$ of unity makes a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square, therefore the sum of the four Squares sought together with four times $\frac{1}{4}$ will make four Squares, but the given number 12 increased with four times $\frac{1}{4}$, to wit, 1, makes 13. Therefore we must divide 13 into four Squares; then if from every one of their sides we subtract $\frac{1}{2}$, there will remain the sides of the four Squares sought. But 13 is compos'd of two Squares 4 and 9, therefore

therefore (by the first Question of this Book) each of these may be divided into two Squares, *viz.* 4 into $\frac{25}{4}$ and $\frac{9}{4}$, and 9 into $\frac{16}{9}$ and $\frac{4}{9}$: now the four Roots of those Squares are $\frac{5}{2}$, $\frac{3}{2}$, $\frac{4}{3}$ and $\frac{2}{3}$, from each of which Roots if $\frac{1}{2}$ be subtracted there will remain the sides of the four Squares sought, to wit, the sides $\frac{3}{2}$, $\frac{5}{6}$, $\frac{5}{6}$ and $\frac{1}{3}$, whose Squares $\frac{9}{4}$, $\frac{25}{36}$, $\frac{25}{36}$ and $\frac{1}{9}$ will solve the Question: For if their sum 7 be added to 5 the sum of their sides, it makes the given number 12; which was required.

Observations upon Quest. 48.

The preceding Resolution depends upon two Propositions, *viz.*

First, if any square number be increased with its side and $\frac{1}{4}$ of unity the sum will be a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square. This may be demonstrated thus;

Let there be a Square

Then to that Square add its side and $\frac{1}{4}$ of unity, to wit, $aa + a + \frac{1}{4}$

So the sum makes this Square, $aa + a + \frac{1}{4}$

Whole Root is $a + \frac{1}{2}$

From which Root if you subtract $\frac{1}{2}$

The Remainder is the side of the first Square, to wit, a

Therefore the Proposition is manifest.

Secondly, the said Resolution takes this Proposition for granted, *viz.* That any given whole number increased with 1 may be divided into four Squares; how this may be generally done *Diophantus* doth not shew: But 'tis evident, that if a given square number increased with 1 makes a Square, or a number composed of two Squares, then the sum may easily be divided into four Squares, or into as many as you please, (by the first or second Question of this Book.) But if 13 be given, how shall 13 increased with 1, that is 14, which is neither a Square nor composed of two Squares, be divided into four Squares? This at first sight seems to be a very hard Task, but if the matter be narrowly considered, the difficulty will soon vanish, for 14 is compos'd of three Squares, to wit, 1, 4, 9; wherefore if one of these be divided into two Squares, then consequently 14 is divided into four Squares. But *Fermat* in his Observation upon this Quest. 31. of the fourth Book of *Diophantus*, affirms that every whole number is either a Square, or else compos'd of two, three or four Squares, and the promise to give the Demonstration of this and other abstruse Mysteries in Numbers in a particular Treatise. *Bachet* confesseth he could not demonstrate the same, but gives Examples of the certainty thereof in all whole numbers from 1 to 120, and saith he had made experiment of all whole numbers to 325. If this Prop. be granted, then the Question may be easily extended to five, six, or as many Squares as you please, without any Determination. But if two Squares only be desired, whose sum with the sum of their sides may make a Square, then after 2 is added to the quadruple of the given number, the sum must be compos'd of two Squares: And if three Squares be sought, then after 3 is added to the quadruple of the given number the sum must be compos'd of three Squares, which conditions are manifest from the first Proposition above express.

QUEST. 49.

[This is the 34th of the fourth Book of *Diophantus*, and the 13th of the fifth Book of *Viete's Zeeticks*.]

To divide a given number x into two such parts, that if the first part be increased with a given number b , and the other part with a given number d ; the Product made by the multiplication of the two sums one into the other may be a square number.

RESOLUTION.

1. For the first part sought put $a - b$
2. Therefore the other part, to the end the sum of the parts may make x , shall be $x - a + b$
3. Then (according to the Question) adding b to the first part, the sum is a
4. And adding d to the second part, the sum is $x - a + b + d$
5. There.

5. Therefore the Product made by the multiplication of those two sums one into the other will be $xa - aa - b - ba - d - da$
6. That is $x - b - d$ into $a - a$
7. Which Product must be equated to a Square, whose side may be feigned ss , and then the Square of ss , to wit, $ssaa$, being equated to the Product in the sixth step, this Equation arith, *viz.* $x - b - d$ into $a - a = ssaa$.

8. Which Equation, after due Reduction, gives $a = \frac{x - b - d}{ss + 1}$

9. Therefore from the eighth and first steps, the first part sought will be made known, to wit, $\frac{x - d - ssb}{ss + 1}$

10. And from the eighth and second steps the second part sought will be discovered, to wit, $\frac{ssx - d - ssb - d}{ss + 1}$

But to the end that each of the two parts in the ninth and tenth steps may be greater than nothing, the number s cannot be taken at pleasure, but within the limits hereafter discovered, *viz.*

11. Forasmuch as the numerator in the ninth step requires that $\frac{x - d}{b} < ss$
12. Therefore by dividing each part in the last step by b , $\frac{x - d}{b} < ss$

13. That is, $ss > \frac{x - d}{b}$

14. Therefore by extracting the square Root out of each part in the last step, $s > \sqrt{\frac{x - d}{b}}$

15. Again, the Numerator in the tenth step shews, that $ssx - d - ssb - d < d$

16. Therefore by dividing each part by $x + b$, it follows that $ss < \frac{d}{x + b}$

17. Therefore by extracting the square Root out of each part in the last step, $s < \sqrt{\frac{d}{x + b}}$

18. Thus in the fourteenth and seventeenth steps it is discovered, that for the number s we may take any number between $\sqrt{\frac{x - d}{b}}$ and $\sqrt{\frac{d}{x + b}}$, and then the two desired parts whose sum is equal to x the number to be divided will be such as are before express'd in the ninth and tenth steps.

An Example in Numbers.

Suppose $\begin{cases} 4 = x \\ 12 = b \\ 20 = d \end{cases}$ numbers given in the Question;

Whence $\frac{4}{3} = s$ a number chosen within the limits in the eighteenth step.

Then by the help of those known numbers, the ninth and tenth steps will give $\frac{11}{3}$ and $\frac{2}{3}$ the two parts sought, whose sum is 4, (or x ;) the former of which parts increased with 12 (or b ;) and the latter with 20 (or d ;) make the two sums $\frac{25}{3}$ and $\frac{22}{3}$; these multiplied one by the other produce a Square whose side is $\frac{55}{3}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 50. (Quest. 35. Lib. 4. *Diophant.*)

To divide a given number into three numbers, such, that if the Product of the multiplication of the first into the second be increased and lessened by the third, as well the sum as the remainder may be a square number.

RESOLUTION.

1. Let the given number be 6
2. For the third number sought put a
3. For the second number sought put some known number less than the given number 6, as 2
4. Therefore, because all the three numbers sought must make 6, the first number shall be $4 - a$
5. Then

5. Then (according to the Question) the Product of the first and second numbers sought, together with the third, must make a Square, viz. $8 - a = \square$
6. Also the same Product lessened by the third number must make a Square, viz. $8 - 3a = \square$
7. So in the two last steps we are fallen upon a Duplicate equality, but 'tis inexplicable; Diophantus therefore seeks out another Duplicate equality wherein the numbers prefix to a may have such proportion to one another as a square number hath to a square number, for then it will be resolvable like that Duplicate equality which hath been already explain'd in *Quest. 35.* of this Book. First, then instead of 2 which was assumed for the second number sought, some other number less than 6 must be taken, such, that if it be increased and lessened by unity, the sum may be to the remainder as a square number to a square number; (for if the rise of 3 and 1, which are prefix to a in the Duplicate equality above express'd, be examined, it will appear that 3 aritheth by adding 1 to 2, and 1 aritheth by subtracting 1 from the same number 2.) Therefore let e represent some number to be taken instead of 2 for the second number sought; then $e - 1$ must be to $e - 1$, as a Square to a Square, suppose as bb to dd . Therefore, as $e - 1 :: bb :: dd$
8. Therefore by comparing the Product of the extremes to the Product of the means, $dde + dd = bbe - bb$
9. Whence after due Reduction, $e = \frac{bb - dd}{bb - dd}$
10. But the value of e must be less than 6, which I shall call f , therefore $\frac{bb - dd}{bb - dd} \Rightarrow f$
11. Therefore by multiplying each part by f , $bb - dd \Rightarrow f \cdot bb - dd$. Hence this Canon to find out the number e , viz.
12. Take any two square numbers whose sum may be less than the Product of their difference multiplied into the number given in the Question; then divide the sum of the said Squares by their difference, so the Quotient shall be the number to be put for the second number sought by the Question proposed, for it shall be less than the number given in the Question; and if it be increased with 1 and lessened by 1, the sum shall be to the remainder as the greater of the Squares taken is to the lesser. Therefore I take the Squares 4 and 1 and divide their sum 5 by their difference 3, so there aritheth $\frac{5}{3}$ for the second number. Now let the Positions be renewed thus, viz.
13. The number given to be divided into three numbers is 6
14. For the second number put $\frac{5}{3}$
15. And for the third number a
16. Then the sum of the second and third numbers subtracted from 6 (the sum of all three) leaves the first number $1\frac{1}{3} - a$
17. Now (according to the Question) the Product of the first and second numbers together with the third, must make a Square, viz. $a\frac{5}{3} - \frac{5}{3}a = \square$
18. And the same Product lessened by the third number must leave a Square, viz. $a\frac{5}{3} - \frac{5}{3}a = \square$
19. So in the two last steps there is a Duplicate equality wherein the numbers prefix to a have such proportion one to another as a Square to a Square, for $\frac{5}{3}$ is to $\frac{5}{3}$ as 1 to 4; and therefore this Duplicate equality may be resolved like that in the preceding *Quest. 35.* in this manner, viz. Forasmuch as a square number multiplied by a Square produceth a Square, therefore to take away the Fractions, multiply all the Quantities in the 17th and 18th steps by the Denominator 9, so this Duplicate equality aritheth, viz. $\begin{cases} 65 - 6a = \square \\ 65 - 24a = \square \end{cases}$
20. Then the former of those two Equations multiplied by 4 produceth $260 - 24a = \square$, so at length this Duplicate equality remains to be resolved, viz. $\begin{cases} 260 - 24a = \square \\ 65 - 24a = \square \end{cases}$
21. Now the difference of these two Equations being 195, I seek by *Canon 1. Quest. 7.* of this Book two such square numbers that their difference may be 195, and that the greater Square may be less than 260. But the only pair of Squares in whole numbers

so qualified are 196 and 1, the greater of which being equated to $260 - 24a$, or the lesser to $65 - 24a$, will give $a = \frac{2}{3}$ for the third number sought; and consequently, by the Positions in the fourteenth and sixteenth steps, the first and second numbers are $\frac{2}{3}$ and $\frac{5}{3}$, which three numbers will solve the Question, as is evident by the Proof, for their sum is 6; also if $\frac{2}{3}$ the Product of the first and second be increased with the third number $\frac{5}{3}$ it makes the Square $\frac{25}{9}$; but if the same Product be lessened by the said $\frac{5}{3}$ it leaves the Square $\frac{1}{9}$.

QUEST. 51. (Quest. 36. Lib. 4. Diophant.)

To find three numbers, whereof the third may be such a Fraction of unity, that if the first number takes from the second such part or parts as the Fraction expresseth, the sum may be to the remainder in a given Reason, suppose as b to d . Also, that the second number taking the same part or parts from the first, the sum may be to the remainder in a given Reason, suppose as f to g . But the Product made by the mutual multiplication of the first term of each Reason must exceed the Product of the latter terms one into the other, viz. bf must be greater than dg .

Preparation.

Let a and e represent the first and second numbers, and u the third, or Fraction sought; then because to take any part or parts of a number, the number must be multiplied by the Fraction expressing the parts; the Question may be stated thus, viz.

1. If $a + ue = ne :: b : d$,
2. And $e + na = na :: f : g$.
What are the numbers, a, e, u ?

RESOLUTION.

3. By comparing the Product of the extremes to the Product of the means in the first Analogy, this Equation is produced, to wit,
 $da + due = be - bue$.
4. Likewise from the latter Analogy this Equation is produced, viz.
 $ge + gna = fa - fna$.
5. From the Equation in the third step by transposition of due , this aritheth,
 $da = be - bue - due$.
6. And by dividing each part of the last Equation by d , this aritheth,
 $a = \frac{be - bue - due}{d}$.
7. Then by exchanging a in the fourth step, for that which is equal to a in the last Equation, and multiplying all into d , the Equation in the fourth step will be converted into this; viz.
 $dge + bgue - bgue = \begin{cases} bfe - 2bfue - dfue \\ - dfue \end{cases} = \begin{cases} bfe - 2bfue - dfue \\ + bfue + dfue \end{cases}$.
8. From which Equation, after due Reduction, this following Equation aritheth, wherein u only is unknown, viz.
 $\frac{bg + 2bf + df}{bg + dg + bf + df} u - nu = \frac{bf - dg}{bg + dg + bf + df}$.
9. Which last Equation may be resolved (by the Canon in *Sett. 10. Chap. 15. Book 1.*) in this manner, viz. half the Coefficient drawn into u in the said Equation is
 $\frac{\frac{1}{2}bg + \frac{1}{2}bf + \frac{1}{2}df}{bg + dg + bf + df}$.
10. The Square of the said half Coefficient is
 $\frac{\frac{1}{4}bbg + \frac{1}{4}bbf + \frac{1}{4}ddff + \frac{1}{2}bbf + \frac{1}{2}bdf + \frac{1}{2}bdf}{bg + dg + bf + df}$ into $bg + dg + bf + df$.
11. Then to reduce the known Absolute quantity which solely possesseth the latter part of the Equation in the eighth step to the same Denominator with the Square in the tenth step, I multiply as well the Numerator as the Denominator of the said Absolute quantity, by its Denominator $bg + dg + bf + df$, and it makes
 $\frac{bbfg + bbff + bddf - bbgg - ddgg - ddff}{bg + dg + bf + df}$ into $\frac{bg + dg + bf + df}{12}$.
12. Which

12. Which Fraction last above express being subtracted from the Fraction in the tenth step will leave this that follows, to wit,

$$\frac{\frac{1}{2}bbg + \frac{1}{2}ddf + dgg + \frac{1}{2}bdfg + bbg + ddf}{bg + dg + bf + df} \text{ into } \frac{bg + dg + bf + df}{bg + dg + bf + df}.$$

13. The Square Root of the Fraction in the twelfth step is

$$\frac{\frac{1}{2}bg + \frac{1}{2}df + dg}{bg + dg + bf + df}.$$

14. Which Square Root being added to and subtracted from the half-Coefficient in the ninth step, the sum and remainder shall be the two values of u in the Equation in the eighth step, viz.

$$u = 1; \text{ also, } u = \frac{bf - dg}{bg + dg + bf + df}.$$

The latter of which values of u , to wit, $\frac{bf - dg}{bg + dg + bf + df}$ is the Fraction sought by the Question.

15. Then according to the said lesser value of u , the compound quantity $\frac{b - bu - du}{d}$ into which e is multiplied in the latter part of the Equation in the sixth step will be reduced into this fractional quantity, to wit, $\frac{bbg + bbg + ddf}{bg + dg + bf + df}$, which multiplied into any number taken at pleasure for the value of e , will give the number a ; and therefore to find out a and e in whole numbers we may take the Numerator of that fractional quantity for a , and the Denominator for e , or any two whole numbers in the same proportion with the said Numerator and Denominator, and the lesser of the two values of u before found for the Fraction sought.

An Example in Numbers.

16. Let there be given $\begin{cases} 3 = b \\ 1 = d \\ 5 = f \\ 1 = g \end{cases}$

17. Then the three numbers sought will be these, viz. $\begin{cases} 16 = bbg + 2bdf + ddf \\ 24 = bbg + ddf + bdf + ddf \\ 12 = \frac{bf - dg}{bg + dg + bf + df} \end{cases}$

18. Which three numbers, to wit, 16, 24 and $\frac{12}{2}$ will solve the Question, as will be manifest by

The Proof.

19. If the first number 16 receive $\frac{1}{2}$ of the second number 24, that is, 12, the sum 30 will be to the remainder 10 as 3 to 1, (viz. as b to d ;) Again, if the second number 24 take $\frac{1}{2}$ parts of the first number 16, to wit, 8, the sum 16 will be to the remainder 8 as 5 to 1, (to wit, as f to g ;) Or instead of 16 and 24 you may take 2 and 3, or any two numbers in the same Proportion.

20. Again, if $b = 3$, $d = 2$, $f = 4$, and $g = 5$, then the literal quantities in the preceding 17th step will give these three numbers, 125, 90 and $\frac{1}{2}$ to solve the Question; or instead of 125 and 90 you may take 25 and 18, or any two numbers in the same Proportion.

Note. If in reducing the Equation in the seventh step, bf were supposed either equal to dg or less than dg , there would come forth an Equation wherein the value of u would be either unity or greater than unity, but according to the import of the Question it ought to be less than unity, and such is the lesser value of u in the Equation in the eighth step, where bf is supposed greater than dg , as the Determination annexed to the Question requires.

QUEST. 52. (Quest. 41. Lib. 4. Diophant.)

To find two numbers, that the Product of their multiplication may be to their sum in a given Reason (or Proportion,) suppose as r to s .

RESO-

RESOLUTION.

- For the first number put a
- And for the second e
- Then their sum is $a + e$
- And the Product of their multiplication is ae
- But according to the Question the Product must be to the sum as r to s , therefore $ae : a + e :: r : s$
- Therefore by comparing the Product of the extremes to the Product of the means this Equation ariseth, $sa = ra + re$
- And by subtracting re from each part of that Equation, this ariseth, viz. $sa - re = ra$
- And by dividing each part of the last Equation by $sa - r$, this ariseth, viz. $e = \frac{ra}{sa - r}$

Which Equation gives this

CANON.

9. For the first number sought take any number greater than the Quotient that ariseth by dividing the former term of the given Reason by the latter, then multiply the first number so taken by the latter term, and from the Product subtract the former term; lastly, by the Remainder divide the Product made by the multiplication of the first number into the former term, and the Quotient shall be the second number sought. For example, if two numbers be desired that their Product may be to their sum as 3 to 2, (that is, as r to s in the Resolution,) you may take any number greater than $\frac{1}{2}$, as 2, for the first number; then (by the Canon) the other number will be found 6; which numbers 2 and 6 are such, that their Product 12 is to their Sum 8, as 3 to 2, as was desired. After the same manner you may find out innumerable Answers to the Question.

10. But if it were desired to find out two numbers that the Product of their multiplication might be equal to their sum, and that the sum or Product might be a square number, the numbers may be found out thus, viz.

- For the first number sought put a
 And for the second number e
 Then according to the Question, $ae = a + e$
 Therefore by transposition of e , $ae - e = a$
 Therefore each part of the last Equation being divided by $a - 1$, there will arise $e = \frac{a}{a - 1}$
 Which last Equation multiplied by a gives $ae = \frac{aa}{a - 1} (= a + e)$

Which $\frac{aa}{a - 1}$ must (according to the Question) be a square number: But the Numerator aa is a Square; it remains then to equate the Denominator $a - 1$ to some square number, let it be dd , viz. suppose $a - 1 = dd$, whence $a = dd + 1$; according to which value of a , the number e which was before found equal to $\frac{a}{a - 1}$, will be $\frac{dd + 1}{dd}$. Therefore for the two numbers sought take $dd + 1$ and $\frac{dd + 1}{dd}$, which in words give this

CANON.

For one of the numbers sought take any square number increased with unity, then divide that sum by the square number taken, and the Quotient shall be the other number sought.

As, for example, you may take $4 + 1$, that is, 5 for one of the numbers sought; then dividing 5 by 4 (the Square taken,) the Quotient $\frac{5}{4}$ shall be the other number sought. So 5 and $\frac{5}{4}$ will solve the Question last proposed, for their Product is $\frac{25}{4}$, their Sum also is $\frac{25}{4}$, which is a square number as was required.

QUEST. 53.

To find two numbers, that their difference may be equal to the difference of their Squares; and that the sum of the Squares of the two numbers may be a Square.

RESO-

RESOLUTION.

1. For the greater number put a
 2. And for the lesser e
 3. Then their difference is $a - e$
 4. And the difference of their Squares is $aa - ee$
 5. Therefore according to the Question, $aa - ee = a - e$
 6. Therefore by dividing each part of that Equation by $a - e$, the Quotient gives $a + e = 1$
 7. Therefore by transposition of a , $e = 1 - a$
 8. Now taking $1 - a$ instead of e , the two numbers sought are a and $1 - a$
 9. The difference of which numbers is either $2a - 1$ or $1 - 2a$, and the same is the difference of their Squares: But the sum of their Squares must make a Square; therefore $2aa - 2a + 1$ must be equated to a Square, yet so, as the value of a may be less than 1. Now to cause that effect the said side may be feigned $3a - 1$, then the Square of $3a - 1$ being equated to the said $2aa - 2a + 1$ the value of a will be found $\frac{2}{3}$, which subtracted from 1 leaves $\frac{1}{3}$; therefore $\frac{2}{3}$ and $\frac{1}{3}$ are the numbers sought. For as well their difference as the difference of their Squares is $\frac{1}{3}$; and the sum of the Squares of $\frac{2}{3}$ and $\frac{1}{3}$ makes a Square, to wit, $\frac{5}{9}$.
- From the premises aritheth this

CANON.

10. Take any number greater than 2; then divide the excess of that number above unity, by the excess of half the Square of the same number above unity, and the Quotient shall be one of the numbers sought, which subtracted from unity leaves the other number sought. As, for example, take the number 3; then dividing the excess of 3 above 1, that is, 2, by the excess of half the Square of 3 above 1, that is, by $\frac{5}{2}$, the Quotient $\frac{2}{5}$ is one of the numbers sought, which subtracted from 1 leaves $\frac{3}{5}$ for the other number.
 11. The same Question may be propounded thus, viz. To find a right-angled Triangle in Rational numbers, that the difference of the sides about the right-angle may be equal to the difference of the Squares of the same sides. For solving this Question, take any two numbers found out by the said Canon, as $\frac{2}{3}$ and $\frac{1}{3}$ for the sides about the right-angle, whence the Hypotenusal (to wit, the square Root of the sum of the Squares of $\frac{2}{3}$ and $\frac{1}{3}$) is $\frac{\sqrt{5}}{3}$.
- Moreover, from the foregoing Resolution of *Quest. 53.* we may deduce this

THEOREM.

12. If unity be divided into any two parts, the difference of the parts is equal to the difference of the Squares of the same parts: And if the Product made by the mutual multiplication of the parts be subtracted from each of them, each Remainder will be a Square: Also the excess of the greater part above its Square is equal to the excess of the lesser part above its Square, and each excess is equal to the Product of the parts. This will easily be manifested by these two numbers a and $1 - a$, whose sum is unity.

QUEST. 54. (Quest. 45. Lib. 4. Diophant.)

To find three numbers, that the excess of the greatest above the mean may be to the excess of the mean above the least in a given Reason; suppose as 3 to 1; and that the sum of every two of the three numbers may be a Square.

RESOLUTION.

1. Forasmuch as the sum of the mean and least of the three numbers must be a Square, let it be 4
 2. Therefore the mean is greater than 2; for if we should put it 2, the least would be also 2, which is absurd; therefore for the mean let there be put $a + 2$
 3. Therefore from the said positions the least number is $2 - a$
 4. And the excess of the mean above the least is $2a$
 5. But the excess of the greatest above the mean must be the triple of the excess of the mean above the least; therefore from the last step the excess of the greatest above the mean is $6a$
6. Which

6. Which last mentioned excess, to wit, $6a$, being added to the mean $a + 2$, gives the greatest of the three numbers, to wit, $7a + 2$
7. But the Question requires two things more, to wit, that the greatest with the mean may make a Square; and that the greatest with the least may make a Square; hence aritheth this Duplicate equality to be resolved, to wit, $8a + 4 = \square$ and $6a + 4 = \square$
8. Which Duplicate equality hath already been resolved in the preceding *Quest. 33.* of this *Book*, where the number $\frac{2}{3}$ (among innumerable other numbers that might be discovered, and each to be less than 2, as the third step of the preceding Resolution requires,) was found out for the value of a . Therefore $\frac{2}{3}$ being taken for the value of a , and the positions in the sixth, second and third steps resolved accordingly, the three numbers sought will be found these, to wit, $\frac{2}{3}$, $\frac{1}{3}$, and $\frac{\sqrt{5}}{3}$, which will solve the Question. For first, the excess of the greatest above the mean, to wit, $\frac{4}{3}$, is the triple of $\frac{1}{3}$, the excess of the mean above the least; secondly, the sum of the greatest and mean is the Square $\frac{4}{9}$, whose side is $\frac{2}{3}$; thirdly, the sum of the greatest and least is the Square $\frac{5}{9}$, whose side is $\frac{\sqrt{5}}{3}$; lastly, the sum of the mean and least is the Square $\frac{4}{9}$, whose side is $\frac{1}{3}$, that is, 2.

QUEST. 55. (Quest. 2. Lib. 5. Diophant.)

To find three numbers in Geometrical proportion, that every one of them increased with a given number d may make a Square.

RESOLUTION.

1. First, seek a Square which added to the given number d may make a Square, and whose $\frac{1}{2}$ part may exceed d ; suppose it be found bb , let this be put for one of the extreme Proportionals sought, to wit, bb
2. For the other extreme put aa
3. Then because the Product of the extremes is equal to the Square of the mean, therefore the Square of the mean is $bbaa$, whose Root is the mean itself, to wit, ba
4. By Construction in the first step the first extreme bb increased with the given number d makes a Square; but according to the Question the other extreme and the mean being severally increased with the same given number d must also make a Square, whence this Duplicate equality aritheth,

$$\text{viz. } \begin{cases} aa + d = \square \\ ba + d = \square \end{cases}$$

5. Now to resolve this Duplicate equality I proceed as in former Questions; viz. First, the difference of those two Equations is $aa - ba$, which is equal to the Product of a into $a - b$; then if the Square of half the sum of a and $a - b$ be equated to $aa + d$, or the Square of half the difference of a and $a - b$ to $ba + d$, from either of those Equations the value of a will be made known: But half the difference of the said a and $a - b$ is $\frac{1}{2}b$, whose Square is $\frac{1}{4}bb$; let this be equated to $ba + d$, and it will be $ba + d = \frac{1}{4}bb$

6. Whence, after due Reduction, the value of a will be discovered, $a = \frac{\frac{1}{4}bb - d}{b}$
 7. And by multiplying each part of the last Equation by b , it gives $ba = \frac{1}{4}bb - d$
- From the premises aritheth the following Canon to find out the three Proportionals sought;

CANON.

8. Take for one of the extreme Proportionals sought such a Square number that if it be increased with the given number it may make a Square, and that a quarter of the first Square may exceed the given number; then from a quarter of the first Square subtract the given number and there will remain the mean Proportional; lastly, divide the mean by the side of the Square first taken, and the Square of the Quotient shall be the other extreme.

An Example in Numbers.

Suppose $19 = d$ the number given in the Question; then find a Square that if it be increased with 19 may make a Square, and that a quarter of the Square found out may exceed

exceed the said 19, (or that the side of the said Square may exceed $\sqrt{76}$.) But such is the Square 81, (found out after the manner of resolving the ninth Question of this Book,) for 81 increased with 19 makes the Square 100, also $\frac{1}{4}$ of 81 is greater than 19; therefore 81 shall be the first of the three Proportionals sought. Then by the Canon above exprest, the other two will be found $\frac{1}{4}$ and $\frac{1}{12}$. I say, 81, $\frac{1}{4}$ and $\frac{1}{12}$ will solve the Question: For first they are continual Proportionals, in regard the Product of the extremes is equal to the Square of the mean; secondly, the first Proportional 81 increased with the given number 19 makes the Square 100; thirdly, the mean Proportional $\frac{1}{4}$ increased with the said 19 makes the Square $\frac{25}{4}$; lastly, the third Proportional $\frac{1}{12}$ increased with the said 19 makes the Square $\frac{169}{36}$, whose side is $\frac{13}{6}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 56. (Quest. 7. Lib. 5. Diophant.)

To find two numbers, that the Product of their multiplication added to the sum of their Squares may make a Square.

RESOLUTION.

1. For one of the numbers sought take any known number, as b
2. For the other number put a
3. The Square of the first is bb
4. The Square of the second is aa
5. The Product of the multiplication of the two numbers is ba
6. Therefore the sum of the said Squares and Product is $aa + ba + bb$
7. Which sum must be equal to a Square, the side whereof may be feigned to be $a -$ any known number greater than b , let it be $a - d$, and then the Square of $a - d$ being equated to the said sum, this Equation aritheth, viz. $aa + ba + bb = aa - 2da + dd$
8. Which Equation after due Reduction gives $a = \frac{dd - bb}{2d + b}$
9. Therefore from the first, second and eighth steps the two numbers sought are equal to these known numbers, viz. b and $\frac{dd - bb}{2d + b}$
10. But to avoid Fractions multiply those two numbers severally by the Denominator $2d + b$, and take the Products for two numbers to solve the Question, viz. $2db + bb$ and $dd - bb$

The Proof.

11. The Square of $dd - bb$ is $dddd - 2ddbb + bbbb$
12. The Square of $2db + bb$ is $4d^2bb + 4dbbb + bbbb$
13. The Product of $dd - bb$ into $2db + bb$ is $2dbdd - 2dbbb + bbbd - bbbb$
14. The sum of the said Squares and Product is $dd^2 + b^2 + 3ddbb + 2db^2 + 2bb^2$
15. Which sum is a Square whose Root is $dd + db + bb$

From the tenth step aritheth

CANON 1.

16. Take any two unequal square numbers, then their difference shall be one of the numbers sought, and the lesser Square increased with the double Product of the multiplication of the sides of those Squares shall be the other number sought.

Moreover, because the Product of the multiplication of the sum of any two numbers into their difference, is equal to the difference of their Squares, therefore from the preceding Canon aritheth

CANON 2.

17. Take any two unequal numbers, then the Product of their sum multiplied into their difference shall be one of the two numbers sought, and the double Product made by the mutual multiplication of the two numbers first taken, together with the Square of the lesser number shall be the other number sought.

From

From the premises 'tis evident that the Question is capable of innumerable Answers in whole numbers, of which (for the Learners exercise) I shall exhibit six, with their Proofs, in the following Table.

d, b	s, r	ss	rr	sr	qq	q
1 2, 1	5, 3	25	9	15	49	7
2 3, 1	8, 7	64	49	56	169	13
3 3, 2	16, 5	256	25	80	361	19
4 4, 3	33, 7	1089	49	231	1369	37
5 5, 4	54, 11	2916	121	604	3025	55
6 5, 3	39, 16	1521	256	624	2401	49

18. The numbers under d and b in this Table are six pairs of numbers taken at pleasure, by which the latter of the two preceding Canons gives six pairs of numbers under s and r to solve Quest. 56. As, for example, if 2 and 1 be taken, then Canon 2. gives 5 and 3, (that is, s and r) to solve the Question: For 25 and 9, the Squares of 5 and 3, together with 15 the Product of 5 and 3, (that is, $ss + rr + sr$) make the Square 49, (that is qq), whose side is 7, (to wit, q .)

But for a further Proof, you may observe from the fourteenth and fifteenth steps of the Resolution, that every number standing under q is equal to its respective $dd + db + bb$; so 7 in the Column of q is equal to the Squares of 2 and 1 standing under d, b , together with the Product of 2 into 1. The like is to be understood of the other five Answers in the Table.

Observat. 1. upon the foregoing Quest. 56.

Whereas it is taken for granted in the tenth step of the preceding Resolution of Quest. 56. that two numbers in the same Reason (or Proportion) with those found out to solve the said Question will likewise satisfy the same, I shall here demonstrate the truth thereof.

Suppositions.

1. Let two numbers capable of solving Quest. 56. suppose a and b
2. 5 and 3, be represented by aa and bb
3. And let their Squares be signified by aa and bb
4. Then according to the import of Quest. 56. $aa + ba + bb = \square$
5. Let two other numbers having the same Proportion to one another as a to b be represented by d and c , viz. $a : b :: d : c$
6. Suppose $dd + cd + cc = \square$ (a Square.)

Demonstration.

7. By Prop. 17. Elem. 7. Euclid. $a : b :: aa : ba$
8. And by Prop. 11. Elem. 8. $a : b :: ba : bb$
9. Therefore out of the two last preceding Analogies $a : b :: aa : ba :: ba : bb$
10. And by the like argumentation, $d : c :: dd : cd :: cd : cc$
11. And because by supposition in the fourth step, $a : b :: d : c$
12. Therefore out of the ninth, tenth and eleventh steps, (per Prop. 11. Elem. 5.) $aa : ba :: ba : bb :: dd : cd :: cd : cc$
13. Therefore alternately (per Prop. 13. Elem. 7.) $aa : dd :: ba : cd :: bb : cc$
14. Therefore (per Prop. 12. Elem. 7.) $aa + ba + bb :: dd + cd + cc :: aa : dd$
15. And because by supposition $aa + ba + bb = \square$
16. And by supposition in the third step, $dd + cd + cc = \square$
17. Therefore from the fourteenth, fifteenth and sixteenth steps, (per Prop. 24. Elem. 8.) $aa + ba + bb = \square$

Which was to be proved.

M

Observat. 2.

Observat. 2. upon Quest. 56.

Albert Girard in pag. 618. of *Simon Stevin's* Arithmetick printed in the French Tongue at Leyden, in 1625. doth from the said seventh Question of the fifth Book of *Diophantus* deduce this following

THEOREM.

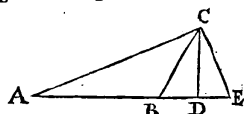
If a plain Triangle be made of three such sides, that the sum of the Squares of two of those sides, together with the Rectangle (or Product of the multiplication) of the same two sides, is equal to the Square of the third side, then the angle opposite to such third side hath for its measure exactly 120 degrees: But if the said Rectangle (or Product) be subtracted from the sum of the said Squares, and the Remainder be equal to the Square of the third side, then the angle opposite to such third side shall have for its measure initially 60 degrees.

This may easily be demonstrated by *Prop. 12, & 13. Elem. 2. Euclid.* but waving the Demonstration, I shall explain the Theorem by Numbers.

First then, if three numbers be desired to express the measures of the sides of a plain Triangle that shall have one angle whose measure is 120 degrees, the preceding Table will furnish you with six such Triangles, for in every rank of numbers in that Table, the three numbers which answer to s , r and q will constitute the Triangle desired. As, for example, 5, 3 and 7; likewise, 8, 7, 13 | 16, 5, 19, &c. will express the Quantities of the sides of Triangles, in every one of which, the measure of the angle opposite to the greatest side is exactly 120 degrees.

Otherwise, without the help of the said Table, if two unequal numbers be taken at pleasure, as $2 = d$, and $1 = b$, then these three following numbers shall express the measures of the sides of a Triangle having an angle of 120 degrees, viz.

$$\left\{ \begin{array}{l} 2db + bb = 5 = AB, \\ dd - bb = 3 = BC = BE = EC, \\ dd + db + bb = 7 = AC. \end{array} \right.$$



Which three numbers; (as is manifest by the preceding Resolution of *Quest. 56.*) have this property, viz. the sum of the Squares of the two lesser numbers together with the Product of their multiplication is equal to the Square of the third or greatest number.

Moreover, if three unequal numbers be desired to express the Quantities of the sides of a plain Triangle that shall have for the measure of one of its angles exactly 60 degrees, you may readily find them out by the help of the preceding Table: For the numbers answering to s , r , and q are the three numbers desired; so from the first rank of numbers in the Table, you may take

$$\left\{ \begin{array}{l} 8 = s + r = AE, \\ 3 = r = EC = EB = BC, \\ 7 = q = AC. \end{array} \right.$$

Which three numbers 8, 3, 7 are the measures of the sides of a Triangle having one of its angles, (to wit, that opposite to 7 or q .) exactly 60 degrees.

Otherwise, without the help of the said Table, if two unequal numbers be taken at pleasure, as $2 = d$, and $1 = b$, then these three following numbers shall be the measures of the sides of a Triangle having one angle of 60 degrees, to wit, that opposite to the side $dd + db + bb$.

$$\left\{ \begin{array}{l} 2db + dd = 8 = AE, \\ dd - bb = 3 = EC, \\ dd + db + bb = 7 = AC. \end{array} \right.$$

In which Triangle the Square of the side $dd + db + bb$ is equal to the sum of the Squares of the other two sides $2db + dd$ and $dd - bb$, less by the Rectangle (or Product) of the same two sides, as is evident by the following

Proof.

$$\begin{array}{lcl} \text{The Square of } 2db + dd \text{ is} & . & . > 4d^2b^2 + 4b^2dd + d^4 \\ \text{The Square of } dd - bb \text{ is} & . & . > d^4 - 2d^2b^2 + b^4 \end{array}$$

The

The sum of those Squares is $2d^2b^2 + 4b^2dd + d^4 + d^4 - 2d^2b^2 + b^4 = 2d^4 + 4b^2dd + b^4$
 The Product of $2db + dd$ into $dd - bb$ is $2d^2b^2 + d^4 - 2d^2b^2 - dbb = d^4 - dbb$
 Which Product being subtracted from the sum of the Squares, leaves $2d^4 + 4b^2dd + b^4 - d^4 + dbb = d^3 + 4b^2dd + b^4 + dbb$
 Which Remainder is the Square of $dd + db + bb$, as was affirmed.

QUEST. 57. (Quest. 8. Lib. 5. Diophant.)

To find three right-angled Triangles in Rational numbers that shall have equal Area's.

RESOLUTION.

1. First, by either of the Canons in *Sett. 16, 17.* of the foregoing *Quest. 56.* find out two numbers capable of solving that Question, suppose s the greater, r the lesser.
2. Therefore, (according to the import of the said *Quest. 56.*) the Squares of s and r , with the Product of s into r is equal to some Rational Square number, let it be q^2 , whence $ss + sr + rr = qq$
3. Therefore by Construction in the first and second steps, $\sqrt{ss + sr + rr} = q$, (Rational.)
4. By the Canon in *Observat. 8. Resolut. 2. Quest. 1.* form a right-angled Triangle from $\sqrt{ss + sr + rr}$: (that is, q) and r , so the three sides will be expressible by these Rational numbers,

$$\text{viz. } \left\{ \begin{array}{l} ss + sr + 2rr = \text{Hypotenusal,} \\ ss + sr = \text{Base,} \\ 2qr = \text{Perpendicular.} \end{array} \right.$$

5. In like manner form a second right-angled Triangle from $\sqrt{ss + sr + rr}$: (that is, q) and s , so the three sides will be expressible by these Rational numbers,

$$\text{viz. } \left\{ \begin{array}{l} 2ss + sr + rr = \text{Hypotenusal,} \\ sr + rr = \text{Base,} \\ 2qs = \text{Perpendicular.} \end{array} \right.$$

6. Likewise, form a third right-angled Triangle from $\sqrt{ss + sr + rr}$: (that is, q) and $s + r$, so the three sides will be expressible by these Rational numbers,

$$\text{viz. } \left\{ \begin{array}{l} 2ss + 3sr + 2rr = \text{Hypotenusal,} \\ sr = \text{Base,} \\ 2qs + 2qr = \text{Perpendicular.} \end{array} \right.$$

7. I say those three Triangles will solve this 57th Question: For first, by Construction they are right-angled Triangles; secondly, all the sides are expressible by Rational numbers, for s , r and q are Rational numbers by Construction; thirdly, if in every one of those three Triangles the Base be multiplied by the Perpendicular, every one of the three Products will manifestly be equal to $2qss + 2qrr$; therefore the halves of those Products, that is, the Area's of those three right-angled Triangles are equal to one another, as was required.

8. And since the foregoing *Quest. 56.* gives innumerable whole numbers answering to s , r and q , you may find out as many Ternions of right-angled Triangles in whole numbers as shall be desired to solve this 57th Question. But to have the nine sides numbers s , r must be the least Terms of a Reason, that is, two such numbers as have no common Divisor besides unity.

9. The premises give the following Canon to find out innumerable Ternions of right-angled Triangles in whole numbers to solve the Question proposed, after the Rational whole numbers represented by s , r and q are first found out by either of the Canons of *Quest. 56.*

CANON.

$$\left\{ \begin{array}{l} ss + sr + 2rr = h = \text{Hypoth.} \\ ss + sr = b = \text{Base,} \\ 2qr = p = \text{Perp.} \end{array} \right\} \text{ of Triangle I,}$$

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$$\begin{array}{lcl}
 2ss + sr + rr = h = \text{Hypoth.} & \} & \text{of Triangle II.} \\
 sr + rr = b = \text{Base,} & \} & \\
 2qs = p = \text{Perp.} & \} & \\
 \hline
 2ss + 3sr + 2rr = H = \text{Hypoth.} & \} & \text{of Triangle III.} \\
 sr = B = \text{Base,} & \} & \\
 2qs + 2qr = P = \text{Perp.} & \} &
 \end{array}$$

Observations upon the Canon last foregoing.

Divers properties, besides the equality of Area's, in the three right-angled Triangles found out by the said Canon, do present themselves to your view, and are worthy of Observation. The principal Properties are these four, viz.

$$\begin{array}{lcl}
 1. & h + b & = h + b. \\
 2. & H + B & = 2b + 2b. \\
 3. & H - B & = h + b = h + b. \\
 4. & P & = p + p.
 \end{array}$$

That is to say, in words,

1. The sum of the Hypotenusal and Base of the first Triangle, is equal to the sum of the Hypotenusal and Base of the second.
2. The sum of the Hypotenusal and Base of the third Triangle, is equal to the double-sum of the Bases of the first and second.
3. The excess of the Hypotenusal above the Base of the third Triangle is equal to the sum of the Hypotenusal and Base of the second, and likewise to the sum of the Hypotenusal and Base of the first.
4. The Perpendicular of the third Triangle is equal to the sum of the Perpendiculars of the first and second.

By the first of those three Triangles is meant that which hath the shortest Hypotenusal; by the second, that whose Hypotenusal is next greater than the shortest; and by the third, that which hath the longest Hypotenusal, in which order they are set in the Table. But the better to explain the Canon and Properties, I shall resume the Table belonging to *Quest.* 56. and call it *Table I.* whence fix Answers in whole numbers to this *Quest.* 57. are deduced, and inserted in the following *Table II.*

Table I. brought from *Quest.* 56.

	d, b	s, r	ss	rr	sr	qq	q
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	32, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

Table II. deduced from Tab. I. by the Canon in Sect. 9. *Quest.* 57.

	h.	b.	p.	h.	b.	p.	H.	B.	P.
1	58	40	42	74	24	70	113	15	112
2	218	120	182	233	105	208	394	56	390
3	386	336	190	617	105	608	802	80	798
4	1418	1320	518	2458	280	2442	2969	231	2960
5	1082	840	682	1537	385	1488	2186	264	2170
6	2657	2145	1568	3922	880	3822	5426	624	5290

The Construction of *Table I.* hath already been exprest in *Quest.* 56. whence the latter *Table* is deduced according to the Canon in Sect. 9. of this 57th Question, and contains six Answers to it; the first of which is to be understood thus,

In

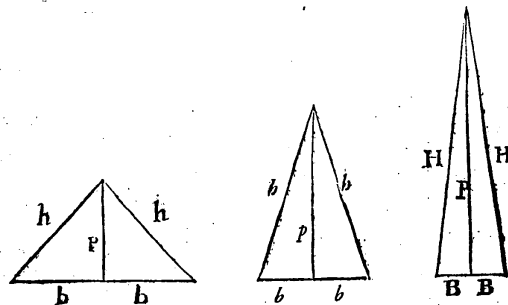
In the first rank, under . . . h, b, p. | b, b, p. | H, B, P.
you will find . . . 58, 40, 42. | 74, 24, 70. | 113, 15, 112.

Which three Triangles have equal Area's, and such other properties as before have been declared, viz. $h + b = b + b$, &c. as will easily appear by comparing the numbers answering to those Equations. The like is to be understood of the other five Answers.

QUEST. 58.

[This is Probl. 29. in pag. 131. of the Introduction to Algebra, translated out of High Dutch into English in 1668. by Tho. Brancker, M. A.]

To find three equicrural Triangles equal to one another in Area; and that the Perimeters of two of those Triangles may be equal to one another; and that the sides and Perpendiculars of every one of those three Triangles may be exprest by rational numbers.



RESOLUTION.

1. By the Canon in the ninth step of the Resolution of the foregoing *Quest.* 57. find out three right-angled Triangles in rational numbers, and equal to one another in Area; such are the three Triangles in any one of the six ranks of numbers in *Table II.* belonging to the said *Quest.* 57. for example, take those in the first rank, viz. $\{ h, b, p. \mid b, b, p. \mid H, B, P. \}$
 $\{ 58, 40, 42. \mid 74, 24, 70. \mid 113, 15, 112. \}$
 2. Then (as is evident by the Diagram belonging to this Question) the sides of the three equicrural Triangles desired shall be these following, viz. $\{ h, h, 2b. \mid b, b, 2b. \mid H, H, 2B. \}$
 $\{ 58, 58, 80. \mid 74, 74, 48. \mid 113, 113, 30. \}$
 3. And the Perpendiculars falling upon the Bases, viz. upon $\{ p, p, P. \}$
 $\{ 2b, 2b, 2B \}$ of those three equicrural Triangles, are these, $\{ 42, 70, 112. \}$
- Which three equicrural Triangles in rational numbers above exprest in the second step will solve the Question, as will be evident by

The Proof.

By Construction in the first step the three right-angled Triangles $h, b, p. \mid b, b, p. \mid H, B, P.$ are equal to one another in Area, therefore their double Area's are equal to one another; but the said double Area's are the Area's of the three equicrural Triangles $h, h, 2b. \mid b, b, 2b. \mid H, H, 2B.$ and therefore the Area's of those three equicrural Triangles are equal between themselves,

$$\text{viz. } \{ bp = bp = BP, \\
 40 \times 42 = 24 \times 70 = 15 \times 112 = 1680.$$

5. Moreover, by the first property in the Observations upon the Canon for resolving *Quest.* 57. this Equation is manifest,

$$\text{viz. } \{ h + b = h + b, \\
 58 + 40 = 74 + 24 = 98.$$

6. There:

6. Therefore the double of the first part of that Equation shall be equal to the double of the latter part,

$$\text{viz. } \begin{cases} 2h + 2b = 2b + 2b, \\ 116 + 80 = 148 + 48 = 196. \end{cases}$$

7. But the said double sums (if you view the Diagram belonging to this Question) are manifestly equal to the Perimeters of the two equicrural Triangles $h, h, 2b$ and $b, b, 2b$, therefore those two equicrural Triangles are equal to one another in their Perimeters as well as in their Area's, and each Area is equal to the Area of the third equicrural Triangle $H, H, 2B$, also all their sides and Perpendiculars are express'd by rational numbers, as the Question required. In like manner five Answers more to this 58th Question may be collected from Table II. in Quest. 57. and 'tis evident from the premises, that innumerable Ternions of equicrural Triangles in rational whole numbers may be found out to solve the said Quest. 58.

QUEST. 59.

The three sides of any plain right-angled Triangle being given in rational numbers; to find out another right-angled Triangle in rational numbers, which shall have the same Area with the former.

[*Monf. de Fermat, in his Observation upon Quest. 8. of the fifth Book of Diophantus, gives a Canon to solve the Question above proposed, but shews not the rise thereof; I shall therefore resolve the Question at large by Literal Algebra, upon the same grounds by which it is resolved by Numerical Algebra in pag. 11. of his Analytical Inventions, prefixed to the late Edition of Diophantus printed at Tholose in 1670.*]

RESOLUTION.

Let there be given a right-angled Triangle in rational numbers, as 3, 4, 5, which may be represented by b, p, h , whose Area is $\frac{1}{2}bp$; then let b and $a + p$ be assumed for the sides about the right-angle of a second right-angled Triangle, whence the Hypotenusal will be $\sqrt{aa + 2pa + pp + bb}$: that is, (because $hb = pp + bb$) $\sqrt{aa + 2pa + hb}$; and the Area of this latter Triangle is $\frac{1}{2}ba + \frac{1}{2}bp$: Now if this latter Area be divided by the

former Area $\frac{1}{2}bp$; and if by the Square Root of the Quotient $\frac{a}{p} + 1$, viz. by $\sqrt{\frac{a}{p} + 1}$: the three sides of the second Triangle be severally divided, the Quotients shall be the three sides of a third right-angled Triangle, whose Area is equal to the Area of the right-angled Triangle first given: For if the sides of the second right-angled Triangle, to wit, $b, a + p$ and $\sqrt{aa + 2pa + hb}$: be severally divided by $\sqrt{\frac{a}{p} + 1}$: the Quotients

$\frac{a + p}{p}$ and $\sqrt{\frac{aa + 2pa + hb}{p}}$ are the sides of a third right-angled Triangle $\sqrt{\frac{a}{p} + 1}$: $\sqrt{\frac{a}{p} + 1}$:

whose Area $\frac{ba + bp}{2}$ is equal to $\frac{1}{2}bp$ the Area of the first right-angled Triangle; (for $\frac{2a}{p} + 2$

$+ 2$ into $\frac{1}{2}bp$ makes $ba + bp$.) So that if $aa + 2pa + hb$ and $\frac{a}{p} + 1$ were square numbers, then the Question were solved: But how to make those two Algebraick quantities to be square numbers, the following Resolution shews.

1. Let there be given, as before, the three sides of a right-angled Triangle, as
2. Then for one of the sides about the right-angle of a second Triangle put
3. And for the other side about the right-angle
4. Then because the sum of the Squares of those two sides must be equal to the Square of the Hypotenusal, the Square of the Hypotenusal of the second right-angled Triangle shall be
5. That is, (because $hb = pp + bb$),

$$aa + 2pa + pp + hb$$

6. From

6. From the second and third steps the Area of the second right-angled Triangle is
7. Which Area divided by $\frac{1}{2}bp$, the Area of the first right-angled Triangle, gives the Quotient
8. Now according to the scope before-mentioned, each of the quantities in the fifth and seventh steps must be equated to a Square, so we are fallen upon this Duplicate equality, viz.
9. In order to resolve that Duplicate equality, I multiply the said $\frac{a}{p} + 1$ by the Square hb , to the end there may be the same known Square hb in each Equation, so it produceth
10. Then the difference of the said $aa + 2pa + hb$ and $\frac{hb}{p}$ is $aa + \frac{2pp - hb}{p}$ and $+ hb$, viz. the difference of the two Squares sought is
11. Which difference is equal to the Product of the multiplication of these two quantities, to wit,
12. The half-sum of the two quantities last mentioned is
13. Then the Square of the said half-sum being equated to $aa + 2pa + hb$, (assumed to be the greater of the two quantities in the eighth step) gives this Equation, viz.
14. Which Equation, after due Reduction, gives
15. Therefore from the second, third and fourteenth steps the two sides about the right-angle of the second right-angled Triangle sought are
16. The sum of the Squares of those two sides, by taking $bb + pp$ instead of the Factor hb , and $bbbb + 2bbpp + pppp$ instead of the Factor $hbhb$, will be found
17. The square Root of the sum of the Squares in the last step is the Hypotenusal of the second right-angled Triangle sought, to wit,
18. Which Hypotenusal, by taking hb instead of $bb + pp$, and $bbbb + 2bbpp + pppp$ instead of $hbhb$, may be expressed thus,
19. Therefore from the fifteenth and eighteenth steps, the three sides of the second right-angled Triangle sought are these, to wit,
20. Therefore the Area of the said second right-angled Triangle is
21. Which Area divided by $\frac{bp}{2}$ the Area of the right-angled Triangle first given, gives the Quotient
22. The square Root of that Quotient is
23. By which square Root, if the three sides of the second right-angled Triangle before express'd in the nineteenth step be severally divided, the Quotients will be the three sides of the third right-angled Triangle sought, to wit,

$$\begin{aligned} 1. & \frac{ppb - \frac{1}{2}hbhb}{ppp - \frac{1}{2}hbhb} \text{, (or } \frac{hpb}{pp - \frac{1}{2}hb} \text{)} \\ 2. & \frac{pppp - \frac{1}{2}hbhb - hbpp}{pph - \frac{1}{2}hbhb} \\ 3. & \frac{\frac{1}{2}hbhb + bbpp}{ppb - \frac{1}{2}hbhb} \end{aligned}$$

24. But

24. But $4hpp - ppp = hbpp$, therefore instead of subtracting $hbpp$ in the Numerator of the second of the three sides last before exprest, we may subtract $hbpp - ppp$, whence that side may be exprest thus $\frac{4hbpp - hbpp}{ppb} = \frac{3hbpp}{ppb}$, and consequently the three sides

of the third right-angled Triangle sought shall be these, to wit,

$$\frac{hbpp}{ppb} = \frac{3hbpp}{ppb} = \frac{4hbpp}{ppb}.$$

25. Or, (to avoid Fractions) we may multiply the Numerator and Denominator of every one of the three sides last above exprest, by 4, so these three following sides (which are of the same value with those) will be produced for the third right-angled Triangle sought, to wit,

$$\frac{4hbpp}{4ppb - 2bhb} = \frac{hbpp - 4bpps}{4ppb - 2bhb} = \frac{hbpp - 4bpps}{4ppb - 2bhb}.$$

Hence arith,

CANON 1.

26. Let b, p, h represent the three sides of any right-angled Triangle given in rational numbers, whereof the Hypotenusal is b , and p the greater of the two sides about the right-angle; then from hb and $2bp$ form a right-angled Triangle, (by the Canon in *Observat.* 8. upon the Resolution by literal Algebra of the first Question of this Book,) that done, divide severally the three sides found out by $4ppb - 2bhb$, so there will arise the three sides of a right-angled Triangle whose Area shall be equal to the Area of the given right-angled Triangle b, p, h .

27. Again, by supposition, $hb - pp = bb - 2pp$, and $p < b$, therefore $pp - bb = 2pp - hb$; whence by multiplying each part into $2b$ it follows that $2ppb - 2bhb = 4ppb - 2bhb$: Therefore instead of the Divisor $4ppb - 2bhb$ in Canon 1. we may take $2ppb - 2bhb$, and so there will arise the following Canon, (which is the same with Monf. *Fermat's* in the place before cited.)

CANON 2.

$$\frac{4hbpp}{2ppb - 2bhb} = \frac{hbpp - 4bpps}{2ppb - 2bhb} = \frac{hbpp - 4bpps}{2ppb - 2bhb}.$$

In words thus,

28. Let b, p, h represent the three sides of any right-angled Triangle given in rational numbers, viz. b the Hypotenusal, p the greater side about the right-angle, and h the lesser; then from hb and $2bp$ form a right-angled Triangle, and divide severally the three sides found out by $2ppb - 2bhb$, so the Quotients shall be the three sides of a right-angled Triangle whose Area is equal to the Area of the given right-angled Triangle b, p, h .

An Example in Numbers.

Let there be given the three sides of a right-angled Triangle $b : p : h$ in rational numbers, to wit,

Then by either of the Canons these three sides of a right-angled Triangle will be found out, to wit,

Which latter Triangle hath the same Area with the former, to wit, 6.

Again,

Let there be given this right-angled Triangle in numbers, $b : p : h$ to wit,

Then by either of the preceding Canons this right-angled Triangle will be found out, to wit,

Which latter Triangle hath the same Area with the former, to wit, 30.

From the premises it is evident, that from any right-angled Triangle given in rational numbers another of the same Area may be found out; and from the second a third; and from the third a fourth, &c. all which right-angled Triangles shall have one common Area, to wit, that of the given right-angled Triangle, and all their sides expressible by rational numbers.

29. Moreover, from the nineteenth step of the preceding Resolution, several Canons may be deduced, to shew how by the help of any right-angled Triangle given in rational numbers,

numbers, to find out another, whose Area divided by the Area of the given Triangle will give the Quotient a square number. But I shall exhibit only this following

CANON.

Let b represent the Hypotenusal, p the greater side about the right-angle, and h the lesser in any right-angled Triangle given in rational numbers; then from hb and $2bp$ form a right-angled Triangle, and divide severally the three sides found out by $4hbpp$; to there will come forth the three sides of a right-angled Triangle, whose Area divided by the Area of the given Triangle, will give the Quotient a square number whose Root is $\frac{p - \frac{1}{2}bb}{hp}$ or $\frac{2pp - hb}{2bp}$.

An Example in Numbers.

Let there be given the three sides of a right-angled Triangle $b : p : h$ in numbers, to wit,

Then by the last Canon, this right-angled Triangle will be found out, to wit,

The Area of the latter Triangle is

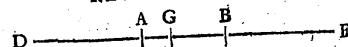
Which Area divided by 6 the Area of the given right-angled Triangle $3, 4, 5$ gives a square number, to wit,

The square Root whereof is

QUEST. 60. (Quest. 12. Lib. 5. Diophant.)

To divide unity into two such parts, that if to each part a given number, suppose 6, be added, the two summs may be square numbers. But the sum of the double of the given number and unity must either be a square number, or else composed of two Squares.

RESOLUTION.



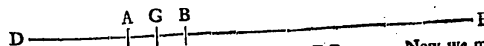
Let AB be 1, and $AD = BE$ be the given number 6; therefore $DE = 13$. Now we must divide AB , to wit, unity, into two parts, suppose in the point G , that GD and GE may be square numbers, so that in effect we must divide DE , that is, 13 into two such Squares that one of them may be greater than 6, but less than 7. But that may be done by the fourth Question of this Book, where 13 is divided into the Squares $\frac{11222}{1681}$ and $\frac{14644}{1681}$, whose sides are $\frac{106}{41}$ and $\frac{121}{41}$, and each of those Squares is greater than 6, but less than 7; therefore taking each Square from 7, the remainders $\frac{7682}{1681}$ and $\frac{1181}{1681}$, (that is GA and GB ;) are the desired parts of unity: For if to each of those parts the given number 6 be added, the two summs will be the Squares $\frac{1181}{1681}$ and $\frac{7682}{1681}$.

It is also evident, that instead of unity, any number may be given to be divided; provided that the sum of this number and of the double of the other number given make a Square, or else a number composed of two Squares.

QUEST. 61. (Quest. 13. Lib. 5. Diophant.)

To divide unity into two such parts, that to the one adding 2, (a number given,) and to the other 6, (another number given,) the summs may be square numbers. But the sum of the given numbers with unity must either make a square number, or else a number composed of two Squares.

RESOLUTION.



Let AB be 1, $AD = 2$, $BE = 6$; therefore $DE = 9$. Now we must divide AB , to wit, 1 into two parts, suppose in the point G , that GD and GE may be square numbers: So that in effect we must divide DE , that is, 9 into two such Squares that one may be greater than 2, but less than 3; or that one Square may be greater than 6, but less than 7. But the third Question of this Book shews how to find out two Squares so qualified, as the Square $\frac{2516}{3025}$ for GD , and the Square $\frac{14644}{3025}$ for GE ; the sides of which

which Squares are $\frac{8}{15}$ and $\frac{11}{15}$; then subtracting 2 from the Square $\frac{28}{15}$, and 6 from the Square $\frac{11}{15}$, the remainders $\frac{10}{15}$ and $\frac{1}{15}$ are the desired parts: G A, G B of unity. For if the former part be added to 2, and the latter to 6, the summs are Squares, to wit, $\frac{28}{15}$ and $\frac{11}{15}$, as was required.

QUEST. 62. (Quest. 14. Lib. 5. Diophant.)

To divide unity into three such parts, that if to every one of them a given number 3 be added, the three summs may be Squares. But the sum of the triple of the given number and unity must either be a Square number, or else a number composed of two or three Squares.

RESOLUTION.

It is easie to apprehend that the sum of the three Squares sought makes 10, and that the scope of the search must be to find out three Squares, every one of which may fall between 3 and 4, and that their sum may be 10; for then the three excesses of those Squares above 3 will be the three desired parts of unity. First then, forasmuch as 10 is compos'd of two Squares 9 and 1, I divide 10 into two other Squares whereof one may be greater than 3, but less than 4, such are the Squares $\frac{25}{9}$ and $\frac{64}{81}$, whose sides are $\frac{5}{3}$ and $\frac{8}{9}$; for the sum of those Squares is 10, and the first Square $\frac{25}{9}$ is between 3 and 4. Then I take the Fraction $\frac{1}{15}$, (to wit, the excess of the first Square above 3) for the first of the three desired parts of unity; for if to that Fraction the given number 3 be added, it makes the Square $\frac{28}{15}$.

Then I divide the latter of the Squares first found, to wit, $\frac{64}{81}$, whose side is $\frac{8}{9}$, into two such Squares that one may be greater than 3, but less than $\frac{25}{9}$, or less than $\frac{1}{15}$; for if such Square be less than $\frac{25}{9}$, it will necessarily be less than $\frac{64}{81}$, because $\frac{1}{15}$ is less than $\frac{64}{81}$. But the sides of two such Squares are $\frac{1}{15}$ and $\frac{1}{15}$, therefore the Squares themselves are $\frac{1}{225}$ and $\frac{1}{225}$; from each of which if $\frac{1}{15}$ be subtracted, the remainders $\frac{1}{1125}$ and $\frac{1}{1125}$, with the Fraction $\frac{1}{15}$ first found, that is, (in the same Denominator with the former) $\frac{28}{15}$ shall be the three desired parts of unity to solve the Question.

After the same manner any number given instead of unity may be divided into three such parts, that a number given being added to every one of them may make three Squares. But the sum of the number given to be divided, and the triple of the number to be added, must be either a Square, or a number compos'd of two or three Squares.

As, if it were desired to divide 2 into three parts, that each part increased with 4 may make a Square: First, forasmuch as 14 the sum of 2 and the triple of 4 is compos'd of three Squares 1, 4, and 9, let 10 the sum of 2 and 1 (two of those three Squares) be divided into two other Squares that the first may exceed 4 the number given to be added, but be less than 6, to the end the excess may be less than 2 the number given to be divided, then add the other of the two Squares found out to 4, (the other of the three Squares before mentioned, whose sum made 14,) and divide the sum into two such Squares that each may be greater than 4; lastly, from each of these two Squares last found out, as also from the first Square before found, subtract 4, so the Remainders shall be the desired parts of 2. But the Operation I leave to the Learner's exercise.

QUEST. 63. (Quest. 15. Lib. 5. Diophant.)

To divide unity into three such parts, that if the first be increased with 2, the second with 3, and the third with 4, the three summs may be square numbers. But the sum of the three numbers given and unity must either be a Square, or compos'd of two or three Squares.

RESOLUTION.

First we must divide 10 (the sum of the three numbers given with unity) that the first may exceed 2, the second 3, and the third 4. To which end, first, (by Quest. 4. of this Book) divide 10 into two Squares that one may fall between 2 and 3; such are the Squares $\frac{16}{4}$ and $\frac{4}{1}$, whose Roots are $\frac{4}{2}$ and $\frac{2}{1}$; then from the first Square $\frac{16}{4}$ subtract 2, and take the Remainder $\frac{10}{4}$ for one of the desired parts of unity.

It remains to divide the other Square $\frac{4}{1}$ into two other Squares, that one may fall between

between 3 and 4: But two such Squares will be found $\frac{25}{9}$ and $\frac{16}{16}$, whose sides are $\frac{5}{3}$ and $\frac{4}{4}$; then from the latter of the said Squares subtract 3, and from the former, 4; so the two Remainders $\frac{1}{9}$ and $\frac{1}{16}$ with $\frac{10}{4}$ before found, that is, (in the same Denominator with the two former) $\frac{10}{4}$ shall be the three desired numbers, which will solve the Question. For first, their sum makes unity; moreover if 2 be added to $\frac{1}{9}$, that is, $\frac{19}{9}$, (the first number) the sum is the Square $\frac{361}{81}$; if 3 be added to the second number $\frac{1}{16}$, it makes the Square $\frac{25}{16}$; lastly, if 4 be added to the third number $\frac{10}{4}$, it makes the Square $\frac{100}{4}$.

QUEST. 64. (Quest. 16. Lib. 5. Diophant.)

To divide a given number 10 into three numbers, that the sum of every two may be a Square; but the double of the given number must be either a Square, or else compos'd of two or three Squares.

RESOLUTION.

Forasmuch as the three numbers sought are to be such that the sum of the first and second must make a Square, also the sum of the first and third must make a Square, likewise the sum of the second and third must make a Square; therefore these three Squares are equal to the said three numbers twice taken. And because the sum of the three numbers is 10, therefore twice their sum, 20, shall be the sum of the three Squares. We must therefore divide 20 into three Squares, each of which may be less than 10; (for every one of these Squares must be equal to two of the three numbers, and consequently less than 10 the sum of all the three numbers.) But 20 is compos'd of two Squares 16 and 4; therefore we may take 4 (which is less than 10) for one of the three Squares sought, and then divide 16 into two Squares, one of which may fall between 10 and 6, for then the other will also be less than 10; (because both must make 16.) And (by Quest. 4. of this Book) the sides of two such Squares will be found $\frac{16}{4}$ and $\frac{4}{1}$, wherefore the three Squares sought are 4, $\frac{16}{4}$, and $\frac{4}{1}$, which subtracted severally from 10 leave three Remainders, 6, $\frac{10}{4}$, and $\frac{10}{4}$ for the three desired numbers, whose sum is 10, and every two of them added together makes a Square, as was desired.

But to make it more clearly evident that three numbers so found out will solve the Question, let bb , cc , dd represent three square numbers, then

$$\begin{array}{l} \text{Suppose } \dots \dots \dots \text{ } \left. \begin{array}{l} bb + cc + dd = 20 \\ \frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd = 10 \\ \text{And consequently } \dots \dots \dots \end{array} \right\} \begin{array}{l} \frac{1}{2}cc + \frac{1}{2}dd = \frac{1}{2}bb \\ \frac{1}{2}bb + \frac{1}{2}dd = \frac{1}{2}cc \\ \frac{1}{2}bb + \frac{1}{2}cc = \frac{1}{2}dd \end{array} \end{array}$$

Then from 10, that is, from $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd$ subtract $\frac{1}{2}bb$, cc and dd severally, so the three Remainders are $\frac{1}{2}bb$, $\frac{1}{2}cc$, and $\frac{1}{2}dd$. I say these three Remainders shall be three numbers to solve the Question; for the sum of the first and second makes the Square dd , the sum of the second and third makes the Square bb , and the sum of the first and third makes the Square cc .

QUEST. 65. (Quest. 17. Lib. 5. Diophant.)

To divide a given number, suppose 10, into four numbers, that the sum of every three may make a Square.

RESOLUTION.

Forasmuch as the sum of every three of the four numbers sought must make a Square, but therefore the four Squares sought are equal to the four desired numbers thrice taken. But the four numbers thrice taken make 30, therefore 30 must be divided into four such Squares that every one of them may be less than 10, (for every one of the four Squares must be equal to three of the numbers sought, and consequently be less than 10 the sum of all four.) But 30 is compos'd of four Squares, 16, 9, 4, and 1, two of which, to wit, 9 and 4 may be taken for two of the Squares sought, and then 17 (the sum of the other two Squares 16 and 1) must be divided into two Squares that one may be less than 10, but greater than 7, and then the other will be also less than 10; but the sides of two such Squares will (by Quest. 4. of this Book) be found $\frac{16}{4}$ and $\frac{1}{1}$, and the Squares themselves are $\frac{16}{4}$ and $\frac{1}{1}$, each of which is less than 10. Therefore the four Squares sought

fought are 9, 4, $\frac{17}{25}$ and $\frac{23}{25}$, which subtracted severally from 10, leave the Remainders 1, 6, $\frac{11}{25}$ and $\frac{5}{25}$ for the four numbers sought, whose sum makes 10, and every three of them added together make the Squares $\frac{16}{25}$, 9, 4 and $\frac{16}{25}$, whose sides are $\frac{4}{5}$, 3, 2 and $\frac{4}{5}$.

Note. If the quadruple of the number given be a whole number, this Question may be extended to five numbers, or as many as you please: for every whole number is compos'd of four Squares, which may be divided into any multitude of Squares within any possible limits, by the help of the third Question of this Book.

QUEST. 66. (A Lemma, used in the following Quest. 67.)

To find three such Cube-numbers, that if from every one of them a given number, suppose 1, be subtracted, the sum of the Remainders may be a Square.

RESOLUTION.

1. For the side of the first Cube put $a-1$ any absolute number, as, $a-1$
2. Then take some square number, as 9, and from its $\frac{1}{3}$, to wit, from 3 subtract 1 the absolute number in the side of the first Cube, and let the Remainder 2 be connected by $-$ with a for the side of the second Cube, the reason whereof will appear in Observation 1. upon this Question, $-a+2$
3. Let the side of the third Cube be any known number whose Cube exceeds the number given in the Question, to wit, the side 2
4. Then the Cubes of those three sides are these, to wit,

1.	$aaa-aa+3a+1$	1.
2.	$-aaa+6aa-12a+8$	2.
3.	$+8$	3.
5. From every one of which Cubes subtract 1 the number given in the Question, and add the Remainders together, so the sum will be $aaa-9a+14$, which must be equated to a Square; but the side thereof must be so feigned that the value of a may be less than 2, to the end that $-a+2$, or $2-a$ the side of the second Cube may be greater than nothing. Now to cause that effect, the said side may be feigned $3a$ —any absolute number between $\sqrt{14}$ and $\sqrt{31}+6$, (which limits are found out after the method before delivered in divers Questions of this Book,) let therefore the side of the said Square be feigned $3a-4$, and then the Square of $3a-4$ being equated to $9aa-9a+14$ above mentioned, this Equation ariseth, to wit,

$$9aa-9a+14 = 9aa-24a+16.$$
6. From which Equation, the value of a will be made known, viz. $a = \frac{2}{3}$
7. Therefore from the sixth, first, second and third steps the sides of the three Cubes sought are $\frac{1}{3}$, $\frac{2}{3}$ and 2; wherefore the Cubes themselves are $\frac{1}{27}$, $\frac{8}{27}$ and 8, which will solve the Question. For if from every one of those Cubes the given number 1 be subtracted, the sum of the three Remainders in its least terms is $\frac{16}{27}$, which is a Square, as was required.

Observations upon Quest. 66.

1. Two things are remarkable in the preceding positions for the sides of the Cubes sought; First, in the first side $a-1$ there is put -1 , and in the second side $-a+2$ there is put $-a$, to the end that in adding together the Remainders mentioned in the fifth step, $-aaa$ and $-aaa$ may destroy one another: Secondly, the unities in the second side must be chosen with such Caution that the number prefix to aa in the sum of the said three Remainders may be a Square, for if $9aa$ had not been a Square, then $9aa-9a+14$ could not have been equated to a Square. Therefore that the number prefix to aa in the said sum of the Remainders may infallibly come forth a square number, the number of unities to be connected with $-a$ by $+$ in the side of the second Cube, must be the excess of one third part of some square number above the unit or unities in the side of the first Cube: The reason whereof may be thus manifested, Suppose $a-1=b$, and $-a+1=d$, and 2, to be the sides of the three Cubes sought; then from the Cubes of those sides subtracting the given number 1 severally, the sum of the Remainders is $3baa-3daa+3bba-3dda+3bbb+3ddd+5$; which sum cannot be equated to a Square, unless the number signified by $3b-3d$, which is multiplied into aa be a square number;

number; suppose therefore $3b-3d=ff$, whence by due Reduction, $d=\frac{1}{3}ff-b$, which shews that d the number of unities in $-a-1$ the side of the second Cube, must be the excess of one third part of some square number, above b the unit or unities in $a-1$ the feigned side of the first Cube, as was directed in the second step of the Resolution.

2. It is easy to apprehend that this Question may be extended to as many numbers as you please: For having put (as before) $a-1$ for the side of the first Cube, and (for the reasons before given) $-a+2$ for the side of the second Cube, you may put 2 for the side of a third Cube, 3 for the side of a fourth, &c. provided that the Cubes of these absolute numbers 2, 3, &c. do every one of them exceed the number given to be subtracted; then from the sum of all those Cubes subtracting the given number, the sum of the Remainders may be equated to a Square, because by Construction the number prefix to aa is a Square.

QUEST. 67. (Quest. 18. Lib. 5. Diophant.)

To find three numbers, that if they be severally added to the Cube of their sum, the three summs made by those additions may be Cubes.

RESOLUTION.

1. For the sum of the three numbers sought put a
2. Then the Cube of their sum is aaa
3. For the three numbers sought put $7aaa, 26aaa$ and $63aaa$
4. Whence it is manifest, that if each of the three last mentioned quantities assumed for the three numbers sought, be increased with aaa which was put for the Cube of their sum, there will come forth the Cubes $8aaa, 27aaa$ and $64aaa$; but the sum of these three quantities $7aaa, 26aaa$ and $63aaa$ must be equal to a ; therefore $96aaa = a$, and consequently $96aa = 1$: where if 96 were a square number, the value of a would be expressible by a rational number, and consequently the Question solved. Whence therefore comes 96? examine the Positions and you will find that by subtracting 1 (to wit, unity) from the three Cubes 8, 27 and 64, the remainders 7, 26 and 63 added together make 96. Therefore we must seek three such Cubes that if 1 be subtracted from every one of them, the sum of the three remainders may be a Square: But the sides of three such Cubes are $\frac{2}{3}, \frac{3}{3}$ and 2, (found out by the preceding Quest. 66.) and the Cubes themselves are $\frac{8}{27}, \frac{27}{27}$ and 8, from every one of which if 1 be subtracted, the three remainders will be $\frac{5}{27}, \frac{26}{27}$ and 7, whose sum in its least terms is the Square $\frac{16}{27}$, whose side is $\frac{4}{3}$. Now by the help of those remainders, let the work be renewed thus, viz.
5. For the sum of the three numbers sought put a
6. Whence the Cube of their sum is aaa
7. Then for the first number put $\frac{1538}{337}aaa$
8. For the second $\frac{1811}{337}aaa$
9. And for the third $\frac{7}{337}aaa$
10. Then the sum of the three numbers last express'd is $\frac{16}{27}aaa$
11. Which sum must be equal to a , which in the fifth step was put for the sum of the three numbers sought, therefore $\frac{16}{27}aaa = a$
12. Which Equation, after due Reduction, discovers the sum of the three numbers sought, viz. $a = \frac{1}{13}$

Therefore from the twelfth, seventh, eighth and ninth steps, the three numbers sought will be $\frac{1538}{337}, \frac{1811}{337}$ and $\frac{7}{337}$, whose sum is $\frac{16}{27}$, the Cube whereof is $\frac{64}{27}$; which being added to every one of the said three numbers, the summs will be Cubes, to wit, $\frac{1538^3}{337^3}, \frac{1811^3}{337^3}$ and $\frac{7^3}{337^3}$, whose sides are $\frac{1538}{337}, \frac{1811}{337}$ and $\frac{7}{337}$; therefore the Question is satisfied, and by the help of the preceding Quest. 66. may be extended to four, five or as many numbers as shall be desired.

QUEST. 68.

To find two Cube-numbers, that if their difference be increased with a given number, suppose 2, it may make a Square, and that the side of the greater Cube may be less than 1 a number given.

RESO.

RESOLUTION.

- For the side of the lesser Cube sought put a
- For the side of the greater Cube put $a + \frac{1}{3}$ one third part of some square number, but such third part must be less than 1 the prescribed limit, therefore let the side of the greater Cube be $a + \frac{1}{3}$
- Therefore the greater Cube is $aaa + aa + \frac{1}{3}a + \frac{1}{27}$
- And the lesser Cube is aaa
- Therefore the difference of the said Cubes is $aa + \frac{1}{3}a + \frac{1}{27}$
- To which difference add 2 the number first given in the Question, and the sum is $aa + \frac{1}{3}a + \frac{1}{27}$
- Which sum must be equal to a Square, the side whereof must be so feigned that the value of a may be less than $\frac{2}{3}$, for then $a + \frac{1}{3}$ (the side of the greater Cube) will be less than 1, as the Question requires. Now to cause that effect, the side of the said Square may be feigned to be either $a + \frac{1}{3}$ any known number between $\frac{2}{3}$ and 1 , or else $a + \frac{1}{3}$ any known number between 1 and 2 , (which limits may be found out by the method directed in the preceding Quest. 12.) let therefore the said side be feigned $a + \frac{1}{3}$, and then the Square of $a + \frac{1}{3}$ being equated to the sum in the sixth step, this Equation ariseth, viz.

$$aa + \frac{1}{3}a + \frac{1}{27} = aa + \frac{1}{3}a + \frac{1}{27}$$

- Which Equation after due Reduction gives $a = \frac{2}{3}$
- Therefore from the eighth, first and second steps the sides of the two Cubes sought are $\frac{2}{3}$ and 1
- The Cubes of which sides $\frac{2}{3}$ and 1 , viz. $\frac{8}{27}$ and 1 will solve the Question; for if to their difference $\frac{1}{27}$ you add 2, the sum $\frac{43}{27}$ is a Square, and $\frac{2}{3}$ the side of the greater Cube is less than 1, as was required.

Example 2.

Let it be required to find two Cube-numbers whose difference added to 1458 (a given number) may make a Square, and that the side of the greater Cube may be less than 9 (a number given.)

Resolution.

- For the side of the lesser Cube put a
- For the side of the greater Cube put $a + \frac{1}{3}$ one third part of some square number, but such third part must be less than 9 the limit above prescribed, therefore let the said side be $a + \frac{1}{3}$
- Therefore the greater Cube is $aaa + 9aa + 27a + 27$
- And the lesser Cube is aaa
- Therefore the difference of those Cubes is $9aa + 27a + 27$
- To which difference adding the number first given in this Example 2. to wit, 1458
- The sum will be $9aa + 27a + 1485$

- Which sum must be equal to a Square, the side whereof must be so feigned that the value of a may be less than 6, for then $a + \frac{1}{3}$ (which in the second step was assumed for the side of the greater Cube sought) will be less than the prescribed limit 9; Now to cause that effect, the side of the said Square may be feigned to be $3a + \frac{1}{3}$ any number between 26 and 39, or $3a + \frac{1}{3}$ any number between 38 and 63; suppose therefore the said side be feigned $3a + \frac{1}{3}$, then the Square of $3a + \frac{1}{3}$ being equated to the sum in the seventh step, this Equation ariseth, viz.

$$9aa + 216a + 1296 = 9aa + 27a + 1485$$

- Which Equation after due Reduction gives $a = 1$
- Therefore from the ninth, first and second steps of this second Example, the sides of the Cubes sought are 1 and 4
- The Cubes of which sides 1 and 4, viz. 1 and 64 will solve the Question; for if their difference 63 be added to 1458 the number given in Example 2. it makes the Square 1521, whose side is 39; and 4 the side of the greater Cube is less than 9, as was required.

Example 3.

Example 3.

Again, the same numbers 1458 and 9 being given as before in Example 2. the side of the Square mentioned in the eighth step may be feigned to be $3a + \frac{1}{3}$ (which is within the limits there express'd,) and then the Square of $3a + \frac{1}{3}$ being equated to $9aa + 27a + 1485$, (before express'd in the seventh step,) after due Reduction the sides of two Cubes to solve Quest. 68. as it is before propos'd in Example 3. will be found $\frac{2}{3}$ and $\frac{1}{3}$; therefore the Cubes themselves are $\frac{8}{27}$ and $\frac{1}{27}$, whose difference $\frac{7}{27}$ added to the given number 1458 makes a Square, to wit, $\frac{43}{27}$, whose side is $\frac{2}{3}$, and $\frac{2}{3}$ the side of the greater Cube is less than the prescribed number 9.

QUEST. 69.

To find two such cube-numbers, that if each of them be subtracted from a given squared cube-number, the sum of the remainders may be a Square.

RESOLUTION.

- Let the given squared cube-number be ddd or d^3
- The Root or side whereof is d
- For the side of the first Cube sought put a
- The Cube thereof is aaa
- For the side of the other Cube sought put $a + \frac{1}{3}$
- The cube-root of the given squared Cube, viz. d
- The Cube thereof is $aaa + 3dda + 3d^2a + d^3$
- Then by subtracting the Cube in the fourth step from the given squared Cube in the first, there will remain $d^3 - aaa$
- And by subtracting the Cube in the sixth step from the given squared Cube in the first, there will remain $d^3 - aaa - 3dda - 3d^2a - d^3$
- The sum of those remainders (in the seventh and eighth steps) is $-3dda - 3d^2a - d^3$
- Which sum must be equal to a Square, the side whereof (in regard d is a Square) we may feign to be either $ea + ddd$, or $ea - ddd$, (where e represents a number yet unknown, and to be chosen according to the limit hereafter discovered.) First then let the said side be feigned $ea + ddd$, and then the Square of $ea + ddd$ being equated to the sum in the ninth step, will give

$$-3dda - 3d^2a + d^3 = eea + 2ed^2a + d^3$$

- Which Equation after due Reduction gives $a = \frac{3d^2 - 2ed^2}{3dd + ee}$
- Therefore from the eleventh, first, second, third and fifth steps the sides of the two Cubes sought $\frac{3d^2 - 2ed^2}{3dd + ee}$ and $\frac{edd + 2ed^2}{3dd + ee}$ will be found equal to these quantities, viz.
- Again, forasmuch as the side of the Square mentioned in the tenth step may be feigned to be $ea - ddd$ (as well as $ea + ddd$), let the said side be $ea - ddd$, and then its Square being equated to the sum in the ninth step, this Equation ariseth, viz.

$$-3dda - 3d^2a + d^3 = eea - 2ed^2a + d^3$$

- Which Equation after due Reduction gives $a = \frac{3d^2 + 2ed^2}{3dd + ee}$
- Therefore from the fourteenth, first, second, third and fifth steps the sides of the two Cubes sought will $\frac{3d^2 + 2ed^2}{3dd + ee}$ and $\frac{ded - 2ed^2}{3dd + ee}$ be found equal to these quantities, viz.

The two quantities express'd by letters in the twelfth step will give

CANON 1.

- Supposing d to be the side or Root of the squared cube-number given, take some known number, with this Caution, That its double may be less than the triple of d , and call the number taken e , then the sides of the two Cubes sought shall be these, viz.

$$\frac{3d^2 - 2ed^2}{3dd + ee} \quad \text{and} \quad \frac{edd + 2ed^2}{3dd + ee}$$

An

An Example in Numbers.

Let there be given any squared cube-number, as $64 = 4^2$
 The Root or side whereof is $2 = d$
 Then take a number for e , according to the Caution in the Canon, as $2 = e$

Then by the Canon you will find $\left. \begin{array}{l} 3d^2 - 2ed^2 = 1 \\ 3dd - ee = 3 \end{array} \right\}$ the sides of the Cubes sought.

The Cubes of which sides 1 and 3, viz. 1 and 27 will solve the Question proposed; for if each of those Cubes be subtracted from the given squared Cube 64, the sum of the remainders 63 and 37 makes a Square, to wit, 100.

The two quantities express'd by letters in the fifteenth step will give

CANON 2.

17. Supposing d to be the side or Root of a given squared cube-number, take some known number with this Caution, 3. That it be greater than the double of d , and call the number taken e , then the sides of the two Cubes sought shall be these, viz.

$$\frac{3d^2 + 2ed^2}{3dd - ee} \quad \text{and} \quad \frac{3dd - ee}{3dd - ee}$$

An Example of Canon 2. in Numbers.

Let there be given any squared Cube, as $1 = 1^2$
 The side or Root whereof is $1 = d$
 Then take a number for e , according to the Caution in Canon 2. as $4 = e$

Then by Canon 2. you will find $\left. \begin{array}{l} 3d^2 + 2ed^2 = 17 \\ 3dd - ee = 13 \end{array} \right\}$ the sides of the Cubes sought.

The Cubes of which sides $\frac{17}{13}$ and $\frac{13}{13}$, viz. $\frac{4913}{2197}$ and $\frac{2197}{2197}$ will solve Quest. 69. For if each of those Cubes be subtracted from 1 the given squared Cube, the sum of the remainders makes a Square, to wit, $\frac{4913}{2197}$, whose side is $\frac{17}{13}$.

QUEST. 70.

To find three such cube-numbers, that if every one of them be subtracted from a given Cube, suppose 1, the sum of the three remainders may be a Square.

RESOLUTION.

1. First, by the foregoing Quest. 68. find two such cube-numbers, that their difference being added to 2, (the double of the Cube given in this Question,) the sum may be a Square, and that the greater of those two Cubes may be less than the given Cube 1. But two such Cubes are $\frac{27}{8}$ and $\frac{27}{8}$, whose sides are $\frac{3}{2}$ and $\frac{3}{2}$, (found out in the first Example of Quest. 68.) for if the difference of the said Cubes be added to 2, the sum is $\frac{17}{4}$, which is a square number whose side is $\frac{17}{2}$.
2. Then for the side of the first of the three Cubes sought put $a - \frac{3}{2}$, ($\frac{3}{2}$ being the side of the greater of the two Cubes found out in the first step.)
3. For the side of the second Cube sought put $a + 1$, (1 being the side of the Cube given in the Question.)
4. And let the side of the third Cube be the side of the lesser of the two Cubes found out in the first step, to wit, $\frac{3}{2}$.
5. Then from the second step the first Cube will be $aaa - \frac{27}{8}aa + \frac{27}{8}a - \frac{27}{8}$.
6. And from the third step, the second Cube will be $aaa + 3aa - 3a + 1$.
7. And from the fourth step the third Cube will be $\frac{27}{8}$.
8. Then according to the Question subtract those three Cubes severally from the given Cube 1, so the three remainders shall be these, to wit,

$$\begin{array}{l} 1. \quad 1 - aaa + \frac{27}{8}aa - \frac{27}{8}a + \frac{27}{8} \\ 2. \quad 1 - aaa - 3aa + 3a \\ 3. \quad 1 - \frac{27}{8} \end{array}$$

9. The

9. The sum of the said three remainders is $-\frac{3}{8}aa + \frac{27}{8}a - \frac{162}{8}$
10. Which sum must be equal to a Square, whose side, to the end the value of a may be greater than $\frac{3}{2}$, but less than 1, (as the Positions in the second and third steps do require) may be feigned to be either $\frac{17}{2} -$ any number of a between $\frac{17}{2}$ and $\frac{17}{2}$, or else $\frac{17}{2} -$ any number of a greater than $3\frac{17}{2}$, (which limits may be found out by the method before delivered in Quest. 13. of this Book;) suppose therefore the said side to be $\frac{17}{2} - \frac{17}{2}$, then the Square of $\frac{17}{2} - \frac{17}{2}$ being equated to the sum of the three remainders in the ninth step, this Equation arithmetically, viz.

$$-\frac{3}{8}aa - \frac{27}{8}a + \frac{162}{8} = \frac{3}{8}aa - \frac{27}{2}a + \frac{162}{8}$$

11. Which Equation after due Reduction will give $a = \frac{17}{2}$
12. Therefore from the eleventh, second, third and fourth steps, the sides of the three Cubes sought will be these, $\frac{17}{2}$, $\frac{17}{2}$, and $\frac{17}{2}$
13. The Cubes of which sides are these $\frac{4913}{2197}$, $\frac{2197}{2197}$, and $\frac{27}{8}$ Which three Cubes will solve the Question, as will appear by

The Proof.

By subtracting severally the said three Cubes found out from 1, (the Cube given in the Question,) the three remainders will be these,

$$\frac{172107}{2981984}, \quad \frac{2966301}{2981984}, \quad \frac{2981888}{2981984}$$

The sum of those remainders is $\frac{8128128}{2981984}$

Which sum reduced to its least terms by the common $\frac{8128128}{2981984}$

Divisor 144, will be $\frac{415222}{2981984}$ is a Square, whose side is $\frac{17}{2}$; therefore the Question is solved, and (as is evident by the tenth step) capable of innumerable Answers, the positions in the second, third and fourth steps standing unaltered.

Observations upon the preceding Resolution of Quest. 70.

1. The chief scope in the said Resolution is, to form the positions for the sides of the three Cubes sought in such manner, that when the said Cubes are severally subtracted from the Cube given in the Question, there may be a possibility of equating the sum of the three remainders to a Square, which sum (as you see in the ninth step) is $-\frac{3}{8}aa + \frac{27}{8}a - \frac{162}{8}$, which could not be equated to a Square if $\frac{162}{8}$ were not an affirmative square number; I shall therefore shew how the said $\frac{162}{8}$ doth necessarily become an affirmative square number by the preceding Operation.
2. If the subtraction of every one of the three feigned Cubes in the fifth, sixth and seventh steps from the given Cube 1, as also the adding of the remainders together be well examined, it will appear, that by adding the Cube $\frac{27}{8}$ to 1, and by subtracting the Cube $\frac{27}{8}$ from 1, and then by adding that sum and remainder together, their sum is $\frac{17}{4}$, which (in regard by Construction the greater of the said Cubes, to wit, $\frac{27}{8}$ is added to 1, and the lesser $\frac{27}{8}$ is subtracted from 1) is the same with the sum that will arise by adding the difference of those Cubes unto 2, (the double of 1.) For if the greater of two numbers be added unto, and the lesser be subtracted from a third number, the sum and remainder added together will make the same sum that arithmetically by adding the difference of those two numbers to the double of the said third number: But by Construction in the first step of the Resolution, the said Cubes $\frac{27}{8}$ and $\frac{27}{8}$ are found such that their difference added to 2 makes a Square, to wit, $\frac{17}{4}$. Whence it is manifest that the Algebraic quantity $-\frac{3}{8}aa - \frac{27}{8}a + \frac{162}{8}$ is capable of being equated to a Square, and that variously, as you see in the tenth step of the Resolution.

Example 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from a given Cube, suppose 729, the sum of the three remainders may be a square number.

Resolution.

1. First, by Quest. 68. find two such Cubes that if their difference be added to 1458, to wit, the double of the given Cube 729, the sum may be a Square, and that the side of the greater of those two Cubes may be less than 9 the side of the given Cube 729: But two such Cubes are 64 and 1, (found out in the second Example of Quest. 68.)

for

for if their difference 63 be added to the prescribed number 1458, the sum 1521 is a Square whose side is 39.

2. Then for the side of the first of the three Cubes fought }
let there be put $a - 4$, (4 being the side of the Cube 64, }
the greater of the two Cubes found out in the first step, } $a - 4$
3. For the side of the second Cube fought put $a - 19$, }
(9 being the side of the given Cube 729,) } $a - 19$
4. Let the side of the third Cube be 1, to wit, the side of }
the lesser of the two Cubes found out in the first step, } 1
5. Then (according to the Question) subtract severally the Cubes of those three sides }
(assumed in the three last steps) from the given Cube 729, and add the three remainders }
together, so the sum will be

$$- 15aa + 195a + 1521.$$

6. Which sum must be equal to a Square, whose side, to the end the value of a may be greater than 4, but less than 9, as the second and third steps require, may be feigned to be either $39 +$ any number of a between $\frac{1}{1000}a$ and $1\frac{1}{1000}a$, or else $39 -$ any number of a between $9\frac{999}{1000}a$ and $21\frac{999}{1000}a$, (which limits may be found out by the method delivered in *Quest.* 13, of this Book.) Suppose therefore the said side be feigned $39 + a$, then the Square thereof being equated to the sum in the fifth step, this Equation will arise, to wit,

$$aa + 78a + 1521 = - 15aa + 195a + 1521.$$

7. Which Equation after due Reduction gives $a = \frac{11}{6}$
8. Therefore from the seventh, second, third and fourth }
steps the sides of the three Cubes fought are these, } $\frac{11}{6}, \frac{25}{6}$ and 1 (or $\frac{11}{6}$)
9. And consequently the Cubes themselves are $\frac{1331}{216}, \frac{15625}{216}$ and 1 (or $\frac{1331}{216}$)

Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting every one of the said three Cubes in the ninth step from the given Cube 729, the three remainders will be these,

$$\begin{array}{r} 2837107 \\ 4096 \end{array}, \quad \begin{array}{r} 2966501 \\ 4096 \end{array}, \quad \begin{array}{r} 2981888 \\ 4096 \end{array}.$$

The sum of those remainders is $\frac{11281216}{4096}$
Which sum being reduced to its least terms by the } $\frac{11281216}{4096}$
common Divisor 16, will be $\frac{1410152}{512}$
Which is a Square, whose side is $\frac{1187}{16}$

Example 3.

1. Again, the same things remaining as before in the second Example from the first to the sixth step, we may feign the side of the Square mentioned in the said sixth step to be $39 - 10a$, and then the Square of $39 - 10a$ being equated to $- 15aa + 195a + 1521$ will give $a = \frac{11}{23}$.
2. Therefore from the second, third and fourth steps of Example 2. the sides of three other Cubes to solve the Question propounded in the said second Example will be found these, to wit, $\frac{11}{23}, \frac{11}{23}$, and 1 (or $\frac{11}{23}$.)
3. And consequently the Cubes themselves are $\frac{1331}{12167}, \frac{1331}{12167}$, and 1 (or $\frac{1331}{12167}$.)
4. Which three Cubes being severally subtracted from the given Cube 729, the sum of the three remainders in its least terms will be $\frac{11281216}{512}$, which is a Square, whose side is $\frac{1187}{16}$, as was required in Example 2.

QUEST. 71.

[Another way of solving the preceding *Quest.* 70. when the given Cube is a squared Cube, or the Cube of a Square.]

Let it be required to find three Cube-numbers, such, that if every one of them be subtracted from a given squared Cube-number, suppose 64, the sum of the three remainders may be a Square.

RESOL.

RESOLUTION.

1. First, by the foregoing *Quest.* 69. find two such cube-numbers, that if each of them be subtracted from the given squared Cube 64, the sum of the remainders may be a Square, such are the Cubes 1 and 27, (found out in the Example of *Canon* 1. of the said *Quest.* 69.) for if each of them be subtracted from 64, the sum of the remainders makes the Square 100.
2. Then for the side of the first of the three Cubes fought }
put $a - 1$ either of the sides of the two Cubes found out } $a - 1$
in the first step, viz. }
3. For the side of the second Cube put $a + 4$, (4 being }
the side of the given squared Cube 64,) } $a + 4$
4. Let the side of the third Cube, be the side of the other of }
the two Cubes found out in the first step, to wit, } 3
5. Therefore from the second step the first Cube is $aaa - 3aa + 3a - 1$
6. And from the third step, the second Cube is $aaa + 12aa - 48a + 64$
7. And from the fourth step, the third Cube is 27
8. Then (according to the Question) subtract those three Cubes severally from the given squared Cube 64, so the three remainders will be these, to wit,

$$\begin{array}{r} 1. \quad -aaa - 3aa + 3a - 1 \quad 63, \\ 2. \quad +aaa - 12aa + 48a, \\ 3. \quad \dots \dots \dots + 37. \end{array}$$

9. The sum of which remainders is $- 15aa + 45a + 100$.
10. Which sum is to be equated to a Square, but the side thereof must be so feigned that the value of a may be less than 3, to the end the side $a - 1$ in the second step may be less than 4, because the Cube of $a - 1$ must be subtracted from the Cube of 4. Now to cause that effect, the side of the said Square may be feigned $10 +$ any number of a less than $2\frac{2}{3}a$, or $10 -$ any number of a greater than $6\frac{2}{3}a$; (which limits may be found out by the method before delivered in *Quest.* 13. of this Book.) Let therefore the said side be feigned $10 + a$, then the Square thereof, to wit, $aa + 20a + 100$ being equated to the sum of the remainders in the ninth step, this Equation ariseth, viz.

$$aa + 20a + 100 = - 15aa + 45a + 100.$$

11. Which Equation, after due Reduction, gives $a = \frac{1}{6}$
12. Therefore from the eleventh, second, third and fourth }
steps, the sides of the three Cubes fought will be found } $\frac{1}{6}, \frac{1}{6}$ and 3
these, to wit, } $\frac{1}{6}, \frac{1}{6}$ and 3
13. Therefore the Cubes themselves are $\frac{1}{216}, \frac{1}{216}$ and 27

Which three Cubes will solve the Question; as will be evident by

The Proof.

By subtracting the said three Cubes severally from 64, (the squared Cube given in the Question,) the three remainders will be these,

$$\begin{array}{r} 193223 \\ 4096 \end{array}, \quad \begin{array}{r} 202825 \\ 4096 \end{array}, \quad \begin{array}{r} 151512 \\ 4096 \end{array}.$$

The sum of those remainders is $\frac{11281216}{4096}$
Which sum being reduced to its least terms by the com- } $\frac{11281216}{4096}$
mon Divisor 16, will be $\frac{1410152}{512}$
Which $\frac{1410152}{512}$ is a Square whose side is $\frac{1187}{16}$, therefore the Question is solved.

Example. 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from 1, the sum of the three remainders may be a Square.

Resolution.

1. First, by the preceding *Quest.* 69. find two such Cube-numbers that if each of them be subtracted from 1, to wit, the given squared Cube, the sum of the three remainders may be a Square, such are the Cubes $\frac{1}{27}$ and $\frac{1}{27}$, for if each of them be subtracted from 1, the sum of the remainders will be $\frac{1}{9}$, which is a Square.

2. Then

Therefore the Question is solved; and if the method of resolving this and the preceding *Quest.* 71. be well examined, it will not be difficult to apprehend how to find out as many Cubes as shall be desired, which being severally subtracted from a given squared cube-number, or from the Cube of a given Square, the sum of the remainders may be a Square.

QUEST. 73.

To find two numbers, that if each of them be subtracted from the Cube of their sum, the remainders may be Cubes.

RESOLUTION.

First, making choice of some Square number as 4, I put $4a$ for the sum of the two numbers sought, the Cube whereof will be $64aaa$; then for the first number I put $56aaa$, and for the other number $37aaa$, for these two being severally subtracted from $64aaa$, the remainders will be $8aaa$ and $27aaa$, which are manifestly Cubes, whereby one part of the Question is satisfied: It remains that the sum of the said assumed numbers $56aaa$ and $37aaa$ be equal to $4a$, viz. $93aaa = 4a$, whence by dividing each part by a , there ariseth $93aa = 4$. Now if the said 93 were a Square number, then the value of a would be a rational number, and consequently the Question solved.

But 93 not being a Square, we must enquire whence it ariseth, and by examining the Operation it will appear, that the two Cubes 8 and 27 having been subtracted severally from 64 , the remainders are 56 and 37 , the sum whereof makes 93 before mentioned. So that our first scope must be to find two such Cubes that if each of them be subtracted from the squared Cube 64 , the sum of the remainders may be a Square: But such are the Cubes 1 and 27 , (found out by *Cannon 1.* of the foregoing *Quest.* 69.) for if each of them be subtracted from 64 , the sum of the remainders 63 and 37 makes the Square 100 : therefore I begin the Resolution a-new thus, viz.

1. For the sum of the two numbers sought I put $4a$
2. The Cube thereof is $64aaa$
3. Then for one of the two numbers sought I put $56aaa$
4. And for the other number sought I put $37aaa$
5. Which two numbers being severally subtracted from $64aaa$ the remainders will be Cubes, to wit, $8aaa$ and $27aaa$
6. But the sum of the numbers assumed in the third and fourth steps must be equal to $4a$ in the first step, therefore $100aaa = 4a$
7. Which Equation duly reduced, gives $a = \frac{1}{25}$
8. Therefore from the seventh, third and fourth steps the two numbers sought are $\frac{63}{125}$ and $\frac{11}{125}$

Which numbers will solve the Question propos'd, as will be manifest by

The Proof.

9. The sum of the two numbers found out to wit, $\frac{63}{125}$ and $\frac{11}{125}$, is $\frac{4}{25}$
10. Therefore the Cube of their sum is $\frac{64}{15625}$
11. From which $\frac{64}{15625}$ subtract each of the numbers $\frac{63}{125}$ and $\frac{11}{125}$, so the remainders are Cubes, to wit, $\frac{8}{15625}$ and $\frac{27}{15625}$

Another Example.

1. First, I take some square number, as 1, then I search out two such Cubes that each of them being subtracted from 1, (the Cube of the Square first taken,) the sum of the remainders may be a Square, such are the Cubes $\frac{1}{27}$ and $\frac{8}{27}$, whose sides are $\frac{1}{3}$ and $\frac{2}{3}$, found out in the Example of *Cannon 2.* *Quest.* 69.) for if each of those Cubes be subtracted from 1, the sum of the remainders $\frac{26}{27}$ and $\frac{19}{27}$ will be $\frac{45}{27}$, or in its least terms $\frac{5}{3}$, which is a Square whose side is $\frac{5}{3}$; then by the help of those Cubes and remainders I form the Resolution as before in the first Example, viz.
2. For the sum of the two numbers sought I put a or $1a$, (1 being the square number first taken,) $1a$
3. The Cube of the said sum is $1aaa$
4. Then for one of the two numbers sought I put $\frac{1}{27}aaa$
5. And for the other number sought, $\frac{8}{27}aaa$
6. Which two numbers being severally subtracted from $1aaa$, the remainders will be Cubes, to wit, $\frac{26}{27}aaa$ and $\frac{19}{27}aaa$

7. But

7. But the sum of the two numbers in the fourth and fifth steps $\frac{4}{25}aaa = 1a$ must be equal to $1a$ in the second step, therefore $a = \frac{1}{25}$
8. Which Equation duly reduced gives $a = \frac{1}{25}$
9. Therefore from the eighth, fourth and fifth steps, the two numbers sought, in their least terms, will be found $\frac{1}{25}$ and $\frac{6}{25}$

Which numbers will solve the Question propos'd, as will be evident by

The Proof.

The sum of the two numbers found out in the ninth step, to wit, $\frac{1}{25}$ and $\frac{6}{25}$ is in the least terms $\frac{7}{25}$
 The Cube of that sum is $\frac{343}{15625}$
 From which Cube if you subtract severally the said numbers found out, the remainders will be the Cubes of these sides, to wit, $\frac{1}{15625}$ and $\frac{216}{15625}$

QUEST. 74. (Quest. 19. Lib. 5. Diophant.)

To find three such numbers, that if every one of them be subtracted from the Cube of their sum, the three remainders may be Cubes.

[The text of Diophantus in the Resolution of this Question is so obscure, that it affords not any satisfactory Answer; I shall therefore shew how to solve it by two different ways of my own, by the latter of which this Question may be extended to four, five, or as many numbers as shall be desired.]

RESOLUTION.

1. First, take any square number, as 1, then search out three such Cubes that if they be severally subtracted from the Cube of the said square number 1, the sum of the three remainders may be a Square: But three such Cubes are $\frac{1}{27}$, $\frac{8}{27}$ and $\frac{64}{27}$, (found out in *Example 1.* of the preceding *Quest.* 70.) for if those Cubes be severally subtracted from 1 (or unity,) the sum of the three remainders $\frac{26}{27}$, $\frac{19}{27}$ and $\frac{13}{27}$ will be $\frac{58}{27}$, which is a Square, whose side is $\frac{\sqrt{58}}{3}$. Now by the help of those three preparatory Cubes and remainders, I proceed thus,
2. For the sum of the three numbers sought I put a , or $1a$, (1 being the square number first taken,) $1a$
3. Therefore the Cube of the said sum is $1aaa$
4. Then for the first of the three numbers sought I put $\frac{1}{27}aaa$, (the said $\frac{1}{27}$ being one of the three remainders before mentioned in the first step,) $\frac{1}{27}aaa$
5. In like manner having multiplied the second remainder into aaa , I put the Product for the second number sought, to wit, $\frac{8}{27}aaa$
6. Likewise multiplying the third remainder into aaa , I put the Product for the third number sought, to wit, $\frac{64}{27}aaa$
7. Which three numbers in the three last steps being severally subtracted from $1aaa$, the three remainders will (by the Construction in the first step) be Cubes, to wit, $\frac{26}{27}aaa$, $\frac{19}{27}aaa$ and $\frac{13}{27}aaa$

8. But the sum of the three numbers in the fourth, fifth and sixth steps must be equal to $1a$ in the second step, whence this Equation ariseth,

$$\frac{148877}{2985984}aaa + \frac{19683}{2985984}aaa + \frac{4096}{2985984}aaa = 1a$$

9. Which Equation duly reduced gives $a = \frac{1}{27}$
10. Therefore from the ninth, second, fourth, fifth and sixth steps, the three numbers sought will be made known, to wit, these,

$$\frac{2837107}{15069223}, \frac{2966301}{15069223}, \frac{2981888}{15069223}$$

Which three numbers will solve the Question, as will be evident by

The Proof.

The sum of the said three numbers is $\frac{8827276}{15069223}$, which reduced to its smallest terms by the common Divisor 61009, makes $\frac{144}{27}$
 The Cube of the said sum is $\frac{2985984}{15069223}$
 From

From which Cube subtracting severally the three numbers found out in the tenth step, the remainders will be these three Cubes, to wit,

$$\frac{148877}{15069223}, \frac{19683}{15069223}, \frac{4096}{15069223}.$$

The sides of which Cubes are these, viz.

$$\frac{53}{247}, \frac{27}{247}, \frac{16}{247}. \text{ Therefore the Question is solved.}$$

But because the operation in finding out the three numbers, as also the three Cubes with their sides as aforesaid will be exceeding laborious by reason of long fractions, unless some Compendiums be used, I shall give a Canon deducible from the premises to lessen the work, respect being first had to these following

Preparatory Directions.

1. First, the Rules for multiplying and dividing Fractions in *Self*. 22, 26. of *Chap. 6. Book 1.* must be diligently observed, that the Products and Quotients may come out in the smallest terms.

2. Secondly, when one, two or more numbers are to be severally multiplied by some number, and the Products are to be severally divided by the same number, that multiplication and division may be quite omitted, for the numbers first propos'd to be multiplied will be the same with the Quotients that arise by the said multiplication and division. Moreover, when one, two or more numbers are to be severally multiplied by some number, and the Products are to be divided by some number greater or less than that multiplying number, reduce the said Multiplier and Divisor into the least terms (when they are not such already) by their greatest common Divisor, and take the Quotients for a new Multiplier and Divisor instead of those first prescribed: As, if 41, 39 and 48 be to be severally multiplied by 32, and the Products be to be severally divided by 16, I first reduce the said 32 and 16 to the smallest terms in the same Reason by the common Divisor 16, so the Quotients or new terms will be 2 and 1; then multiplying 41, 39 and 48 severally by 2, (instead of 32,) and dividing the Products severally by 1, (instead of 16,) the Quotients will be 82, 78 and 96, which are found out much speedier and in smaller terms than those that would be found out by multiplying the said 41, 39 and 48 by 32, and dividing the Products by 16 as was first prescribed. This Rule will oftentimes be very useful in the fourth branch of the following Canon.

3. Thirdly, let the square number first taken in the first step of the foregoing Resolution of this *Quest.* 74. be called *bb*, and its side *b*.

4. Fourthly, let the other square number which is equal to the sum of the three remainders found out in the said first step of the Resolution be called *cc*, and its side *c*.

These things premised, I proceed to the

CANON.

1. Divide the known number *b* by the known number *c*, and call the Quotient *a*, which is now a known number.

2. Divide the Cube of *b* by *c*, and let the Quotient be called *d*, which known number is the sum of the numbers sought by the Question.

3. Reduce the numbers *a* and *d* to their smallest common Denominator.

4. Reduce likewise the sides of the preparatory Cubes (found out in the first step of the Resolution) to their smallest common Denominator, then multiply severally the Numerators of those sides by the Numerator of the number *a*, and divide the Products severally by the said common Denominator of the sides of the said preparatory Cubes, and reserve the Quotients for Dividends.

5. Divide severally those Dividends referred, by the Denominator of *a* or *d*, (for these were above reduced to a common Denominator,) so shall the Quotients be the sides of the Cubes sought.

6. Lastly, by subtracting severally the Cubes of the sides last found out, from the Cube of the sum of the numbers sought, (which sum was above found by the second step of the Canon,) the remainders shall be the numbers sought, and the smallest that have a common Denominator with the Cubes found out in the fifth step of the Canon.

This Canon with the preceding preparatory Directions may be practically illustrated by the Examples of the preceding *Quest.* 73. and of this and the following 75 and 76 Questions.

Example 2.

Example 2.

Let it be required to find three such numbers, that if every one of them be subtracted from the Cube of their sum, the three remainders may be Cubes.

1. First take some square number, as 9, then find three such Cubes, that if they be severally subtracted from 729 (the Cube of the said Square 9) the sum of the three remainders may be a Square: But three such Cubes are $\frac{122222}{4096}$, $\frac{122222}{4096}$ and $\frac{4096}{4096}$, (or 1,) whose sides are $\frac{11}{16}$, $\frac{11}{16}$ and 1, (found out in the second Example of *Quest.* 70.) for if those Cubes be severally subtracted from 729, (or $\frac{222222}{4096}$), the sum of the three remainders $\frac{222222}{4096}$, $\frac{222222}{4096}$ and $\frac{222222}{4096}$ being reduced to its least terms will be $\frac{222222}{256}$, which is a square number whose side is $\frac{11}{16}$.

2. Then by proceeding according to the foregoing preparatory Directions and Canon, the numbers and Cubes sought will be found to be the same as were before found out in the first Example of this 74th Question.

Example 3.

1. Taking again the same square number 9 as in the second Example, I seek three other Cubes, that every one of them being subtracted from 729 (the Cube of the said Square 9) the sum of the three remainders may be a Square: But three such Cubes are $\frac{122222}{11167}$, $\frac{122222}{11167}$ and 1, whose sides are $\frac{11}{11167}$, $\frac{11}{11167}$ and 1, (or $\frac{11}{11167}$), found out in the third Example of *Quest.* 70. for if those three Cubes be severally subtracted from the said 729, the sum of the three remainders in its least terms will be $\frac{111111}{11167}$, which is a Square, whose side is $\frac{11}{11167}$.

2. Then by proceeding according to the preparatory Directions and the Canon, (which follow the first Example of this 74th Question,) the sides of the three Cubes sought will be found $\frac{11}{11167}$, $\frac{11}{11167}$ and $\frac{11}{11167}$, and the three numbers sought are these, to wit, $\frac{111111}{11167}$, $\frac{111111}{11167}$ and $\frac{111111}{11167}$, whose sum in its least terms is $\frac{111111}{11167}$, from the Cube whereof if the said three numbers be severally subtracted, the three remainders will be Cubes, whose sides are those above found out. Therefore the Question is solved.

QUEST. 75. (Another way of solving the preceding Quest. 74.)

To find three such cube-numbers, that if every one of them be subtracted from the Cube of their sum, the remainders may be Cubes.

RESOLUTION.

1. First take some square number, as 4, then find three such Cubes that if they be severally subtracted from 64, (the Cube of the said Square 4,) the sum of the three remainders may be a Square: But three such Cubes are $\frac{22222}{4096}$, $\frac{22222}{4096}$ and 27, whose sides are $\frac{11}{16}$, $\frac{11}{16}$ and 3, (found out in the first Example of *Quest.* 71.) for if those Cubes be severally subtracted from the said 64, the three remainders will be these, to wit, $\frac{22222}{4096}$, $\frac{22222}{4096}$ and $\frac{22222}{4096}$, whose sum in its least terms is $\frac{22222}{4096}$, which is a Square whose side is $\frac{11}{16}$.

2. Then by proceeding according to the preparatory Directions and Canon which follow the first Example of the preceding *Quest.* 74. the sides of the three Cubes sought will be found these, to wit,

$$\frac{82}{185}, \frac{78}{185}, \frac{96}{185}.$$

3. And consequently the Cubes themselves are

$$\frac{551368}{6331625}, \frac{474552}{6331625}, \frac{884736}{6331625}.$$

4. And the three numbers sought are these,

$$\frac{1545784}{6331625}, \frac{1622600}{6331625}, \frac{1212416}{6331625}.$$

Which will solve the Question, as will be manifest by

The Proof.

5. The sum of the said three numbers is $\frac{222222}{6331625}$, which reduced to its smallest terms by the common Divisor 34225, gives $\frac{111111}{185}$.

6. The Cube of the said sum is $\frac{222222}{6331625}$. From which Cube if you subtract severally the three numbers before found out in the fourth step, the remainders will be the three Cubes above express'd in the third step.

Another

Another Example.

1. First take some square number, as 1, then find three such Cubes that if they be severally subtracted from 1 (the Cube of the Square first taken) the sum of the three remainders may be a Square: But three such Cubes are $\frac{1}{27}, \frac{1}{8}, \frac{1}{27}$, (found out in the third Example of *Quest. 71.*) for if whose sides are $\frac{1}{3}, \frac{1}{2}$ and $\frac{1}{3}$, (found out in the third Example of *Quest. 71.*) for if those Cubes be severally subtracted from 1, (the Cube of the Square first taken,) the sum of the three remainders $\frac{26}{27}, \frac{7}{8}, \frac{26}{27}$ being reduced to its least terms will be $\frac{44}{3072}$, which is a Square whose side is $\frac{11}{16}$.

2. Then by proceeding according to the preparatory Directions and Canon (which follow the first Example of the preceding *Quest. 74.*) the sides of the three Cubes sought will be found these, to wit, $\frac{11}{16}, \frac{11}{16}$ and $\frac{11}{16}$; and the three numbers sought will be these, to wit, $\frac{1093861}{167936}, \frac{1093861}{167936}$ and $\frac{1093861}{167936}$, which will solve the said 75th Question, as will appear by the Proof.

QUEST. 76.

To find four such numbers, that if every one of them be subtracted from the Cube of their sum, the four remainders may be Cubes.

RESOLUTION.

1. First take some square number, as 4, then find four such Cubes that if they be severally subtracted from 64 (the Cube of the said Square 4) the sum of the four remainders may be a Square: But four such Cubes are these $\frac{1}{64}, \frac{1}{16}, \frac{1}{16}, \frac{1}{64}$, (found out in the Example of the said fifth Question,) for if those Cubes be severally subtracted from the said 64, the four remainders will be these, to wit, $\frac{63}{64}, \frac{15}{16}, \frac{15}{16}$ and $\frac{63}{64}$, which remainders being added together, the sum in its least terms is $\frac{149}{64}$, which is a Square whose side is $\frac{12}{8}$.

2. Then by proceeding according to the preparatory Directions and Canon, (which follow the first Example of the foregoing *Quest. 74.*) the sides of the four Cubes sought will be found these, to wit,

$$\frac{9268}{16027}, \frac{9672}{16027}, \frac{9672}{16027} \text{ and } \frac{6630}{16027}.$$

3. Therefore the four Cubes sought are these,

$$\frac{821130284032}{4116771011683}, \frac{904792212448}{4116771011683}, \frac{338608873000}{4116771011683}, \frac{2914344247000}{4116771011683}.$$

4. Which Cubes being severally subtracted from the Cube of $\frac{12}{8}$, (which by the second branch of the Canon will be found out for the sum of the four numbers sought,) the remainders will be the numbers sought, to wit, these,

$$\frac{465783187968}{4116771011683}, \frac{949304199000}{4116771011683}, \frac{183121129552}{4116771011683}, \frac{996479225000}{4116771011683}.$$

Which four numbers solve the Question, as will appear by the Proof.

From the manner of solving this and the preceding 75th Question, it is easy to apprehend, how five, six or as many numbers as shall be desired, may be found out, which being severally subtracted from the Cube of their sum, may leave as many Cubes: But so many numbers as are desired, so many preparatory Cubes must be first found out, such, that if they be severally subtracted from the Cube of some square number chosen at pleasure, the sum of the remainders may be a Square; which preparatory Cubes may be found out by the method before delivered in *Quest. 71.* and 72.

QUEST. 77. (Quest. 20. Lib. 5. Diophant.)

To find three such numbers that if the Cube of their sum be subtracted from every one of them, the remainders may be Cubes.

RESOLUTION.

1. For the sum of the three numbers sought put a
2. And let the three numbers be $2aaa, 9aaa, 28aaa$
3. It remains that their sum $39aaa$ be equated to a , whence $39aa = 1$; where if 39 were a Square the Question would be solved by Rational numbers. But 39 is not

not

not a Square, whence therefore is it produced? Examine the Positions, and you will find that 1 being added severally to the three Cubes 1, 8 and 27, the sum of those three additions makes 39. We must therefore search out three Cubes whose sum increased with 3 may make a Square, to which end

4. For the sides of the three Cubes put $e, -e+3$; and 1
5. Then the sum of the Cubes of those three sides increased with 3, makes $9ee - 27e + 31$
6. Which sum is to be equated to a Square, but the side thereof must be so feigned that the value of e may be less than 3; now to cause that effect the side may be variously feigned within limits easy to be discovered from the method in divers preceding Questions of this Book, let it be $3e - 7$, then the Square of $3e - 7$ being equated to $9ee - 27e + 31$ will give $e = \frac{2}{3}$; therefore the sides of the three Cubes are $\frac{2}{3}, \frac{1}{3}$ and 1, and the Cubes themselves are $\frac{8}{27}, \frac{1}{27}$ and 1, by the help whereof the work is to be renewed thus, viz.
7. Add 1 to every one of the three Cubes before found, and the sums will be $\frac{11}{27}, \frac{4}{27}$ and $\frac{10}{27}$; then instead of $2aaa, 9aaa$ and $28aaa$ (in the second step) put for the three numbers sought,

$$\frac{11}{27}aaa, \frac{4}{27}aaa, \frac{10}{27}aaa.$$

8. Then the sum of those three numbers being equated to a , (which in the first step was put for the sum of the three numbers sought,) gives this Equation, to wit, $\frac{21}{27}aaa = a$
 9. Whence, after due Reduction, $a = \frac{27}{21}$
- Therefore from the ninth and seventh steps the three numbers sought are $\frac{11}{27}, \frac{4}{27}$ and $\frac{10}{27}$; for if from every one of them, the Cube of their sum, to wit, $\frac{1}{27}$ be subtracted, there will remain the Cubes $\frac{10}{27}, \frac{3}{27}$ and $\frac{10}{27}$, whose sides are $\frac{2}{3}, \frac{1}{3}$ and $\frac{2}{3}$. From the premises it is evident that the Question is capable of innumerable Answers, and may easily be extended to four, five, or as many numbers as you please.

QUEST. 78. (Quest. 21. Lib. 5. Diophant.)

To find three such numbers that their sum may be a Square, and that if to the Cube of the said sum the three numbers be severally added, the three sums may be square numbers.

RESOLUTION.

1. For the sum of the three numbers sought, that it may be? aa
2. Then for the first number put $3aaaa$
3. For the second, $8aaaa$
4. And for the third, $15aaaa$
5. Whence it is evident that every one of them added to the Cube of their sum makes a Square; But the sum of the three numbers (in the second, third and fourth steps) must be equal to aa which was first put for their sum; therefore $26aaaa = aa$, and consequently, (by dividing each part by aa), $26aaaa = 1$. In which last Equation if 26 were a squared square number, the value of a would be a rational number: Whence therefore comes 26? Examine the Positions, and you will find that 'tis the sum of the three numbers 3, 8 and 15, every one of which increased with 1 makes a Square; therefore the scope of our search must be to find three numbers, every one of which increased with 1 may make a Square, and that the sum of the said three numbers may be a Biquadrate: To which end let the three numbers be $aaaa - 2aa$; $aa - 1 - 2a$ and $aa - 2a$; for every one of these increased with 1 makes a Square, and their sum makes a Biquadrate, to wit, $aaaa$; and 'tis evident the value of a may be any number greater than 2, (for the third number $aa - 2a$ shews that aa must be greater than 2, and consequently a greater than 2.) Suppose therefore $a = 3$, whence the three numbers $aaaa - 2aa$; $aa - 1 - 2a$; $aa - 2a$ will be 63, 15 and 3; now with the help of these three numbers the work may be renewed thus, viz.
6. Let the sum of the three numbers sought be (as before) aa
7. And the first number $63aaaa$
8. The second $15aaaa$
9. And the third $3aaaa$

P 2

10. There-

5. Therefore the Square of the side in the third step is $\rightarrow 9aa - 12 + \frac{4}{aa}$
6. And the Square of the side in the fourth step is $\rightarrow 16aa - 12 + \frac{2}{aa}$
7. The former of those two Squares increased with 24 makes a Square, to wit, $\rightarrow 9aa + 12 + \frac{4}{aa}$
8. And the latter Square in the sixth step increased with 24 makes a Square, to wit, $\rightarrow 16aa + 12 + \frac{2}{aa}$
9. The sides of which two Squares in the two last steps are $\rightarrow 3a + \frac{2}{a}$ and $4a + \frac{1}{a}$
10. It remains that the sum of the Squares in the fifth and sixth steps, together with the given number 15 may make a Square, but it makes $\rightarrow 25aa - 9 + \frac{2}{aa}$
11. Which sum must be equated to a Square, viz. either to $25aa$, or to $\frac{2}{aa}$, let it first be equated to $25aa$, $\rightarrow 25aa - 9 + \frac{2}{aa} = 25aa$
- viz. suppose
12. Which Equation, after due Reduction, gives $\rightarrow a = \frac{1}{5}$
13. Therefore from the twelfth, third and fourth steps the sides of the second and third Squares sought are $\frac{1}{5}$ and $\frac{1}{5}$, and consequently $9, \frac{1}{25}$ and $\frac{1}{25}$ are the three Squares sought; for every two with 15 make Squares, to wit, $\frac{1}{100}, \frac{1}{225}, \frac{1}{36}$, whose sides are $\frac{1}{10}, \frac{1}{15}, \frac{1}{6}$.
- But if $25aa - 9 + \frac{2}{aa}$ be equated to $\frac{2}{aa}$, then the value of a will be $\frac{1}{5}$, yet the same Solution will be found as before, because here we must conceive $\frac{2}{a} - 3a$ and $\frac{1}{a} - 4a$

to be the sides of the two Squares, which sides being resolved according to the latter value of a , to wit, $a = \frac{1}{5}$, there will come forth (as before) $\frac{1}{10}$ and $\frac{1}{15}$.

This Question is capable of innumerable Answers upon a double ground; for first, the first square may be any known square number at pleasure; then the sides of the second and third Squares may be variously feigned from divers numbers, which may be the sides about the right-angle of unlike right-angled Triangles; as instead of 3 and 4 we may take 8 and 15, 5 and 12, and innumerable others.

QUEST. 88. (Quest. 32. Lib. 5. Diophant.)

To find three Squares that the sum of their Squares may make a Square; or, (which is the same thing) to find three numbers that the sum of their squared Squares may be a Square.

RESOLUTION.

- For one of the square numbers sought put $\rightarrow aa$
- And for the two others put $\rightarrow bb$ and cc
- Then the sum of the Squares of those three Squares is $\rightarrow aaaa + bbbb + cccc$
- Which sum must be equated to a Square, let its side be $aa - dd$, the Square whereof equated to the said sum gives $\rightarrow aaaa + bbbb + cccc = aaaa - 2ada + dddd$
- Which Equation, after due Reduction, gives $\rightarrow aa = \frac{ddd - bbb - ccc}{2d}$
- In which last Equation, because the Numerator and Denominator are not perfect Squares, the value of a is not expressible by a rational number; but to cause it to be rational, we may discover by the said Fraction that a certain number and two Squares must be found out, such, that if from the Square of the number the sum of the two Squares be subtracted, the remainder may be to the double of the said number as a square number to a square number: Now to find out such a number and two Squares, let $rr + ss$ be put for the number, and rr for one Square, and ss for the other; then from the Square of $rr + ss$, that is, from $rrr + 2rss + sss$ subtract the sum of the Squares of rr and ss , that is, $rrr + sss$, and the remainder $2rss$ must be to the double of the number $rr + ss$, that is, to $2rr + 2ss$, as a Square to a Square; therefore also the

the halves of $2rrs$ and $2rs - 2ss$, that is, rrs and $rr - ss$ shall be as a Square to a Square, for the Proportion is not changed: And since rrs is a Square, whose side is rs , it remains only to make $rr - ss$ a Square; but such it will be if r and s represent the sides about the right-angle of a right-angled Triangle whose three sides are expressible by rational numbers, for then $rr - ss$ will be the Square of the Hypotenuse. Therefore from the premises the following Canon is deducible to solve the Question proposed.

CANON.

Take for two of the three Squares sought the Squares of the sides about the right-angle of some right-angled Triangle in rational numbers; then divide the Product made by the mutual multiplication of those two Squares, by the Square of the Hypotenuse, and there will come forth the third Square sought.

As, for example, let there be exposed the right-angled Triangle 3, 4, 5, then two of the Squares sought shall be 9 and 16, (to wit, the Squares of 3 and 4 the sides about the right-angle;) and if the Product of 9 into 16, that is, 144, be divided by 25 (the Square of the Hypotenuse,) the Quotient $\frac{144}{25}$ shall be the third Square sought. I say, 9, 16 and $\frac{144}{25}$ are three Squares, which will solve the Question, for the sum of their Squares makes $\frac{225}{625}$, which is a Square, whose side is $\frac{15}{25}$.

QUEST. 89. (Quest. 33. Lib. 5. Diophant.)

A certain Vintner made a mixture of two sorts of Wines, whereof one cost eight pence the quart, and the other five pence, at which prices the whole mixture was worth a square number of pence, unto which 60 being added the sum would also make a square number, whose side was the number of quarts contained in the mixture. The Question is, to find the number of quarts of each sort of Wine in the mixture.

RESOLUTION.

- For the price of the whole mixture put $\rightarrow aa - 60$
- Then if 60 be added to that total price the sum will be the square aa , whose Root (as the Question requires) must be equal to the number of quarts of both sorts of Wine in the mixture, to wit,
- From the premises we may rightly infer, that $aa - 60$ (the total cost of the mixture) is greater than $5a$, but less than $8a$; (that is, greater than the Product of the price of the worse sort of Wine in the mixture by the price of the worse sort of Wine multiplied into the number of quarts of the mixture, but less than the Product of the same number of quarts multiplied into the dearer sort of Wine.) But if $aa - 60$ be greater than $5a$, and less than $8a$, then the value of a (by Quest. 10. of this Book) is greater than $10\frac{1}{3}$, &c. but less than $12\frac{2}{3}$, &c. Therefore $aa - 60$ (which the Question requires to be a Square) must be equated to some Square whose side must be so feigned that the value of a may be within those limits? Now to cause that effect, the side of the said Square may be feigned $\rightarrow a - 1$ any absolute number between $17\frac{1}{3}$, &c. and $37\frac{1}{3}$, &c. (as $22\frac{1}{3}$, &c. or $a - 1$ any absolute number between $10\frac{1}{3}$, &c. and $12\frac{2}{3}$, &c. as $11\frac{1}{3}$, &c. have been shown in Quest. 11. of this Book.) Let then the said side be feigned $\rightarrow a - 1$, the whole Square $aa - 4a + 1$ equated to $aa - 60$ will give $a = 12\frac{2}{3}$ for the desired number of quarts in the whole mixture.
- Then from $\frac{144}{25}$, the Square of the said $12\frac{2}{3}$, subtract 60, and the remainder $\frac{144}{25} - 60 = \frac{144 - 1500}{25} = \frac{-1356}{25}$ is the price of the whole mixture, which is a Square whose side is $\frac{37}{5}$, and because $\frac{144}{25}$ is the number of pence expressing the value of the mixture, it must be equal to the Product of 8 multiplied by a certain part of $12\frac{2}{3}$, (the number of quarts in the mixture,) together with the Product of 5 multiplied by the remaining part of $12\frac{2}{3}$, we must therefore divide $12\frac{2}{3}$ into two such parts, that if the one be multiplied by 5, and the other by 8, the sum of the two Products may make $\frac{144}{25}$; but that may be done thus,
- For one of the desired parts of $12\frac{2}{3}$ put $\rightarrow 12\frac{2}{3} - e$
- Then the other shall be $\rightarrow e$
- And if the former part be multiplied by 5, and the latter by 8, the sum of the Products will be $\rightarrow 91\frac{1}{3} - 3e$
- Which sum must be equated to $\frac{144}{25}$, viz. $\rightarrow 91\frac{1}{3} - 3e = \frac{144}{25}$

9. Which Equation, after due Reduction, makes known one of $\frac{1}{2}e = \frac{11}{2}a$
the desired parts, to wit, $\frac{11}{2}a$.
10. Which subtracted from $12\frac{1}{2}a$, leaves the other part, to wit, $\frac{11}{2}a$.
11. I say the total mixture of Wine might be compos'd of $\frac{11}{2}$ quarts of five pence the quart, and $\frac{11}{2}$ quarts of eight pence the quart; whence the value of the whole mix quantity, to wit, of $12\frac{1}{2}$ quarts is $\frac{11}{2} \times 5 + \frac{11}{2} \times 8$ pence, which is a square number whose side is $\frac{11}{2} \times 3$; and if to the said Square $\frac{11}{2} \times 3$ you add 60, the sum is also a Square, to wit, $\frac{11}{2} \times 3$, whose side $12\frac{1}{2}$ is the number of quarts in the mixture.
12. But because the side of the Square to be equated to $aa - 60$ may be feign'd a — any absolute number between $2\frac{1}{2}$ and $3\frac{1}{2}$, &c. let the said side be $a - 3$, the Square whereof equated to $aa - 60$ will give $a = 11\frac{1}{2}$ for the number of quarts in the mixture, then the Square of $11\frac{1}{2}$ is $\frac{11}{2} \times 3$, from which subtracting 60, the remainder $\frac{11}{2} \times 3$ is the square number of pence expressing the value of the mixture. Now the said $11\frac{1}{2}$ is to be divided into two such numbers that if one of them be multiplied by 5, and the other by 8, the sum of the Products may make the Square $\frac{11}{2} \times 3$; but two such numbers (by working as before) will be found $\frac{11}{2}$ and $\frac{11}{2}$.
- I say again, the mixture may be compos'd of $\frac{11}{2}$ quarts of five pence the quart, and $\frac{11}{2}$ quarts of eight pence the quart; whence the value of the whole mix quantity, to wit, of $12\frac{1}{2}$ quarts, is $\frac{11}{2} \times 5 + \frac{11}{2} \times 8$ pence, which is a Square, to which if you add 60 the sum is also a Square, to wit, $\frac{11}{2} \times 3$, whose side $12\frac{1}{2}$ is the number of quarts in the whole mixture. From the premises 'tis evident that the Question is capable of innumerable Answers in rational numbers.

QUEST. 90.

To find a right-angled Triangle in rational numbers, that one of the sides about the right-angle may be to the Area in a given Reason, suppose as r to s .

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, viz. a the greater and e the lesser, so $aa + ee = aa - ee$; and the three sides will be these, to wit, $aa + ee$, $aa - ee$, $2ae$.
2. The Area of this Triangle is $aa^2 - ae^2$.
3. Then (according to the Question) let these four quantities be supposed to be Proportional, viz. $r : s :: aa - ee : aa^2 - ae^2$.
4. And because if the two latter terms of that Analogy be severally divided by $aa - ee$, the Quotients are 1 and ae , therefore that Analogy may be reduced to this, viz. $r : s :: 1 : ae$.
5. And by comparing the Product of the extremes to the Product of the means, this Equation ariseth, viz. $rae = s$.
6. And by dividing each part of the last Equation by ra , this ariseth $e = \frac{s}{ra}$.
- Hence this.

CANON.

7. Take any number at pleasure, which may be called a , then divide s the latter term of the given Reason, by the Product of the first term r multiplied into the number a , and call the Quotient the number e ; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find a right-angled Triangle, such, that one of the sides about the right-angle may be to the Area as 1 to 10.

Suppose am to be $\frac{1}{10}$ the terms of the given Reason;
 $\frac{1}{10} : 1 :: a : e$
 Then by the Canon, $e = \frac{1}{10a}$
 taken at pleasure,

Lastly, from 5 and 2 form a right-angled Triangle, and the three sides will be 29, 21 and 20, which Triangle will solve the Question, for 21, one of the sides about the right-angle, is to the Area 210, as 1 to 10. Which was required.

Likewise

Likewise by the Canon, this right-angled Triangle, to wit, 101, 99 and 20 will be found to solve the Question, for 99 is to the Area 990, as 1 to 10, and innumerable right-angled Triangles in Fractions may be found to perform the same effect.

QUEST. 91.

To find a right-angled Triangle in rational numbers, that the Hypotenusal may be to the Area in a given Reason, suppose as r to s .

RESOLUTION.

1. For one of the sides about the right-angle put a .
2. And for the other side about the right-angle put e .
3. Then the square Root of the sum of the Squares of those sides shall be the Hypotenusal, to wit, $\sqrt{aa + ee}$.
4. And the Area of the said Triangle is $\frac{1}{2}ae$.
5. Now according to the Question, the Hypotenusal must be to the Area as r to s , therefore from the third and fourth steps this Analogy ariseth, viz. $r : s :: \sqrt{aa + ee} : \frac{1}{2}ae$.
6. But the Squares of those Proportionals are also Proportionals, therefore $rr : ss :: aa + ee : \frac{1}{4}ae^2$.
7. And from the last Analogy, by comparing the Product of the multiplication of the extremes to the Product of the means, this Equation ariseth, viz. $\frac{1}{2}raae = ssaa + ssee$.
8. From which Equation, by transposition of $ssee$, this ariseth, $\frac{1}{2}raae - ssee = ssaa$.
9. And by dividing each part of the last Equation by $\frac{1}{2}raa - ss$, $e = \frac{ssaa}{\frac{1}{2}raa - ss}$ there will arise $e = \frac{ssaa}{\frac{1}{2}raa - ss}$.
10. In which last Equation the Numerator $ssaa$ is a Square whose side is sa , and if the Denominator were a Square, then the whole Fraction would be also a Square, and consequently the side thereof, to wit, the number e would be rational; it remains therefore to equate the Denominator $\frac{1}{2}raa - ss$ to a Square, to which end, let the side thereof be feign'd $\frac{1}{2}ra - b$, then the Square of $\frac{1}{2}ra - b$ being equated to $\frac{1}{2}raa - ss$, this Equation ariseth, viz. $\frac{1}{4}raa - ss = \frac{1}{4}raa - rba + bb$.
11. Whence, after due Reduction, you will find $a = \frac{ss + bb}{rb}$.
12. Now if we suppose r , s and b to represent known rational numbers, then a , e and $\sqrt{aa + ee}$ which in the three first steps were put for the three sides of the right-angled Triangle sought, will also (from the eleventh, ninth and tenth steps,) be expressible by rational numbers, to wit, these $\frac{ss + bb}{rb}$, $\frac{2ss + 2bb}{rbb - rbb}$, $\frac{ss + 2ssb + bbb}{rbb - rbb}$.
13. Or the two first of the three sides last express'd may be reduced to the same Denominator with the third, and then the three sides of the right-angled Triangle sought will be these, to wit, $\frac{ss + bb}{rbb - rbb}$, $\frac{2ss + 2bb}{rbb - rbb}$, $\frac{ss + 2ssb + bbb}{rbb - rbb}$.
- Which three sides, if they be express'd by words, will give this

CANON.

14. Take for b any number less than s the latter term of the given Reason, then from the numbers s and b form a right-angled Triangle, and multiply the three sides severally by the Hypotenusal; lastly, divide those three Products severally by the Product made by the multiplication of the difference of the Squares of the two numbers s and b , (which formed the said Triangle,) into the Product of b the lesser of the same two numbers and r the first term of the given Reason; so shall the Quotients be the three sides of a right-angled Triangle, which will solve the Question proposed.

An

An Example in Numbers.

Let it be required to find out a right-angled Triangle whose Hypotenusal may be to the Area as 2 to 3.

Suppose $\left. \begin{array}{l} r = 2 \\ s = 3 \\ b = 1 \end{array} \right\}$ the Terms of the given Reason, less than 3, (or 1.)

Then form a right-angled Triangle from 3 and 1, (to wit, s and b), and the three sides will be 10, 8 and 6; these multiplied severally by the Hypotenusal 10 will produce 100, 80 and 60, which divided severally by the Product which answers to $ss - bb$ into 16, that is, by 16, will give $\frac{25}{4}$, $\frac{5}{2}$ and $\frac{3}{4}$ for the Triangle sought; for the Hypotenusal $\frac{25}{4}$, or $\frac{125}{16}$, is to the Area $\frac{125}{16}$, as 2 to 3. Which was required.

In like manner, if it were desired to find a right-angled Triangle whose Hypotenusal might be to the Area as 3 to 2; then by the Canon, the three sides will be found $\frac{13}{2}$, $\frac{12}{2}$ and $\frac{5}{2}$.

QUEST. 92.

To find a right-angled Triangle in rational numbers, that the sum of all the three sides may be to the Area in a given Reason, suppose as r to s .

RESOLUTION.

1. For the right-angled Triangle sought let a Triangle be formed from any two numbers, suppose from a the greater and e the lesser, so the three sides will be these, $aa + ee$, $aa - ee$, $2ae$
2. Then the Area will be $AAAE - AEEe$
3. And the sum of the three sides is $2AA + 2Ae$
4. Now (according to the Question) the sum of the three sides must be to the Area, as r to s , therefore,

$$\text{As } r : s :: 2AA + 2Ae : AAEE - AEEe.$$

5. Or, by dividing each of the two latter terms of that Analogy by a , this arith; viz.

$$\text{As } r : s :: 2A + 2e : AAe - Eee.$$

6. Whence, by comparing the Product of the extremes to the Product of the means, this Equation arith, viz. $reAA - reee = 2ia + 2ie$
7. Therefore by due transposition, $reAA - 2ie = ree + 2ie$
8. And by dividing all in the last Equation by re , this arith, $aa - \frac{2e}{r} = \frac{ree + 2ie}{r}$
9. Which last Equation being resolved by the Canon in Sect. 15. Book 1. the value of a will be discovered, viz. $a = e + \frac{2e}{r}$

Hence this

CANON.

10. Take any number at pleasure, which may be called e ; then to the number e add the Quotient that arith by dividing the double of the latter term of the given Reason, by the Product of the first term multiplied into the number e , and call the sum the number a ; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find out a right-angled Triangle, that the sum of all the three sides may be to the Area as 1 to 5.

Suppose $\left. \begin{array}{l} r = 1 \\ s = 5 \\ e = 2 \end{array} \right\}$ the Terms of the given Reason, then taken at pleasure,

By the Canon, $a = 7 = e + \frac{2e}{r}$

Then form a right-angled Triangle from 7 and 2, and the three sides will be 53, 45, 28, which will solve the Question, for the sum of all the three sides, to wit, 126, is to the Area 630, as 1 to 5. Which was required to be done.

Likewise if a right-angled Triangle be formed from 11 and 1, (to wit, a and e , found out by the Canon,) the three sides will be 122, 120, 22, whose sum 264 is to the Area 1320, as 1 to 5.

Again,

Again, if a right-angled Triangle be formed from 11 and 10, the three sides will be 121, 21 and 220, whose sum 462 is to the Area 2310, as 1 to 5.

Lastly, you may find out as many right-angled Triangles in Fractions as you please to solve the Question. Also, upon the same ground it will not be difficult to find out innumerable Isosceles-Triangles, in every one of which the Perimeter shall be to the Area in a given Reason.

QUEST. 93.

To find a right-angled Triangle in rational numbers, that as well the Hypotenusal as the difference of the sides about the right-angle may be a Square.

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, suppose from a and e , and let a be the greater, so the three sides will be these, viz. $aa + ee$, $aa - ee$, $2ae$
2. Now (according to the Question) as well the Hypotenusal as the difference of the sides about the right-angle must be a Square, so we are fallen upon this Duplicate equality, viz. $aa + ee = \square$, $2ae - aa + ee = \square$
3. The difference of those two quantities is $2AA - 2Ae$
4. Which difference is equal to the Product of these two quantities, to wit, $2A - 2e$ and A
5. The half-sum of those two Factors in the last step is $\frac{1}{2}A - e$
6. Then the Square of the said half-sum being equated to the greater of the two quantities in the second step, $aa + ee = 2AA - 3Ae + e^2$
7. Whence after due Reduction there will arise $12e = 5A$
8. And by reducing the last Equation into Proportionals, it shall be $\text{As } 12 : 5 :: a : e$.

Hence this

CANON.

9. If from 12 and 5, or any two numbers in that proportion; a right-angled Triangle be formed, it will solve the Question.

As, for example, in the right-angled Triangle 169, 119 and 120, which is formed from 12 and 5, the Hypotenusal 169 is a Square; also the difference of the sides about the right-angle; to wit, 1 is a Square. The same effect will be produced in a right-angled Triangle formed from any two numbers which have such proportion one to another as 12 to 5.

QUEST. 94.

To find a right-angled Triangle, that one of the sides about the right-angle may be a Square, which added to a given multiple, suppose the triple, of the Square of the difference of the sides about the right-angle may make a Square.

RESOLUTION.

1. For one of the sides about the right-angle, that it may be a Square, put 4
2. And for the other put a
3. The sum of their Squares must make a Square, to wit, $aa + 16 = \square$
4. The difference of the sides about the right-angle is the Square of the Hypotenusal, therefore $a \sim 4$
5. The triple of the Square of that difference is $3aa - 24a + 16$
6. To which (according to the Question) add 4, the square number first assumed for one of the sides about the right-angle, and the sum must be equal to a Square, viz. $3aa - 24a + 16 + 4 = \square$
7. So in the third and sixth steps we are fallen upon a Duplicate equality, but the numbers prefix to aa in the quantities to be equated, are not Squares, neither are the two known numbers in the same quantities both Squares, for 52 is not a Square, whereby the said Duplicate equality is inexplicable; but if the said 52 were a Square, then the Duplicate equality

equality might be resolv'd; therefore instead of the square number 4 which in the first step was assum'd for one of the sides about the right-angle, we must seek such a Square that if it be added to the triple of its Square the sum may be a Square. Suppose therefore that Square sought to be ee , this added to the triple of its Square makes $3ee + ee$ to be equated to a Square, the side whereof may be variously feigned, let it be $ee + e$, then the Square of $ee + e$, to wit, $ee + 2ee + ee$ being equated to $3ee + ee$, after due Reduction the value of e will be found 1, and ee is also 1. So we have found a Square, to wit, 1, which added to the triple of its Square makes the Square 4; therefore now the Resolution may be renewed thus, *viz.*

8. For one of the sides about the right-angle put the Square . . . 1
9. And for the other side . . . a
10. The sum of their Squares must be equal to a Square, *viz.* . . . $aa + 1 = \square$
11. The difference of the sides about the right-angle is . . . $a - 1$
12. The triple of the Square of that difference is . . . $3aa - 6a + 3$
13. To which adding the side 1 in the eighth step, the sum must be equal to a Square, *viz.* . . . $3aa - 6a + 4 = \square$
14. Also from the tenth step, . . . $aa + 1 = \square$
15. So in the two last steps we have a new Duplicate equality which may be resolv'd thus; first, to the end there may be one and the same known square number in each of the two quantities to be equated to Squares, I multiply the quantity in the fourteenth step, to wit, $aa + 1$ by 4, and it makes $4aa + 4$; now each of these quantities is to be equated to a Square, *viz.* . . . $3aa - 6a + 4 = \square$
 $4aa + 4 = \square$
16. The difference of those two quantities is . . . $aa + 6a$
17. Which difference is equal to the Product of these two Factors, to wit, . . . $3a + 4$, and a
18. Half the sum of those Factors is . . . $\frac{1}{2}a + 2$
19. The Square of the said half-sum is . . . $\frac{1}{4}a^2 + 2a + 4$
20. Which Square equated to $4aa + 4$, (the greater of the two quantities in the fifteenth step,) will alter due Reduction give $a = \frac{8a}{3}$.
21. Therefore from the twentieth, ninth and eighth steps the sides about the right-angle are $\frac{8a}{3}$ and 1, the sum of whose Squares is $\frac{64a^2}{9} + 1$, whose Square Root $\frac{8a}{3}$ is the Hypothenusal sought.

I say $\frac{8a}{3}$, 1 and $\frac{8a}{3}$ are the sides of a right-angled Triangle, which will solve the Question; for one of the sides about the right-angle is a Square, to wit, 1, and if this be added to the triple of the Square of the difference of the sides about the right-angle, it makes the Square $\frac{64a^2}{9} + 1$, whose side is $\frac{8a}{3}$. From the premises it is evident that innumerable right-angled Triangles may be found to solve the Question.

QUEST. 95.

To find out a right-angled Triangle in rational numbers, that the Square of one of the sides about the right-angle may be equal to the other of the same sides.

[This is Problem. 15. in pag. 8c. of the Introduction to Algebra before cited in Quest. 8. but I shall resolve it after another manner.]

RESOLUTION.

1. For one of the sides about the right-angle put ra , (r representing some known number, and a some number unknown,) . . . ra
2. Then (according to the Question) the Square of that side must be the other of the sides about the right-angle, to wit, . . . $rraa$
3. The sum of the Squares of those sides is . . . $rrraaaa + rraa$
4. Which sum must be equal to the Square of the Hypothenusal, and therefore it remains to equate the said $rrraaaa + rraa$ to a Square, to which end take some known number s greater than r , and then the side of the said Square may be feigned $sa - rraa$, the Square whereof being equated to $rrraaaa + rraa$, and due Reduction made, you will find

$$a = \frac{ss - rr}{2rr}$$

5. But

5. But s and r were assum'd to represent two known numbers whereof s is the greater, therefore from the premises the three sides of the Triangles sought shall be known also, and may be express'd thus,

$$\left. \begin{array}{l} \frac{ss - rr}{2rr}, \text{ or } \frac{2ssr - 2rrr}{4ssr} \\ \frac{ssr - 2rrr}{4ssr} \end{array} \right\} \text{The sides about the right-angle.}$$

The Hypothenusal.

6. Moreover, because the Square of $\frac{2sr}{ss - rr}$ is $\frac{4ssr}{ssr - 2rrr}$; and the sum of the Squares of the two last Fractions is equal to the Square of $\frac{2ssr + 2rrr}{ssr - 2rrr}$, (as will easily appear by Multiplication and Addition;) therefore the three quantities last express'd shall also be the sides of a right-angled Triangle to solve the Question propos'd, *viz.*

$$\left. \begin{array}{l} \frac{2sr}{ss - rr}, \text{ or } \frac{2ssr - 2rrr}{ssr - 2rrr} \\ \frac{4ssr}{ssr - 2rrr} \end{array} \right\} \text{The sides about the right-angle}$$

The Hypothenusal.

7. If the three sides of the Triangle sought, as they be above express'd in the fifth and sixth steps be compar'd together, it will be easie to deduce from thence this following

CANON.

First, form a right-angled Triangle from any two unequal numbers; then multiply the three sides of that Triangle severally by either of the sides about the right-angle; lastly, divide severally those three Products by the Square of the other of the sides about the right-angle; so the three Quotients shall be the sides of the right-angled Triangle sought.

Examples in Numbers.

First, find out three numbers to express the sides of a right-angled Triangle, suppose these, . . . $3, 4, 5$
 Then multiply those three sides severally by 4, the greater of the sides about the right-angle, and the three Products will be these, *viz.* . . . $12, 16, 20$
 Lastly, divide those three Products severally by 9, the Square of the lesser of the sides about the right-angle; so the Quotients will be the sides of a right-angled Triangle to solve the Question propos'd, to wit, these, . . . $\frac{4}{3}, \frac{16}{9}, \frac{20}{9}$

Again,

Let the right-angled Triangle first found out be here repeated, . . . $4, 3, 5$
 to wit, . . .
 Then multiply those three sides severally by 3, the lesser of the sides about the right-angle, and the Products will be these, . . . $12, 9, 15$
 Lastly, divide those three Products severally by 16, the Square of the greater of the sides about the right-angle; so the Quotients will give these three sides of a right-angled Triangle to solve the Question, to wit, . . . $\frac{3}{4}, \frac{9}{16}, \frac{15}{16}$

QUEST. 96. (Quest. 1. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the sides about the right-angle being severally subtracted from the Hypothenusal may leave cube-numbers.

RESOLUTION.

1. First, form a right-angled Triangle from two unequal numbers, suppose from a and e , and let a be the greater, . . . $aa + ee, aa - ee, 2ae$
 to the three sides will be these, *viz.* . . . R
2. Then

Hypoth. Base, Perp.
 $aa + ee, aa - ee, 2ae$

2. Then by subtracting the Base from the Hypothenusal, the remainder is $2ee$, which should be a Cube, but it is not; yet it shews that e (to wit, one of the numbers from which the desired right-angled Triangle is to be formed,) must be such that the double of its Square may make a Cube. Now that we may chuse the number e with that condition, let $2ee$ be equated to the Cube $bbbee$, viz. suppose $2ee = bbbee$, (where bbb may represent any known cube-number,) whence after due Reduction the value of e will be made known, viz.

$$e = \frac{2}{bbb}.$$

3. It remains, that the Perpendicular subtracted from the Hypothenusal may leave a Cube; but the remainder is the Square $aa - ee = 2ae$, which is not a Cube, but if its Root $a - e$ were a Cube, then that Square would be a Cube; (for a Cube multiplied into a Cube produceth a Cube, by Prop. 3, & 4. Elem. 9. Euclid.) Let therefore the said Root $a - e$ be equated to some Cube, viz. suppose $a - e = ddd$; hence, and from the third step it follows, that

$$a = e + ddd = \frac{2}{bbb} + ddd.$$

Now, if from $\frac{2}{bbb} + ddd$ and $\frac{2}{bbb}$ (the values of a and e) a right-angled Triangle be formed, it will solve the Question. Hence this

C A N O N.

4. Divide 2 by any cube-number, and reserve the Quotient; then to the said Quotient add any cube-number; lastly, from the sum and Quotient form a right-angled Triangle and it shall solve the Question proposed.

As, for example, I divide 2 by the Cube 8, and reserve the Quotient $\frac{1}{4}$ for one of the numbers by which the right-angled Triangle is to be formed; then to the Quotient $\frac{1}{4}$ add some Cube, as 1, and the sum is $\frac{5}{4}$; lastly, from $\frac{5}{4}$ and $\frac{1}{4}$ form a right-angled Triangle and find the sides $\frac{5}{2}$, $\frac{3}{2}$, and $\frac{4}{2}$, which will solve the Question; for the sides about the right-angle, to wit, $\frac{3}{2}$ and $\frac{4}{2}$ being severally subtracted from the Hypothenusal $\frac{5}{2}$, the remainders are the Cubes $\frac{1}{8}$ and 1.

But after one right-angled Triangle is found out to solve the Question, if you multiply or divide every one of the sides thereof by one and the same cube-number, the Products or Quotients will give another right-angled Triangle to solve the Question: As, if the three sides before found out, to wit, $\frac{3}{2}$, $\frac{4}{2}$ and $\frac{5}{2}$ be severally multiplied by the Denominator 8, they will produce 12, 16 and 20, which will solve the Question. Likewise, if 12, 16 and 20 be severally multiplied by 8, there will be produced the right-angled Triangle 104, 96 and 40, where the differences between the Hypothenusal and the other two sides are Cubes, to wit, 8 and 64. The reason is evident; for,

First, by Construction $\frac{1}{4}$, $\frac{5}{4}$, $\frac{3}{4}$ are the sides of a right-angled Triangle that will solve the Question, viz. $\frac{1}{4} + \frac{3}{4} = \frac{5}{4}$ a Cube; $\frac{1}{4} - \frac{3}{4} = -\frac{2}{4}$ a Cube.

And because a Cube multiplied by a Cube produceth a Cube, those two differences or Cubes multiplied by the $\frac{1}{4}$ will produce $13 - 12 = 1$ a Cube, $13 - 5 = 8$ a Cube.

Wherefore the right-angled Triangle 13, 12, 5 shall necessarily solve the Question, as well as $\frac{13}{8}$, $\frac{12}{8}$, $\frac{5}{8}$.

QUEST. 97. (This is a Lemma, used by Dioph. in resolving the following Quest. 98.)

To find a right-angled Triangle and a Square in rational numbers, that if the Area of the Triangle be subtracted from the Square, the remainder multiplied by a given number (d) may produce a square number.

R E S O L U T I O N.

1. Let a right-angled Triangle be formed from a and $\frac{1}{a}$; $\left. \begin{array}{l} aa + \frac{1}{aa} = \text{the Hypoth.} \\ aa - \frac{1}{aa} = \text{the Perpend.} \\ 2 = \text{the Base.} \end{array} \right\}$
so the three sides will be
2. Therefore the Area, (by multiplying the Perpendicular into half the Base,) is
3. Then

3. Then feign the side of the Square sought to be $a + \frac{2d}{a}$
4. Therefore the Square it self is $aa + 4d + \frac{4dd}{aa}$
5. From which Square, the Area above express'd being subtracted, there will remain $4d + \frac{4dd + 1}{aa}$
6. That is, $\frac{4daa + 4dd + 1}{aa}$
7. Then the said remainder being multiplied into the given number (d) produceth $\frac{4ddaa + 4ddd + d}{aa}$
8. Which Product must (according to the Question) be a Square: But the Denominator aa is a Square, it remains therefore to equate the Numerator to a Square, viz. $4ddaa + 4ddd + d$ must be equated to a Square, the side whereof may be variously feigned, let it be $2da + d$; and then the Square of $2da + d$ being equated to the said $4ddaa + 4ddd + d$, this Equation aritheth; to wit,
 $4ddaa + 4ddd + dd = 4ddaa + 4ddd + d.$
9. Which Equation after due Reduction gives $a = \frac{4dd + 1 - d}{4d}$

From the ninth, first and third steps aritheth this

C A N O N.

10. From $\frac{4dd + 1 - d}{4d}$ and $\frac{4d}{4dd + 1 - d}$ form a right-angled Triangle, which shall be that sought by the Question; and the side of the Square sought shall be $\frac{4dd + 1 - d}{4d} + \frac{8dd}{4dd + 1 - d}.$

An Example in Numbers.

Suppose $5 = d$ the number given; then form a right-angled Triangle from $\frac{4}{5}$ and $\frac{1}{5}$, so the Hypothenusal will be $\frac{17}{5}$, the Perpendicular $\frac{7}{5}$, and the Base 2; moreover, the side of the Square sought will be $\frac{41}{5}$, and the Square it self $\frac{1681}{25}$; or (in the same Denominator with the said Hypothenusal and Perpendicular) $\frac{289}{25}$; which Square and Triangle will solve the Question, for if the Area of the Triangle, to wit, $\frac{14}{25}$ be subtracted from the said Square $\frac{1681}{25}$, the remainder $\frac{1667}{25}$, that is, $\frac{1667}{25}$ multiplied into the given number 5, produceth the Square $\frac{8335}{5}$, whose Root is $\frac{91}{5}$.

Another Example.

Suppose $3 = d$ the number given in the Question; then let a right-angled Triangle be formed from $\frac{1}{3}$ and $\frac{2}{3}$, (which numbers are discovered by the preceding Canon,) so the Hypothenusal will be $\frac{5}{3}$, the Perpendicular $\frac{4}{3}$, and the Base 2; moreover, the side of the Square sought will be found $\frac{17}{3}$, and the Square it self $\frac{289}{9}$; from which Square subtracting the Area of the Triangle, to wit, $\frac{2}{9}$, the remainder $\frac{287}{9}$, or in its least terms $\frac{287}{9}$, multiplied into the given number 3, produceth the Square $\frac{861}{3}$, whose side is $\frac{29}{3}$.

Note. Instead of 2 which is prefix to d in the Numerator of the Fraction $\frac{2d}{a}$ in the third step of the preceding Resolution, you may take the half of any square number and prefix it to d for a Numerator, over the Denominator a ; as, $\frac{4}{2}d$, $8d$, $12d$, &c. and then by prosecuting the work as before from the third step to the end of the Resolution, various Answers to the Question from one and the same given number will be discovered.

QUEST. 98. (Quest. 3. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the Area thereof increased with a given number, suppose 5, may make a Square.

R E S O L U T I O N.

1. Let the sides of some known right-angled Triangle, as 4 and 3, be severally multiplied by d , and take the Products to represent the sides of the Triangle sought, to wit,
2. Then

2. Then the Area thereof increased with 5 (the number given in the Question) makes $6aa + 5$
3. Which sum must be equal to a Square, suppose it be $9aa$, $6aa + 5 = 9aa$ therefore
4. Whence by equal subtraction of $6aa$, there remains $5 = 3aa$
5. Let each part of the last Equation be multiplied by 5, and $25 = 15aa$ it produceth
6. Now if 15 which is prefix'd to aa were a Square, then the value of a would be rational: Whence therefore comes 15? Examine the work, and you will find, that by subtracting 6 the Area of the Triangle 5, 4, 3 from the square number 9, and then by multiplying the remainder 3 into the given number 5, there is produced 15; whereby it is manifest that the scope of our search must be to find a right-angled Triangle and a square number, that the Area of the Triangle being subtracted from the Square and the remainder multiplied by the given number 5 may make a Square. But the preceding 97th Question shews how to find out such a Triangle and Square; take if you please those there found in the first Example, to wit, the right-angled Triangle whose Hypotenuse is $\frac{13}{12}$, Perpendicular $\frac{5}{12}$, and Base 2, and the Square $\frac{169}{144}$, whose side is $\frac{13}{12}$: Then renew the Resolution thus,
7. For the three sides of the Triangle sought put $\frac{13}{12}a, \frac{5}{12}a, 2a$
8. The Area thereof increased with 5 makes $\frac{13}{12}a \cdot \frac{5}{12}a + 5$
9. Which sum must be equated to the Product of $\frac{13}{12}a \cdot \frac{5}{12}a + 5 = \frac{65}{144}a^2$ (the Square before found) multiplied into aa , so this Equation aritheth,
10. Therefore by subtracting $\frac{65}{144}a^2$ from each part of that Equation this remains, to wit, $5 = \frac{65}{144}a^2$
11. And by multiplying each part of the last Equation into 5 it produceth $25 = \frac{325}{144}a^2$
12. And by extracting the square Root out of each part of the last Equation, there will arise $5 = \frac{13}{12}a$
13. Whence by dividing each part by $\frac{13}{12}$, the value of a will be made known, viz. $a = \frac{60}{13}$
14. Wherefore from the thirteenth and seventh steps the three sides of the Triangle sought will be found these, to wit, $\frac{13}{12}a = \frac{13}{12} \cdot \frac{60}{13} = 5$, the Area whereof is $\frac{13}{12} \cdot \frac{5}{12} = \frac{65}{144}$, to which adding 5, the sum will be the Square $\frac{65}{144} + 5 = \frac{805}{144}$, whose Root is $\frac{29}{12}$. Therefore the Question is solved.

Vieta, in the 9th of the 5th Book of his *Zeteticæ*, shews how to find out a right-angled Triangle whose Area increased with a number compos'd of two Squares may make a Square; whereby 'tis probable the thought this Question to be applicable only to a number compos'd of two Squares, because *Diophantus* propos'd the given number 5, which is compos'd of two Squares; but 'tis evident from the precedent Resolution that the Question may be extended to any given number whatsoever: And for greater illustration, let it be required to find out a right-angled Triangle whose Area increased with 3 may make a Square.

First a right-angled Triangle is to be sought, and also a Square, that the Area of the right-angled Triangle being subtracted from the Square and the remainder multiplied by the given number 3, the Product may be a square number: But by the latter Example of the preceding 97th Question such a Triangle and Square are found out, viz. the Triangle whose Hypotenuse is $\frac{13}{12}$, the Perpendicular $\frac{5}{12}$, and the Base 2; and the Square $\frac{169}{144}$, whose side is $\frac{13}{12}$; then for the three sides of the right-angled Triangle sought let there be put $\frac{13}{12}a, \frac{5}{12}a$ and $2a$, the Area whereof is $\frac{65}{144}a^2$, to which adding 3 it makes $\frac{65}{144}a^2 + 3$; which sum equated to $\frac{65}{144}a^2 + 3 = \frac{169}{144}a^2$ being the Square above found, will give $\frac{169}{144}a^2 - \frac{65}{144}a^2 = 3$ for the value of a , by which $\frac{13}{12}$ the three sides above put being resolved, you will find the three sides of the right-angled Triangle sought to be these, to wit, $\frac{13}{12}a = \frac{13}{12} \cdot \frac{12}{13} = 1$, the Area of which Triangle is $\frac{13}{12} \cdot \frac{5}{12} = \frac{65}{144}$, to which adding 3 it makes the Square $\frac{65}{144} + 3 = \frac{493}{144}$, whose side is $\frac{22}{12}$.

QUEST. 99.

QUEST. 99. (Quest. 6. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof increased with one of the sides about the right-angle may make a given number, suppose n .

RESOLUTION.

1. For the sides about the right-angle of the Triangle sought put a and e
2. Then is the Area $\frac{1}{2}ae$
3. Which increased with one of the sides about the right-angle must make the given number n , hence this Equation, $\frac{1}{2}ae + a = n$
4. From which Equation, after due Reduction, you will find $e = \frac{n-a}{\frac{1}{2}a}$
5. And because ae must be equal to the Square of the Hypotenuse, it follows from the first and fourth steps, that the Square of the Hypotenuse must be equal to the sum of the Squares of a and $\frac{n-a}{\frac{1}{2}a}$; that is,

$$\frac{1}{4}aaa + aa - na + nn = \frac{1}{4}aa$$

6. Of which fractional quantity the Denominator $\frac{1}{4}aa$ is a Square, whose side is $\frac{1}{2}a$; therefore the Numerator were a Square the whole Fraction would be a Square: It remains then to equate the Numerator $\frac{1}{4}aaa + aa - na + nn$ to some Square, to which end, let its side be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the said Numerator, this Equation aritheth, viz.

$$\frac{1}{4}aaa + aa - na + nn = \frac{1}{4}aaa - naa + nn$$

7. Whence the value of a will be made known, viz. $a = \frac{2n}{n-1}$
8. And according to that value of a , the value of e will also be made known by the fourth step, viz. $e = n-1$
9. And because the square Root of ae is equal to the Hypotenuse, therefore the square Root of the sum of the Squares of the two quantities in the latter parts of the Equations in the seventh and eighth steps, will give for the Hypotenuse sought
10. Therefore from the seventh, eighth and ninth steps the three sides of the right-angled Triangle sought are these, to wit,

$$\frac{2n}{n-1}, n-1, \left(\text{or } \frac{nn-1}{n+1} \right) \text{ and } \frac{nn+1}{n+1}.$$

Which three sides if they be express'd by words will give the following Canon, which is the same with that delivered by *Fermat* in his Observation upon the sixth Question of the sixth Book of *Diophantus*.

CANON.

11. When the given number is greater than unity let a right-angled Triangle be formed from those two numbers, and then divide the three sides severally by the sum of the given number and unity; so shall the Quotients be the three sides of the right-angled Triangle sought.

An Example in Numbers.

Let it be required to find a right-angled Triangle whose Area increased with one of the sides about the right-angle may make 7.

First (as the Canon directs) let a right-angled Triangle be formed from the given number 7 and 1, so the three sides will be these, 50, 48, 14. Then divide those three sides severally by 8, (to wit, 7+1,) so the Quotients shall be the sides of the right-angled Triangle sought, to wit, $6\frac{1}{2}$, 6, and $1\frac{1}{2}$ are the sides of a right-angled Triangle, which will solve the Question; for the Area $\frac{1}{2}ae$ increased with $\frac{1}{2}a$ (one of the sides about the right-angle) makes $\frac{1}{2}ae + \frac{1}{2}a$, that is, 7, as was required.

But because the said Canon takes not place unless the given number exceed unity, I shall in the next place explain *Diophantus's* Resolution of this Question, by which way, whatever the given number be, a right-angled Triangle may be found out to solve the Question proposed.

Another

Another way of resolving the foregoing 99th Question.

1. Let b, b, p represent the Hypotenusal, Base and Perpendicular of some right-angled Triangle known in numbers, then multiply those three sides severally by a , (which represents a number unknown,) and take the Products for the three sides of the Triangle sought, to wit,
2. The Area of which Triangle increased with one of the sides about the right-angle, suppose with ba , must be equal to the given number n , therefore
3. Which Equation divided by $\frac{1}{2}bp$ gives
4. Now to the end that the value of a in the last Equation may be a rational number, the Square of half the Coefficient which is drawn into a , together with the absolute quantity which possesseth the latter part of the Equation must make a Square, (as is evident by the Canon in Sect. 6. Chap. 15. Book 1.)
5. And because the Denominator $\frac{1}{2}bpb$ is a Square, it remains only to equate the Numerator to a Square, viz.
6. Or, to avoid Fractions, let the said $\frac{1}{2}bb + \frac{1}{2}bpn$ be multiplied by 4, and then
7. Which last Equation shews, that in order to the solving of the Question proposed, a right-angled Triangle must first be found, such, that the Square of one of the sides about the right-angle, together with the Product of the quadruple Area multiplied by the given number n , may make a Square. Now to find out such a Triangle,
8. For one of the sides about the right-angle put
9. And for the other side put some square number, as,
10. Then the quadruple of the Area is
11. Which multiplied by the given number n , suppose by $\frac{1}{2}$, makes
12. To which Product add the Square of one of the sides about the right-angle, to wit, the Square of 1, which is also 1, and the sum must be equal to a Square, viz.
13. Also the sum of the Squares of the sides about the right-angle must be equal to a Square, to wit, the Square of the Hypotenusal, therefore
14. Now in order to resolve the Duplicate equality in the two last steps, first the difference of the two quantities which are to be equated to Squares is
15. Which difference is equal to the Product of the multiplication of these two quantities, or Factors, to wit,
16. The half-sum of those two Factors is
17. The Square of which half-sum being equated to $ee + 1$, will give
18. And because e and 1 were put for the sides about the right-angle, therefore the square Root of the sum of the Squares of $\frac{2n}{1-n}$ (that is e) and 1, shall be the Hypotenusal, to wit, $\frac{2n}{1-n}$; so we have found out a right-angled Triangle whose three sides are $\frac{2n}{1-n}$, $\frac{2n}{1-n}$ and 1, which are fit for renewing the search of the right-angled Triangle sought by the Question, in this manner, viz.
19. For the three sides of the right-angled Triangle sought put
20. The Area of which Triangle increased with one of the sides about the right-angle, suppose with a , must be equal to the given number $\frac{1}{2}$, therefore
21. Which Equation being resolved by the Canon in Sect. 6. Chap. 15. Book 1. will give
22. According to which value of a , the three sides in the nineteenth step being resolved, there will be produced the three sides of the right-angled Triangle sought, to wit,

I say $\frac{4n}{1-n}$, $\frac{4n}{1-n}$ and $\frac{2n}{1-n}$ are the three sides of a right-angled Triangle which will solve the Question,

Question; for the Area $\frac{1}{2}$ increased with $\frac{1}{2}$, (one of the sides about the right-angle,) makes $\frac{3}{2}$, that is, $\frac{1}{2}$; as was required.

QUEST. 100.

To find a right-angled Triangle in rational numbers, that the Area subtracted from one of the sides about the right-angle may leave a given number; let the given number be n .

RESOLUTION.

1. For the sides about the right-angle of the Triangle sought
2. Then is the Area
3. Which subtracted from one of the sides about the right-angle, suppose from a , leaves
4. Whence, after due Reduction,
5. And because $aa + ee$ must be equal to the Square of the Hypotenusal, it follows from the first and fourth steps, that the Square of the Hypotenusal must be equal to the sum of the Squares of a and $\frac{a-n}{2}$, that is,
6. And since the Denominator $\frac{1}{2}aa$ is a Square, whose side is $\frac{1}{2}a$, it remains only to equate the Numerator $\frac{1}{2}aaaa + aa - 2na + nn$ to a Square, whose side may be signed either $\frac{1}{2}aa - n$ or $\frac{1}{2}aa + n$; first then, let the side of the said Square be signed $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the said $\frac{1}{2}aaaa + aa - 2na + nn$, this Equation ariseth, viz.
7. From which Equation the value of a will be made known,
8. According to which value of a , the number e will be discovered from the fourth step, viz.
9. And the square Root of the sum of the Squares of
10. Therefore from the three last preceding steps, the three sides of the right-angled Triangle sought are these, to wit,
11. But if instead of $\frac{1}{2}aa - n$, which in the sixth step was feigned for the side of a Square, we assume $\frac{1}{2}aa + n$, and equate the Square of this side to the before-mentioned $\frac{1}{2}aaaa + aa - 2na + nn$, there will arise $a = \frac{2n}{1-n}$; according to which value, the sides of the Triangle sought will be found these, to wit,

The two last steps give this

CANON.

12. When the given number is less than unity, let a right-angled Triangle be formed from unity and the given number; then divide the three sides severally by the sum or difference of unity and the given number, so shall the Quotients be the sides of the Triangle sought.

As, for example, if it be desired to find out a right-angled Triangle, that the Area subtracted from one of the sides about the right-angle may leave $\frac{1}{2}$, the Canon will discover the sides of two Triangles, to wit, $\frac{4}{3}$, $\frac{4}{3}$, $\frac{5}{3}$ and $\frac{6}{5}$, $\frac{6}{5}$, $\frac{8}{5}$, each of which will satisfy your desire; for in the first Triangle the Area $\frac{1}{2}$ subtracted from $\frac{4}{3}$ (one of the sides about the right angle,) leaves the given number $\frac{1}{2}$; likewise in the latter Triangle, the Area $\frac{1}{2}$ subtracted from $\frac{6}{5}$ leaves $\frac{1}{2}$.

But how to solve this Question when the given number is any number whatever, I shall hereafter shew by Fermat's method, in Quest. 130. of this Book.

QUEST. 101.

QUEST. 101. (Quest. 12. Lib. 6. Diophant.)

To find a right-angled Triangle, that as well the difference of the sides about the right-angle as the greater of the same sides may be a Square; and that the Area, with the lesser of the sides about the right-angle may make a Square.

The Resolution of this Question depends upon three *Lemma's*, which I shall first explain.

LEMMA 1.

1. If a right-angled Triangle be formed from two numbers whereof the greater is the double of the lesser, as well the difference of the sides about the right-angle as the greater of the same sides shall be a Square. Moreover, if the Area of the said Triangle be multiplied by the Square of a Fraction having unity for its Numerator, and the lesser of the two numbers by which the said Triangle was formed for a Denominator, the Product increased with the lesser of the sides about the right-angle, will make a Square containing nine times the Square of the lesser of the two numbers by which the said Triangle was formed.

To make this manifest, let a right-angled Triangle be formed $\left\{ \begin{array}{l} 5aa, 3aa, 4aa \\ \text{from } a \text{ and } 2a, \text{ so the three sides will be these, to wit, } \end{array} \right.$
 Whence 'tis evident, first, that the difference of the sides about the right-angle is a Square, to wit, aa ; secondly, that the greater of the sides about the right-angle is a Square, to wit, $4aa$; and lastly, if the Area $6aaaa$ be multiplied by the Square of $\frac{1}{a}$, that is, by $\frac{1}{aa}$, the Product $6aa$ increased with $3aa$, (that is, the lesser of the sides about the right-angle,) makes the Square $9aa$, which contains nine times the Square of the lesser of the two numbers by which the said right-angled Triangle was formed. Which was to be shewn.

LEMMA 2.

2. Two numbers being given whose sum is a Square, to find innumerable Squares, every one of which being multiplied by one of the given numbers, and taking to the Product the other number, may make a Square.

Let there be two given numbers 6 and 3, and let it be desired to find a Square, such that if it be multiplied by 6, and 3 be added to the Product, the sum may make a Square.

Let the side of the Square sought be $\left\{ \begin{array}{l} a+1 \\ aa+2a+1 \\ 6aa+12a+6 \\ 6aa+12a+9 \end{array} \right.$
 Whence the Square it self is
 Which multiplied by 6 produceth
 To which Product add 3 and it makes

Which $6aa+12a+9$ is to be equated to a Square, whose side, (because the absolute number 9 is a Square,) may be variously feigned; let it then be $3a-3$, the Square whereof, to wit, $9aa-18a+9$ being equated to the said $6aa+12a+9$ will give $a=10$; wherefore $a+1$ the side of the Square sought is 11, and the Square it self is 121, which multiplied by 6 produceth 726, to which adding 3 it makes 729, which is a Square whose side is 27. It is also evident, that instead of 121 innumerable other Squares may be found out to perform the like effect, because the side of the Square to be equated to $6aa+12a+9$ may be feigned infinitely.

In like manner if it were desired to find a Square which multiplied into 3, and taking 6 to the Product, may make a Square, let the Square sought be feigned $aa+2a+1$, this multiplied by 3 and the Product increased with 6, gives $3aa+6a+9$ to be equated to a Square whose side may be feigned $3a-3$, whence $a=4$, and therefore $a+1$ the side of the Square sought is 5.

LEMMA 3.

3. If two numbers b and c whose sum makes not a Square be given, such, that by multiplying one of them, suppose b , by a given Square dd , and by adding to the Product the other number c , the sum $bdd+c$ makes a Square; we may find innumerable other Squares instead of the given Square dd to produce the like effect.

Suppose $\left\{ \begin{array}{l} b=2, \\ c=8, \\ d=2, \\ dd=4; \end{array} \right.$ whence $bdd+c = 16$.

Now

Now it is required to find another Square instead of dd , or 4, that if the Square found out be multiplied by 2, to wit, b ; and the Product be increased with 8, to wit, c , the sum may be a Square.

For the side of the Square sought put $a+\frac{1}{2}$ the side of the given $\left\{ \begin{array}{l} \text{Square } dd, \text{ viz. } \dots \dots \dots a+\frac{1}{2} \\ \text{Whence the Square sought is } \dots \dots \dots aa+\frac{1}{4}a+\frac{1}{4} \\ \text{Which multiplied by 2, (to wit, } b,) \text{ produceth } \dots \dots \dots 2aa+\frac{1}{2}a+\frac{1}{2} \\ \text{To which Product add 8, (to wit, } c,) \text{ and the sum is } \dots \dots \dots 2aa+\frac{1}{2}a+\frac{1}{2}+8 \end{array} \right.$

Which sum is to be equated to a Square, the side whereof (in regard 16 is a Square) may be variously feigned, let it then be $2a-4$, the Square whereof being equated to the said $2aa+\frac{1}{2}a+\frac{1}{2}+8$ will give $a=12$; wherefore $a+\frac{1}{2}=14$ the side of the Square sought, and the Square it self is 196, which multiplied by 2, and taking 8 to the Product makes the Square 400, whose side is 20. I shall now proceed to

The RESOLUTION of the preceding QUEST. 101.

1. Let a right-angled Triangle be formed from two such numbers, that the greater may be the double of the lesser, as from $\left\{ \begin{array}{l} 5aa, 3aa, 4aa \\ a \text{ and } 2a, \text{ so the three sides will be these, to wit, } \end{array} \right.$
 Whence it is evident that the two first parts of the Question are satisfied, for as well the difference of the sides about the right-angle as the greater of them is a Square. It remains that we see whether the Area of the Triangle, with the lesser of the sides about the right-angle makes a Square: But it makes $6aaaa+3aa$, and by dividing all by aa there ariseth $6aa+3$ to be equated to a Square, which is possible to be done, because the sum of 6 and 3 is a Square; for if a be 1, then aa is also 1, and consequently $6aa+3$ makes the same Square as $6+3$, to wit, 9. Since then a is found 1, a right-angled Triangle formed from 1 and 2 (that is, a and $2a$) will solve the Question. Wherefore the sides of the Triangles sought are 5, 3, 4.

2. Having found out one Square for the value of aa , to wit, 1, we may by the help of the preceding Lemma 2. find out innumerable other Squares to perform the same effect; so instead of 1, the Square 121 is found out in the Example of the said Lemma. And therefore let a Triangle be formed from 11 and 22, and the sides will be 605, 363, 484, which will solve the Question; for the difference of the sides about the right-angle is the Square 121, and the greater of the sides about the right-angle, to wit, 484, is a Square whose side is 22; also the Area added to the lesser side makes 88209, which is a Square whose side is 297.

6. But for removing an Obstruction which the Learner may meet with, let a right-angled Triangle be formed from $2a$ and $4a$, whence the sides are $20aa$, $12aa$ and $16aa$; where as well the difference of the sides about the right-angle, as the greater of them is a Square. But the Area increased with the lesser side makes $96aaaa+12aa$, and by dividing all by aa it makes $96aa+12$, which must be equated to a Square. But how shall that be done, since 96 which is multiplied into aa is not a Square, neither is the absolute number 12 a Square, nor is the sum of 96 and 12 a Square? This knot, though it seems to be a very hard one, may yet by the help of the last clause of the preceding Lemma 1. be easily untied: For since 96 is the Area of a right-angled Triangle formed from 2 and 4, and 12 is the lesser of the sides about the right-angle, it will appear by Lemma 1. that if 96 be multiplied by $\frac{1}{4}$, and the Product 24 be increased with 12 it shall necessarily make a Square, to wit, 36; and consequently by Lemma 3. we may find out innumerable Squares instead of $\frac{1}{4}$, every one of which being multiplied by 96, and taking 12 to the Product will make a Square. As, for example,

For the side of a Square to be found out instead of $\frac{1}{4}$, let $\left\{ \begin{array}{l} \text{there be put } a = \text{the square Root of } \frac{1}{4}, \text{ viz. } \dots \dots \dots a - \frac{1}{2} \\ \text{Then the Square of } a - \frac{1}{2} \text{ is } \dots \dots \dots aa - a + \frac{1}{4} \\ \text{Which multiplied by 96 produceth } \dots \dots \dots 96aa - 96a + 24 \\ \text{To which Product adding 12 it makes } \dots \dots \dots 96aa - 96a + 36 \end{array} \right.$

Which $96aa - 96a + 36$ must be equated to a Square, (the side whereof in regard 36 is a Square,) may be variously feigned, let it then be $4a-6$, the Square whereof being equated to $96aa - 96a + 36$ will give $a=\frac{1}{2}$, therefore $a-\frac{1}{2}$, (that is, $\frac{1}{2}-\frac{1}{2}$) will be $\frac{1}{2}$ for the side of the Square sought and the Square it self is $\frac{1}{4}$, by which if you multiply

multiply 96, and to the Product add 12, it makes the Square $\frac{121}{4}$, whose side is $\frac{11}{2}$. Now so far as the said side $\frac{11}{2}$ is to be taken for the value of a in the $2a$ and $4a$ by which the Triangle was first formed, let a Triangle be now formed from $\frac{11}{2}$ and $\frac{7}{2}$, and the three sides are $\frac{11}{2}$, $\frac{7}{2}$, $\frac{25}{2}$; where 'tis evident that the difference of the sides about the right-angle, as also the greater of them is a Square: But the Area is $\frac{77}{4}$, to which if you add the lesser side $\frac{7}{2}$, it makes the Square $\frac{81}{4}$, whose side is $\frac{9}{2}$.

QUEST. 102.

To find a right-angled Triangle, and a Square number, such, that if the Square be multiplied by the lesser of the sides about the right-angle, and to that Product there be added the Product made by the multiplication of the Area of the said Triangle into the difference of the sides about the right-angle, the sum may be a Square. Moreover, that the greater of the sides about the right-angle may be a Square.

CANON.

1. By the preceding *Quest.* 101. find a right-angled Triangle, that as well the difference of the sides about the right-angle, as the greater of the same sides, may be a Square: Moreover, that the Area increased with the lesser of the said sides may make a Square, so shall such Triangle be that which is sought by this *Quest.* 102. and the difference of the sides about the right-angle shall be the first Square sought.

But that this Canon will solve the Question proposed, I demonstrate thus;

2. Suppose $\begin{cases} h = \text{the Hypotenusal;} \\ b = \text{the greater side,} \\ p = \text{the lesser side,} \end{cases}$ about the right-angle.
3. Suppose also that by the preceding *Quest.* 101. $\begin{cases} b-p, \\ b, \\ p \end{cases}$ are three square numbers. $\frac{b-p}{b}, \frac{b}{p}$ are found such, that $\frac{b-p}{b} + p$ is a square number.
4. I say $b-p$ is such a square number, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product made by the multiplication of the Area $\frac{1}{2}bp$, into $b-p$ the difference of the sides about the right-angle, the sum shall be a Square: For,
5. The Product of $b-p$ into p is $bp - pp$.
6. To which if you add the Product of the Area into the difference of the sides about the right-angle, to wit, $\frac{1}{2}b^2p - \frac{1}{2}b^2p$.
7. The sum is $bp - pp + \frac{1}{2}b^2p - \frac{1}{2}b^2p$.
8. Which sum is a square number, for it is equal to the Product of $b-p$ multiplied into $\frac{1}{2}bp + p$: But each of these Factors $b-p$ and $\frac{1}{2}bp + p$ is a Square by Construction; wherefore the Product of their multiplication, to wit, $bp - pp + \frac{1}{2}b^2p - \frac{1}{2}b^2p$ is a Square.

An Example in Numbers.

9. Take any right-angled Triangle which will solve the preceding *Quest.* 101. as, $\begin{cases} 5; 3; 4 \end{cases}$

In which Triangle the difference of the sides about the right-angle, to wit, 1, is such a Square that if it be multiplied by 3 the lesser side, and the Product be increased with 6, to wit, the Product of the Area multiplied by the difference of the sides about the right-angle it makes the Square 9. Moreover, the greater side about the right-angle, to wit, 4, is a Square; as was required.

10. But the same right-angled Triangle 5, 3, 4 being retained, we may instead of the Square 1, (to wit, the difference of the sides about the right-angle) find out innumerable Squares, (by the help of *Lemma* 2. in *Quest.* 101.) every one of which shall solve this Question, and be within given limits if need require. So if it were desired to find out a Square greater than 6, and such as together with the said right-angled Triangle 5, 3, 4 may solve this 102^d Question, the said *Lemma* 2. will discover the Squares 25 and 361, (among innumerable others,) which are such, that if each of them be multiplied by 3, (the lesser of the sides about the right-angle of the said Triangle,) and the Products 75 and 1083 be severally increased with 6, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Squares 81 and 1089, whose sides are 9 and 33.

11. And

11. And in like manner, by the help of any other right-angled Triangle which will solve the preceding *Quest.* 101. as the Triangle 605, 363, 484, we may find out innumerable Answers to this *Quest.* 102. For, first, the difference of the sides about the right-angle, to wit, 121 is a Square, and such, that if it be multiplied by the lesser side 363; and to the Product 43923 there be added 10629366, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Square 10673289, whose side is 3267. And therefore by the help of the third *Lemma* which belongs to the Resolution of the preceding *Quest.* 101. you may instead of the Square 121 find out infinite other Squares to perform the same thing, and each Square shall be within given limits if need require.

QUEST. 103. (Quest. 13. Lib. 6. Diophant.)

To find a right-angled Triangle, that the Area thereof being increased severally with each of the sides about the right-angle may make Squares.

RESOLUTION.

1. Let the Hypotenusal, Base and Perpendicular of some right-angled Triangle in numbers be represented by $b; b; p$.
2. Then multiply those sides severally by a , and suppose the Products to be the sides of the right-angled Triangle sought, to wit, ba, ba, pa .
3. Then; (according to the Question,) the Area of the Triangle in the last step being increased with each of the sides about the right-angle must make a Square, hence this Duplicate equality ariseth to be resolved, viz. $\frac{1}{2}b^2pa + ba = \square$
 $\frac{1}{2}b^2pa + pa = \square$
4. Now suppose $\frac{1}{2}b^2pa + ba = e^2aa$
 $\frac{1}{2}b^2pa + pa = e^2aa$
5. Whence, after due Reduction, you will find $a = \frac{ee - \frac{1}{2}bp}{b^2p}$
6. According to which value of a , the latter of the two quantities in the third step being resolved, instead of that quantity this that follows ariseth to be equated to a Square, viz. $\frac{b^2p^2 - \frac{1}{2}bbp - \frac{1}{2}bbp}{e^2aa - b^2p^2} = \square$
7. But because the Denominator of the Fraction last express'd is a Square, whose side is $ee - \frac{1}{2}bp$, it remains only to equate the Numerator to a Square: And because a Square divided by a Square gives the Quotient a Square, therefore if we suppose b in the said Numerator $b^2p^2 - \frac{1}{2}bbp - \frac{1}{2}bbp$ to be a square number, then the said Numerator divided by b gives $pe - \frac{1}{2}bp - \frac{1}{2}bp$; that is, $pe - \frac{1}{2}bp \times b - p$ to be equated to a Square. So that the matter is reduced to this, we must find out a right-angled Triangle, such, that the greater of the sides about the right-angle, suppose b , may be a square number; and we must also find another square number, suppose ee , that may be greater than the Area of the said Triangle, (as may be infer'd from the Denominator of the Fraction in the fifth step,) and such, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the sides about the right-angle, the sum may make a Square: But such a right-angled Triangle and Square, the preceding *Quest.* 102. shews how to find out. Suppose then the Hypotenusal, Base and Perpendicular of the said Triangle so found out to be b, b, p ; and the Square to be ee , all which being known in numbers, the number represented by a shall consequently be known from the fifth step: And lastly, if you multiply the numbers b, b, p severally by the number a , the Products shall be the three sides of the Triangle sought. From the premises there ariseth the following

CANON.

First, by the Canon of the preceding *Quest.* 102. find out a right-angled Triangle, (whose three sides in the Resolution of this *Quest.* 103. are represented by b, b, p ;) and besides, a Square, call it ee , that may be greater than the Area of the said Triangle, and such that if it be multiplied by the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the same sides, the sum may be a Square; then divide the greater of the sides about the right-angle of the said Triangle, by the excess of the said Square ee above the Area, and by the Quotient multiply severally the three sides b, b, p , so shall the Products be the three sides of the Triangle sought.

S 2

An Example in Numbers.

It hath been shewn in *Sect. 10.* of the preceding *Quest. 102.* that the right-angled Triangle 5, 3, 4 and the Square 25 will solve that Question; and besides, the said Square 25 is greater than 6 the Area of the said Triangle, therefore according to the directions of the Canon above-delivered, I divide 4 the greater of the sides about the right-angle, by 19 the excess of the said Square 25 above 6 the Area of the said Triangle, and the Quotient is $\frac{4}{19}$, by which I multiply severally 5, 3, 4 (the sides of the Triangle first found) and there comes forth $\frac{20}{19}$, $\frac{12}{19}$, $\frac{16}{19}$, which shall be the sides of the Triangle sought; for the Area is $\frac{36}{19}$, to which if we add severally $\frac{12}{19}$ and $\frac{16}{19}$, (the sides about the right-angle,) there will be made the Squares $\frac{4}{19}$, $\frac{9}{19}$, and $\frac{16}{19}$, whose sides are $\frac{2}{19}$ and $\frac{4}{19}$.

After the same manner, the same right-angled Triangle 5, 3, 4 and the Square 361; (found out also by the preceding *Quest. 102.*) will discover $\frac{4}{355}$, $\frac{12}{355}$ and $\frac{16}{355}$ for the three sides of another right-angled Triangle to solve this Question, as may easily be perceived. And because innumerable right-angled Triangles and Squares may be found out to solve the said *Quest. 102.* infinite Answers may be given to this.

QUEST. 104. (Quest. 15. Lib. 6. Diophantus.)

To find a right-angled Triangle, such, that if from its Area the Hypotenusal and one of the sides about the right-angle be severally subtracted, each remainder may be a Square.

The Resolution of this and the following 105th Question depends upon a Lemma, which I shall here premise and demonstrate.

LEMMA.

1. If a right-angled Triangle be formed from two square numbers, or from two numbers in proportion one to another as a square number to a square number; I say first, the Square of the difference of those two square numbers being multiplied by the Product of their multiplication will produce a Square less than the Area of the said Triangle; secondly, if from the solid Product made by the multiplication of these three numbers, to wit, the square number above produced less than the Area, the Hypotenusal, and that side about the right-angle which is the double Product of the multiplication of the two square numbers forming the Triangle, there be subtracted the solid Product made of these three numbers, to wit, the Area, the said side about the right-angle, and the excess by which the Hypotenusal exceeds the same side, the remainder shall be a square number; thirdly and lastly, the sum of those two solid Products shall also be a square number.

Demonstration.

2. Let a right-angled Triangle be formed from two square numbers, suppose bb the greater, and cc the lesser, whose sides are b and c , so the three sides of the said Triangle will be these,

$$\begin{aligned} \text{viz. } \left\{ \begin{array}{l} bbbb + cccc = \text{the Hypotenusal,} \\ bbbb - cccc = \text{the Base,} \\ 2bbcc = \text{the Perpendicular.} \end{array} \right. \end{aligned}$$

3. The Area of the said Triangle is $b^2cc - bcc^2$.

4. The Product of the multiplication of bb and cc , to wit, $bbcc$, being multiplied by the Square of $bb - cc$, produceth this Square, to wit, $b^4cc - 2b^3c^2 + b^2c^4$.

5. Which Square is less than the Area $b^2cc - bcc^2$.

6. For it is evident that $bbcc - bccc \rightarrow bbbc + bcc^2$.

7. Therefore by multiplying each part into $bbcc$, $b^4cc - 2b^3c^2 + b^2c^4 \rightarrow b^5cc - b^4c^2$, $b^4cc - b^3c^3 \rightarrow b^5cc - b^4c^2$, it necessarily follows that $b^5cc - b^4c^2$.

Which was affirmed in the first part of the Lemma.

8. In the next place, if these three following quantities be multiplied one into another,

$$\begin{aligned} \text{viz. } \left\{ \begin{array}{l} b^4cc - 2b^3c^2 + b^2c^4, \quad (\text{the Square in the fourth step,}) \\ b^4 + c^4, \quad (\text{the Hypotenusal,}) \\ 2bbcc, \quad (\text{the Perpendicular,}) \end{array} \right. \end{aligned}$$

The solid Product of their multiplication will be $2b^5c^2 - 4b^4c^3 + 4b^3c^4 - 4b^2c^5 + 2b^4c^4$.

9. And

9. And if these three following quantities be multiplied one into another, to wit,

$$\begin{aligned} & b^4cc - b^3c^3, \quad \left\{ \begin{array}{l} \text{the Area,} \\ \text{the Perpendicular,} \end{array} \right. \\ & + 2bbcc, \quad \left\{ \begin{array}{l} \text{the Area,} \\ \text{the Perpendicular,} \end{array} \right. \\ & bbb + cccc - 2bbcc, \quad (\text{the excess of the Hypotenusal above the Perpendicular,}) \end{aligned}$$

The solid Product of their multiplication will be $2b^5c^2 - 4b^4c^3 + 2b^3c^4 + 4b^2c^5$.

10. Which latter solid Product subtracted from the former, $4b^3c^3 - 8b^2c^4 + 4b^2c^5$ leaves this Square, to wit, $4b^3c^3 - 8b^2c^4 + 4b^2c^5$.

The side whereof is $2b^2c^2 - 2bb^2c$.

Wherefore the second part of the Lemma is manifest.

11. Lastly, the sum of the two solid Products in the eighth and ninth steps makes this Square, to wit, $4b^3c^3 - 8b^2c^4 + 4b^2c^5$.

The side whereof is $2b^2cc - 2b^2c^2$.

Wherefore the Lemma is every way proved.

An Example in Numbers.

Suppose $4 = bb$, and $1 = cc$; then let a right-angled Triangle be formed from 4 and 1, the three sides will be 17, 15, 8. Now I say, first, that if 9 the Square of the difference between the said 4 and 1, be multiplied by 4 the Product of 4 and 1, there will be produced the square number 36, which is less than 60 the Area of the said Triangle.

Secondly, if from 4896, which is the solid Product made by the multiplication of these three numbers, to wit, 36 the Square before found, the Hypotenusal 17, and 8 the double Product of 4 and 1; there be subtracted 4320, which is the solid Product of these three numbers, to wit, the Area 60; the said double Product 8, and 9 the excess of the Hypotenusal above the said 8; there will remain the Square 576, whose side is 24.

Thirdly and lastly, the sum of the said solid Products 4896 and 4320 makes the square 9216, whose side is 96.

Now followeth the RESOLUTION of QUEST. 104. before proposed.

1. Let the Hypotenusal, Base and Perpendicular of some right-angled Triangle in numbers be represented by h, b, p .

2. Then multiply those sides severally by a , and suppose the Products to be the sides of the Triangle sought, to wit, ha, ba, pa .

3. Then, (according to the Question,) if the Area of the Triangle in the last step be lessened by the Hypotenusal and one of the sides about the right-angle severally, each remainder must be a Square; hence this Duplicate equality arithmetically to be resolved, viz. $\frac{1}{2}bpa - ba = \square$, $\frac{1}{2}bpa - pa = \square$.

4. Now in order to resolve that Duplicate equality, suppose $\frac{1}{2}bpa - pa = ccaa$.

5. Whence after due Reduction you will find $a = \frac{p}{2p - ce}$.

6. According to which value of a , if the former of the two quantities in the third step be resolved, instead of $\frac{1}{2}bpa - ba$, $\frac{1}{2}bpa - ba = \square$, $\frac{1}{2}bpa - pa = \square$, it shall be equated to a Square, viz. $\frac{1}{2}bpa - pa = ccaa$.

7. But because the Denominator of the Fraction last express'd is a Square, for its side is $2p - ce$, it remains only to equate the Numerator to a Square. We must therefore inquire into the rise of the Numerator, so we shall find that it imports the search of a right-angled Triangle h, b, p in numbers, and a square number ce less than the Area of the said Triangle; (for the Denominator of the Fraction in the fifth step of the Resolution shews that $2bp$ must exceed ce .) Moreover, the said Triangle and Square must be such, that if from the solid Product made by the multiplication of the said Square, the Hypotenusal, and one of the sides about the right-angle, there be subtracted the solid Product made by the multiplication of the Area, the said side about the right-angle, and the excess of the Hypotenusal above the same side, the remainder may be a Square. But the last preceding Lemma shews how to find out such a Triangle and Square. Suppose then the Hypotenusal, Base and Perpendicular of the said Triangle to be found out, to wit, h, b, p , and the Square ce , to be all known in numbers, then the number represented by a shall consequently be known from the fifth step of the Resolution; and lastly, if you multiply the numbers h, b, p severally by the number a , the Products shall be the three sides of the Triangle sought. From the premises there ariseth the following

CANON.

CANON.

8. First, let a right-angled Triangle be formed from two square numbers, or from two numbers which have such proportion to one another as a square number to a square number; then multiply the Square of the difference of those two numbers by the Product of their multiplication and reserve the Product, which is a square number, and may be called ee ; that done, divide that side about the right-angle which is the double Product of the two numbers that formed the said Triangle, by the excess of its Area above the Square ee before reserved, and by the Quotient multiply severally the three sides of the same Triangle, so shall the Products be the three sides of the right-angled Triangle sought.

An Example in Numbers.

First, I form a right-angled Triangle from the square numbers 4 and 1, so the three sides are 17, 15, 8; then I multiply 9 (the Square of the difference between 4 and 1) by 4, the Product of 4 into 1, and there comes forth the Square 36, (to wit, ee ;) then I divide 8, (to wit, that side about the right-angle of the said Triangle which is the double Product of 4 into 1,) by 24, which is the excess of 60 the Area of the said Triangle above the Square 36, (to wit, ee ;) and the Quotient is $\frac{1}{3}$, by which I multiply severally above the Square 36, (to wit, ee ;) and the Products $\frac{4}{3}$, $\frac{16}{3}$, $\frac{64}{3}$, (the sides of the right-angled Triangle first found,) and the Products $\frac{4}{3}$, $\frac{16}{3}$, and $\frac{64}{3}$ are the sides of the right-angled Triangle sought: For, if from the Area $\frac{60}{3}$, and the $\frac{4}{3}$ be severally subtracted, there will remain the Squares 1 and 4.

This Question is capable of innumerable Answers in a double respect; for first, instead of 4 and 1 we may take any two square numbers, or any two numbers which are in such proportion to one another as a square number to a square number, for the forming of a right-angled Triangle as the Canon directs: secondly, the same right-angled Triangle 17, 15, 8 being retained, we may instead of the Square 36, to wit, ee , find infinite others, every one of which shall be less than the Area 60, and such, that if it be multiplied into 136, (to wit, bp ;) and from the Product there be subtracted 4320, (to wit, $\frac{1}{2}bp \times b - p$;) the remainder shall be a square number. (The finding out of which Squares may easily be deduced from Lemma 3, in the preceding Quest. 101.)

QUEST. 105.

To find a right-angled Triangle, that if its Area be subtracted from the Hypotenuse, and from one of the sides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let the Hypotenuse, Base and Perpendicular of some right-angled Triangle in numbers be represented by h ; b ; p
2. Then multiply those sides severally by a , and assume the Products to be the sides of the right-angled Triangle sought, to wit, ha ; ba ; pa
3. Then, (according to the Question,) each of these two quantities must be equated to a Square, viz. $ha - \frac{1}{2}bpaa = \square$
 $pa - \frac{1}{2}bpaa = \square$
4. Now in order to resolve that Duplicate equality, suppose $pa - \frac{1}{2}bpaa = ecaa$
5. Whence after due Reduction you will find $a = \frac{p}{ee + \frac{1}{2}bp}$
6. According to which value of a , if the former of the two quantities in the third step be resolved, instead of that quantity, (to wit, $ha - \frac{1}{2}bpaa$;) this that follows ariseth to be equated to a Square, viz. $hpee + \frac{1}{2}bpp \times b - p = \square$
 $eece + bpce + \frac{1}{2}bbpp = \square$
7. But because the Denominator of the Fraction last express'd is a Square, for its side is $ee + \frac{1}{2}bp$, it remains only to equate the Numerator to a Square, and the Numerator well examined, shews that we must find a right-angled Triangle h , b , p in numbers, and a square number ee , such, that if to the solid Product made by the multiplication of the said Square, the Hypotenuse and one of the sides about the right-angle, there be added the solid Product made by the multiplication of the Area, the said side about the right-angle, and the excess of the Hypotenuse above the same side, the sum may be a Square: But the Lemma prefixt to the Resolution of the preceding Quest. 104, shews how to find out such a Triangle and Square: Suppose then the Hypotenuse, Base

Base and Perpendicular of the said Triangle so found out, to wit, h , b , p , and the Square ee to be all known in numbers, then consequently the number a shall be known also from the fifth step of the Resolution: And lastly, the numbers h , b , p being multiplied severally by the number a will give the three sides of the Triangle sought. From the premises there ariseth the following

CANON.

1. First, let a right-angled Triangle be formed from two square numbers, or from two numbers which have such proportion to one another as a square number to a square number; then multiply the Square of the difference of those two numbers by the Product of their multiplication, and reserve the Product, which is a square number, and may be called ee ; then divide that side about the right-angle which is the double Product of the two numbers that formed the said Triangle, by the sum of its Area and the Square ee before reserved; lastly, by the Quotient multiply severally the three sides of the Triangle first formed: So shall the Products be the three sides of the right-angled Triangle sought.

An Example in Numbers.

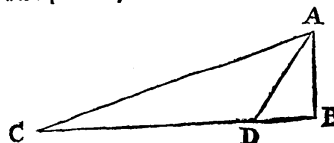
First, a right-angled Triangle being formed from the square numbers 4 and 1; the three sides will be 17, 15, 8; then I multiply 9 the Square of the difference between 4 and 1, by 4 the Product of 4 into 1, and it produceth the Square 36, (to wit, ee ;) then I divide the side 8, by 96 which is the sum of the Area 60 and the Square 36, (to wit, ee ;) so the Quotient is $\frac{1}{12}$; lastly, the sides 17, 15, 8 being multiplied severally by $\frac{1}{12}$ will give $\frac{17}{12}$, $\frac{15}{12}$, and $\frac{8}{12}$ for the right-angled Triangle sought. For if from the Hypotenuse $\frac{17}{12}$, and from the side $\frac{8}{12}$, the Area $\frac{1}{12}$ be subtracted, there will remain the Squares 1 and $\frac{1}{4}$.

After the same manner, among innumerable right-angled Triangles that might be found by the said Canon to solve this Question, these three will be discovered, to wit, $\frac{11}{12}$, $\frac{35}{12}$, $\frac{32}{12}$; $\frac{41}{12}$, $\frac{41}{12}$, $\frac{41}{12}$; $\frac{11}{12}$, $\frac{11}{12}$, $\frac{11}{12}$; every one of which Triangles besides the first found, to wit, $\frac{17}{12}$, $\frac{15}{12}$, $\frac{8}{12}$ is express'd by smaller numbers than that Triangle found by Fermat's method, in the following Quest. 127.

This Question is also capable of innumerable Answers upon another ground, as may easily be collected from what hath been said at the latter end of the preceding Quest. 104.

QUEST. 106. (Quest. 18. Lib. 6. Diopham.)

To find a right-angled Triangle in rational numbers, suppose ABC right-angled at B, that one of its acute-angles BAC being cut into two equal parts by the line AD, the said line AD may be also expressible by a rational number.



RESOLUTION.

1. Let the Hypotenuse, Base and Perpendicular of some right-angled Triangle known in numbers be represented by h ; b ; p
 $ha = AD$
 $ba = BD$
 $pa = AB$
 $b - ba = DC$
2. Then multiply those sides severally by a , (which represents a number unknown,) and put $BD \cdot AB :: DC \cdot CA$
3. Now forasmuch as (per 3. Prop. 6. Elem. Euclid.) these lines in the preceding Diagram are Proportionals, viz. $ba \cdot pa :: b - ba \cdot p - pa$
4. Therefore from the premises these numbers shall be also Proportionals, viz. $ba \cdot pa :: b - ba \cdot p - pa$
5. And because (per 47. Prop. 1. Elem. Euclid.) the Square of AB together with the Square of BC is equal to the Square of CA; therefore the Square of pa together with the

the Square of b , (for $ba + b - ba = b$;) shall be equal to the Square of $p - pa$, whence this Equation aritheth, to wit,

$$ppaa + bb = ppaa - 2ppa + pp.$$

6. Which Equation, after due Reduction, gives $a = \frac{pp - bb}{2pp}$

$$\begin{aligned} \frac{ppp - pbb}{2pp} &= AB \\ \frac{2bpp}{2pp} &= BC \\ \frac{ppp + pbb}{2pp} &= CA \\ \frac{bpp - bbb}{2pp} &= AD \\ \frac{bpp - bbb}{2pp} &= BD \\ \frac{bpp + bbb}{2pp} &= DC \end{aligned}$$

7. Therefore from the first, second and sixth steps, all the lines sought shall now be known in rational numbers, viz. . . .

8. And by multiplying all the Fractions in the last step by the common Denominator $2pp$, the Products will give these numbers which will also solve the Question, and may serve as a Canon for that purpose, viz.

$$\begin{aligned} ppp - pbb &= AB \\ 2bpp &= BC \\ ppp + pbb &= CA \\ bpp - bbb &= AD \\ bpp - bbb &= BD \\ bpp + bbb &= DC \end{aligned}$$

An Example in Numbers.

Take any right-angled Triangle in numbers, as $5, 3, 4$, then by putting $5 = b$, $3 = p$ and $4 = p$, (the greater of the sides about the right-angle,) you will find

$$\begin{aligned} ppp - pbb &= 28 = AB \text{ the Perpendicular,} \\ 2bpp &= 96 = BC \text{ the Base,} \\ ppp + pbb &= 100 = CA \text{ the Hypotenusal,} \\ bpp - bbb &= 35 = AD \text{ the line bisecting the acute-angle opposite to the Base,} \\ bpp - bbb &= 21 = BD \text{ the segments of the Base,} \\ bpp + bbb &= 75 = DC \end{aligned}$$

The Proof.

First, these numbers are Proportional, $\frac{21}{BD} : \frac{28}{AB} :: \frac{75}{DC} : \frac{100}{CA}$

Therefore (per Prop. 3. Elem. 6. Euclid.) the angle BAD is equal to the angle DAC. The rest of the Proof is obvious.

QUEST. 107.

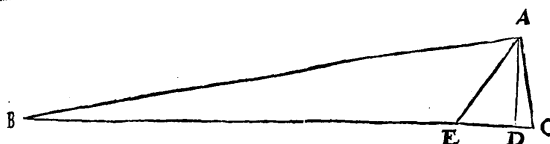
To find out an oblique-angled Triangle, whose three sides, as also the Perpendicular, and a line cutting one of the angles into two equal parts may be express severally by rational numbers.

[Jac. de Billy in probl. 5. cap. 4. of the latter part of his Diophant. redivi. printed at Lyons in 1670. resolves this Question briefly by Numeral Algebra; but to the end a Canon may be raised to solve the Question proposed, I shall form the Resolution thereof by Literal Algebra.]

Preparation.

Let there be an oblique-angled Triangle ABC; then supposing AD to be perpendicular to the Base BC, and the line AE to cut the angle BAC into two equal angles

EAB and EAC, let it be required to find out rational numbers to express the quantities of the sides AB, BC, AC, as also of AD, AE, BE, ED, DC.



RESOLUTION.

1. First, let b, p, b represent the Base, Perpendicular and Hypotenusal of any right-angled Triangle known in numbers, and suppose the Perpendicular p to be greater than the Base b , then

$$\text{Put } \dots \left\{ \begin{aligned} b &= DE \text{ the Base,} \\ p &= AD \text{ the Perp.} \\ b &= AE \text{ the Hyp.} \end{aligned} \right\} \text{ of } \triangle ADE.$$

2. Secondly, making p (that is, AD) to be the Perpendicular of a second right-angled Triangle, suppose ADB, find out the Base DB and the Hypotenusal BA in rational numbers, which may be done thus, viz. Forasmuch as the Square of the Perpendicular AD is equal to the difference of the Squares of the Hypotenusal BA and the Base DB, let the Square of the given number p be esteem'd to be the difference of two square numbers, and find out the Squares themselves, then put the side of the lesser Square for DB, and the side of the greater for BA; but the said Squares must be found out with this Caution, that DB the side of the lesser Square may have greater proportion to DE, (that is, b) than $2pp$ hath to $pp - bb$, as may easily be infer'd from the Canon of the preceding Quest. 106. where it appears, that when the angle DAC in the Diagram belonging to that Question is equal to the angle DAB, then the Base BC hath such proportion to BD, as $2pp$ to $pp - bb$; but in the Diagram of this Question the angle EAB must be greater than the angle EAD, in regard by supposition the angle EAB is equal to the angle EAC. Now since by Quest. 7. of this Book, innumerable pairs of Squares having one common difference may be found out, such, that the side of one Square of each pair shall be greater or less than a given number, let us suppose the sides of two square numbers to be found out agreeable to the said Caution,

$$\text{viz. } \left\{ \begin{aligned} b + d &= DE + EB = DB, \\ d &= EB, \\ g &= BA. \end{aligned} \right.$$

3. Thirdly, the next scope is to find out EC and AC in rational numbers, which must have the same proportion one to another as EB and BA; (for by supposition the angle EAB is equal to the angle EAC, and therefore (per Prop. 3. Elem. 6. Euclid.) $EB : BA :: EC : AC$.) Moreover, the Square of $EC - ED$, that is, of DC, together with the Square of AD must be equal to the Square of AC. But EB and BA were before found out in numbers, to wit, $d = EB$, and $g = BA$; now to find out two numbers in the same proportion as d and g , multiply these severally by a (which represents a number yet unknown,) and put the Products da and ga for EC and AC; whence these are Proportional,

$$\text{viz. } \left\{ \begin{aligned} d &: g :: da : ga, \\ EB &: BA :: EC : AC. \end{aligned} \right.$$

4. And because the Square of $EC - ED$, that is, the Square of DC, together with the Square of AD makes the Square of AC, therefore in the letters of the Resolution, the Square of $da - b$ together with the Square of p must be equal to the Square of ga ; hence this Equation aritheth, viz.

$$daa - 2bda + bb + pp = gga.$$

5. And because (by the first step of the Resolution,) $bb = bb + pp$
6. Therefore $daa - 2bda - hh = gga$
7. Whence, after due Reduction, this following Equation aritheth, viz. $aa + \frac{2bd}{gg - da}a = \frac{gg - dd}{8}$ which

8. Which last Equation being resolv'd by the Canon in *Sett. 6. Chap. 15. Book 1.* the value of a will be made known, *viz.*

$$a = \sqrt{\frac{bbdd + ggbh - ddbb}{gg - dd} \div \frac{bd}{gg - dd}}$$

9. Now if the Numerator $bbdd + ggbh - ddbb$ be a rational Square number, then the value of a is manifestly rational, for the Denominator is a Square, whose side is $gg - dd$; but that the said Numerator is a rational Square I prove thus: In the Triangle BAE obtuse-angled at E , the Square of BA less by the sum of the Square of BE and twice the rectangle made of BE and ED is equal to the Square of AE , (*per prop. 2. Elem. Euclid.*) therefore in the letters of the Resolution,

$$gg - dd - 2db = bh.$$

10. But if $gg - dd - 2db$ instead of bh be multiplied into $gg - dd$, and to the Product there be added $bbdd$, then instead of the aforesaid Numerator $bbdd + ggbh - ddbb$, this following Square will arise, to wit,

$$gggg - 2ggdd - 2ggdb + dddd + 2dddb + ddbb;$$

Whole side is $gg - dd - 2db$.

11. Therefore from the eighth and tenth steps, the value of a is expressible by a rational number, *viz.*

$$a = \frac{gg - dd - 2db}{gg - dd}$$

12. Therefore from the third and eleventh steps,

$$\frac{dgg - ddd - 2ddb}{gg - dd} = EC$$

13. And from the third and eleventh steps,

$$\frac{ggg - ggd - 2gdb}{gg - dd} = AC$$

14. And by subtracting b from the quantity in the twelfth step, (*viz.* ED from EC ;) it gives

$$\frac{gg - dd}{gg - dd} = DC$$

15. Lastly, if the three Fractions in the three last preceding steps, as also b, p, h, d, g be severally multiplied by the Denominator $gg - dd$, there will come forth the following quantities in Integers, which may serve as a Canon to solve the Question proposed; provided that the numbers b, p, h, d, g be first found out agreeable to the Canon before prescribed.

CANON.

$$\begin{array}{lcl} ggd - ddd - 2ddb & = & EC \\ ggd - ddd & = & EB \\ 2ggd - 2ddd - 2ddb & = & BC \\ ggg - ggd & = & BA \\ ggg - ggd - 2gdb & = & AC \\ ggp - ddp & = & AD \\ ggb - ddb & = & AE \\ ggb - ddb & = & DE \\ ggd - ddd - ggb - bdd & = & DC \\ ggd + ggb - ddd - ddb & = & DB \end{array} \quad \text{in the Diagram belonging to this Question.}$$

An Example in Numbers.

First, take any right-angled Triangle in whole numbers, as the Triangle 18, 24, 30; then

$$\text{Put } \begin{cases} b = 18 & ED; \\ p = 24 & AD, \\ h = 30 & AE. \end{cases}$$

Secondly, making 24, (to wit, $p = AD$) to be the Perpendicular of a second right-angled Triangle, as well as of the first ADE , find out the Base DB , and the Hypotenuse AB in rational numbers; but for the reason before given, the number of the Base DB must have greater proportion to the number of the Base DE , than $2pp$ to $pp - bb$, *viz.* in this Example, greater proportion than 32 to 7, and consequently DB must exceed DE taken $4\frac{1}{2}$ times: But by supposition $DE = 18$; therefore DB must exceed $82\frac{1}{2}$. Now because the Square of the Perpendicular AD is equal to the difference of the Squares of AB and DB , therefore 576 the Square of the Perpendicular 24 ($= AD$) being taken for the difference of two Squares, find out the Squares severally, but with this condition, that the side of the lesser of them may exceed $82\frac{1}{2}$: But two such Squares (among innumerable other pairs of Squares that may be found out by the seventh Question of

of this Book,) are 20449 and 21025, whose sides are 143 and 145; therefore $143 = DB$, $145 = AB$, and $125 = EB$, (that is, $DB - DE$;) then

$$\text{Put } \begin{cases} d = 125 = EB, \\ g = 145 = BA. \end{cases}$$

Lastly, by the help of the numbers before found out for the values of b, p, h, d and g , the preceding Canon will discover rational numbers, which reduced to their least terms by their greatest common Divisor, will give the whole numbers here-under express'd, for the measures of the sides of the oblique-angled Triangle sought; as also of the Perpendicular, and the line cutting the angle opposite to the Base into two equal parts, and of the segments of the Base made as well by the Perpendicular as by the line bisecting the said angle, *viz.*

$$\begin{array}{l} EC = 125 \\ EB = 750 \\ BC = 875 \\ BA = 870 \\ AC = 145 \\ AD = 144 \\ AE = 180 \\ DE = 108 \\ DC = 17 \\ DB = 858 \end{array}$$

agreeable to the Diagram and Canon belonging to this Question.

The Proof.

First, these numbers are Proportionals, *viz.* $750 : 870 :: 125 : 145$
 $EB : BA :: EC : AC$
 Therefore, (*per prop. 6. Elem. Euclid.*) the angle EAB is equal to the angle EAC .
 The rest of the Proof is obvious.

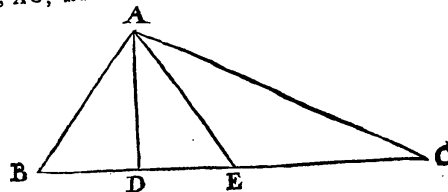
QUEST. 108.

To find out an oblique-angled Triangle, whose three sides, as also the Perpendicular, and a line drawn from the angle opposite to the Base, and cutting the Base into two equal parts, may be express'd severally by rational numbers.

[*Jac. de Billy in his Appendix to the Problem cited in the preceding 107th Question resolves this also, but very briefly; I shall therefore form the Resolution thereof at large by Numeral Algebra, by the steps whereof the more curious Analyst may easily resolve it by Specious Algebra, but the Canon thence arising will be exceeding tedious.*]

Preparation.

Let there be an oblique-angled Triangle ABC ; then supposing AD to be perpendicular to the Base BC , and the line AE to cut the Base BC into two equal parts in the point E , let it be required to find out rational numbers to express the quantities of the sides AB, BC, AC ; as also of the lines AD, AE, BD, DE .



RESOLUTION.

1. First, take any right-angled Triangle in numbers, as 3, 4, 5; then;

$$\text{Put } \begin{cases} 3 = BD \text{ the Base,} \\ 4 = AD \text{ the Perpendicular,} \\ 5 = AB \text{ the Hypotenuse,} \end{cases} \quad \text{of the right-angled Triangle BDA.}$$

2. Then for the distance between the foot of the Perpendicular and the middle of the Base BC , put a , *viz.* suppose $a = DE$

3. And

3. And because by supposition, $BE = EC$, if to a you add 3, (to wit, BD), the sum shall be equal to half the Base, viz. $a+3 = BE = EC$
 4. And because $DE \perp EC = DC$, the sum of the two Equations in the second and third steps shall be equal to the greater segment of the Base made by the Perpendicular, viz. $2a+3 = DC$
 5. And because the Square of DC together with the Square of AD is equal to the Square of AC , the Square of $2a+3$ together with the Square of 4 must make a Square, viz.

$$4aa + 12a + 25 = \square.$$

6. And because the Square of DE together with the Square of AD is equal to the Square of AE , the Square of a together with the Square of 4 must be equated to a Square, viz.

$$aa + 16 = \square.$$

7. So in the two last steps we are fallen upon a Duplicate equality, which, in regard the Squares 25 and 16 are unequal, I reduce to another that shall have equal known Squares, viz. (after the manner used in divers preceding Questions of this Book,) I divide the greater Square 25 by the lesser 16, and by the Quotient $\frac{5}{4}$ I multiply the quantity in the sixth step, to wit, $aa + 16$, so the Product $\frac{5}{4}aa + 25$ is to be equated to a Square, and therefore this Duplicate equality comes now to be resolved,

$$\text{viz. } \begin{cases} 4aa + 12a + 25 = \square \\ \frac{5}{4}aa + 25 = \square \end{cases}$$

8. Now in order to resolve the Duplicate equality last express'd, first, by subtracting the lesser quantity from the greater, I find their difference to be

$$\frac{3}{4}aa + 12a.$$

9. Then I search out two quantities, the Product of whose multiplication may make the said difference $\frac{3}{4}aa + 12a$; and that as well in the difference as in the sum of the same quantities there may be found 10, (to wit, the double of the side of the Square 25,) so I find those two quantities to be

$$\frac{3}{2}a + 10 \text{ and } \frac{1}{2}a.$$

10. Then the Square of half the difference of the two quantities last express'd, viz. the Square of $\frac{3}{4}a + \frac{1}{2}$ being equated to $\frac{3}{4}aa + 25$, gives

$$\frac{9}{16}aa + 25 = \frac{9}{16}aa + \frac{1}{4}aa + \frac{1}{2}a + 25.$$

11. Which Equation, after due Reduction, will discover the number a , viz.

$$a = \frac{4 \times 1600}{142311} = DE$$

12. Then by adding the Square of the said number a to 16, and extracting the square Root of the sum, there ariseth

$$\frac{120156}{142311} = AE$$

13. And by adding 3 (to wit, BD) to the number a , (to wit, DE), the sum shall be the measure of half the Base BC , viz.

$$\frac{812111}{142311} = BE = EC$$

14. Therefore the double of the number in the last step is the measure of the Base BC , viz.

$$\frac{120156}{142311} = BC$$

15. And by adding the number a in the eleventh step to half the Base in the fourteenth, the sum is the greater segment of the Base made by the Perpendicular, viz.

$$\frac{3828111}{142311} = DC$$

16. Then by adding the Square of the said greater segment DC to 16 the Square of the Perpendicular AD , and extracting the square Root of the sum, there comes forth

$$\frac{3722161}{142311} = AC$$

17. Lastly, by multiplying the numbers in the first, eleventh, twelfth, thirteenth, fourteenth, fifteenth and sixteenth steps severally by the Denominator 142311, there will come forth these following whole numbers for the measures of the lines sought, viz.

$$\left. \begin{array}{l} 426933 = BD \\ 569244 = AD \\ 711555 = AB \\ 425600 = DE \\ 710756 = AE \\ 852533 = BE = EC \\ 1705066 = BC \\ 1278133 = DC \\ 1399165 = AC \end{array} \right\} \text{ in the Diagram belonging to this Question.}$$

The

The Proof is easie to be made, by comparing the sum of the Squares of the numbers answering to the sides about the right-angle of every right-angled Triangle in the Diagram, with the number answering to the Square of the Hypotenuse of such Triangle respectively.

QUEST. 109. (Quest. 21. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the Area thereof increased with one of the sides about the right-angle may make a Square; and that the sum of all the three sides may be a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $a+1$; then divide the three sides severally by $a+1$, and take the Quotients for the sides of the Triangle sought, viz.

$$\begin{cases} \frac{2aa+2a+1}{a+1} = \text{the Hypoth.} \\ \frac{2aa+2a}{a+1} = \text{the Base;} \\ \frac{2a+1}{a+1} = \text{the Perpend.} \end{cases}$$

2. The sum of those three sides is $4a+2$ must be equal to a Cube; which therefore (according to the Question) the said $4a+2$ must be equal to a Cube; which in the following ninth step I shall shew how to find out.

3. Moreover, the Area together with one of the sides about the right-angle of the Triangle in the first step must make a Square. But the Area (by multiplying half the Base into the Perpendicular) will be found

$$\frac{2aa+3aa+a}{aa+2a+1}$$

4. And one of the sides about the right-angle (to wit, the Perpendicular) is

$$\frac{2a+1}{a+1}$$

5. Which side reduced to the same Denominator with the Area, (by multiplying as well the Numerator

$$\frac{2aa+3a+1}{aa+2a+1}$$

6. Now if the side in the last step be added to the Area in the fourth, the sum will be

$$\frac{2aaa+5aa+4a+1}{aa+2a+1}$$

7. Which sum reduced to its least terms, (by dividing the Numerator by the Denominator) will be

$$2a+1$$

8. Therefore (according to the Question) $2a+1$ must be equated to a Square; and before in the third step it was found that $4a+2$ must be equated to a Cube; now because $4a+2$ is the double of $2a+1$, we must find out a Cube that may be the double of a Square; but the Cube 8 is the double of the Square 4, therefore let $4a+2$ be equated to 8, or $2a+1$ to 4; whence you will find $a = \frac{3}{2}$, and consequently $a+1 = \frac{5}{2}$. Wherefore according to the Positions in the first step, let a right-angled Triangle be formed from $\frac{3}{2}$ and $\frac{5}{2}$, and divide the three sides severally by $\frac{5}{2}$, so there will come forth the sides of the right-angled Triangle sought, to wit, $\frac{12}{5}$, $\frac{14}{5}$, $\frac{17}{5}$, which solve the Question; for the Area $\frac{12}{5} \times \frac{14}{5}$ together with $\frac{12}{5}$ (to wit, one of the sides about the right-angle) makes the Square 4 ; and the sum of all the three sides makes a Cube, to wit, 8.

It is also evident by the premises, that the Question is capable of innumerable Answers, in regard you may find out as often as you please a Cube and a Square, such, that the former shall be the double of the latter: As, by equating aaa to $8aa$, it will give the Cube 512, and the Square 256, the former of which is the double of the latter.

QUEST. 110. (Quest. 23. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the sum of all the three sides may be a Square; and that the said sum increased with the Area may make a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and 1, so that the three sides will be these, to wit.

$$\left. \begin{array}{l} aa+1, aa-1, 2a \end{array} \right\}$$

2. The

2. The sum of all the three sides is $2aa + 2a$, which must be equal to a Square, let it therefore be equated to $ccaa$, viz. suppose
3. Whence, after due Reduction, you will find $a = \frac{2}{cc-2}$
4. Therefore by squaring the last Equation, there ariseth $aa = \frac{4}{cccc-4cc+4}$
5. Which last Equation doubled, is $2aa = \frac{4cc-4cc+4}{cccc-4cc+4}$
6. And by adding the double of the Equation in the third step, to wit, $2a = \frac{4}{cc-2}$ to the Equation in the fifth step, the sum gives this Equation, to wit,
- $$2aa + 2a = \frac{4cc}{cccc-4cc+4}$$
7. The latter part of which Equation is manifestly a Square, and by Construction 'tis equal to the sum of all the three sides of the Triangle in the first step. It remains that the said sum with the Area make a Cube; but from the first step the Area is $aaa - a$, and according to the value of a in the third step, the Area $aaa - a$ will be reduced to this, to wit,
- $$\frac{8cc-2cccc}{cccc-6cccc+12cc-8}$$
8. To which add the sum of all the three sides, to wit, the latter part of the Equation in the fifth step, that is, $\frac{4cc}{cccc-4cc+4}$, (which, by multiplying the Numerator and Denominator severally by $cc-2$, will be reduced to $\frac{4cccc-6cccc+12cc-8}{cccc-6cccc+12cc-8}$), and there will come forth this sum, to wit,
- $$\frac{8cccc-6cccc+12cc-8}{cccc-6cccc+12cc-8}$$
9. Which sum last produced must be equal to a Cube; and because by Construction the Denominator is a Cube, to wit, the Cube of $cc-2$, it remains that we equate the Numerator $8cccc-6cccc+12cc-8$ to some Cube; or by dividing $2cccc$ by ccc it gives $2c$ to be equated to a Cube, which is easie to be done, for we may put $2c$ equal to any known cube-number, as ddd ;
10. Suppose therefore $2c = ddd$
11. Then because the Denominator of the Fraction in the third step shews that $c = \sqrt[3]{2}$
12. And consequently by doubling each part, $2c = \sqrt[3]{8}$
13. It follows from the tenth and twelfth steps, that $ddd = \sqrt[3]{8}$
14. Again, one of the sides about the right-angle is by Construction in the first step $aa - 1$, therefore $aa = 1$
15. But by the fourth step, $cccc-4cc+4 = aa$
16. Therefore from the two last steps, $\frac{4}{cccc-4cc+4} = 1$
17. And by multiplying each part in the sixteenth step by the Denominator $cccc-4cc+4$, 'tis manifest that $4 = cccc-4cc+4$
18. Whence, by adding $4cc$ to each part, $4+4cc = cccc+4$
19. And by subtracting 4 from each part in the last step $4cc = cccc$
20. And by dividing each part in the nineteenth step by cc , $4 = cc$
21. And by extracting the Square Root out of each part in the 20th step, 'tis evident that $2 = c$
22. And by doubling each part in the twenty-first step, $4 = 2c$
23. But by supposition in the tenth step, $ddd = 2c$
24. Therefore from the two last steps, $ddd = 4$
25. And consequently, $ddd = 4$
26. Thus in the 13th and 25th steps it is found that the cube-number represented by ddd must be greater than the Square Root of 8, but less than 4; and then the half of the cube-number taken within those limits shall be the number e , which being known, the number a will also be discovered by the third step: Lastly, a right-angled Triangle formed from the number a and unity, shall be that which is sought. From the premises ariseth this

CANON.

CANON.

17. First, take any cube-number greater than the Square Root of 8, (viz. greater than $\sqrt[3]{8}$, but less than 4; then take the half of that cube-number, and call it e ; that done, divide 2 by the excess of the Square of the number e above 2, and call the Quotient a ; lastly, let a right-angled Triangle be formed from the number a and 1, so shall this Triangle be that which is sought.

An Example in Numbers.

First, I take some cube-number within the limits prescribed in the Canon; as $\frac{27}{8}$, whole half, to wit, $\frac{27}{8}$ is the number e ; then I divide 2 by the excess of the Square of $\frac{27}{8}$ above 2, so the Quotient $\frac{128}{121}$ is the number a ; lastly, from $\frac{128}{121}$ and 1 I form a right-angled Triangle and find the three sides to be these, to wit, $\frac{102213}{47089}$, $\frac{222152}{47089}$, and $\frac{222152}{47089}$, which solve the Question: For the sum of those three sides make the Square $\frac{222152}{47089}$, whose side is $\frac{47089}{102213}$; and the said sum of the sides together with the Area $\frac{102213}{102213}$ makes the Cube $\frac{27}{8}$, whose side is $\frac{3}{2}$.

QUEST. III. (Quest. 24. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the sum of all the three sides may be a Cube; and that the said sum increased with the Area may make a Square.

RESOLUTION.

1. For the sum of all the three sides of the Triangle fought put s
2. And for the Area a
3. Then because the double of the Area is equal to the Product made by the mutual multiplication of the sides about the right-angle, the said Product is $2a$
4. For one of the sides about the right-angle put $\frac{1}{c}$
5. Then because the Product of the said sides is $2a$, 'tis manifest that $2ac$ by dividing $2a$ by $\frac{1}{c}$, the Quotient shall be the other side, to wit, $2ac + \frac{1}{c}$
6. Therefore from the fourth and fifth steps the sum of the sides about the right-angle is $s - 2ac - \frac{1}{c}$
7. Which sum subtracted from s the sum of all the three sides, leaves the Hypotenusal, to wit, $\frac{1}{c}$
8. The Square of the said Hypotenusal is $\frac{1}{c^2} + 4aacc + \frac{1}{cc} + 4a - 4sac - \frac{2s}{c}$
9. And the sum of the Squares of the sides about the right-angle, to wit, of the Square of $\frac{1}{c}$ and $2ac$ is $\frac{1}{cc} + 4aacc$
10. But the Square of the Hypotenusal is equal to the sum of the Squares of the sides about the right-angle, therefore from the eighth and ninth steps this Equation ariseth, viz. $\frac{1}{cc} + 4aacc + \frac{1}{cc} + 4a - 4sac - \frac{2s}{c} = \frac{1}{cc} + 4aacc$
11. From which Equation, after due Reduction in order to find out the value of e , by the letters s and a , there will arise the following Equation, viz.
- $$\frac{ss + 4a}{4sa} e - cc = \frac{2s}{4sa}$$
12. And by resolving the last Equation according to the Canon in Sect. 10. Chap. 153 Book 1. the two values of e will be found these, to wit,
- $$e = \frac{\frac{1}{2}ss + 2a}{4sa} + \sqrt{\frac{\frac{1}{2}ss + 4a - 6sa}{16s1aa}}$$
- $$e = \frac{\frac{1}{2}ss + 2a}{4sa} - \sqrt{\frac{\frac{1}{2}ss + 4a - 6sa}{16s1aa}}$$

13. But 'tis evident that those values of e will not be rational unless the Numerator $\frac{1}{2}ss + 4a - 6sa$ be equal to a Square. Moreover, the Question requires that

the sum of all the three sides with the Area, to wit, $s + a$ may be equal to a Square; so we are fain upon this Duplicate equality,

$$\text{viz. } \begin{cases} 4ss + 4aa - 6ss = 0 \\ s + a = 0 \end{cases}$$

14. Now if in that Duplicate equality any known Square number be taken for the value of s , then we may discover the number a . But the Question requires s , (that is, the sum of all the three sides of the Triangle sought,) to be a Cube, therefore some number which is both a Square and a Cube must be taken for the value of s ; let therefore the Squared Cube 64 be put equal to s , and then the said Duplicate equality will be resolved into this,

$$\text{viz. } \begin{cases} 4aa - 24576a + 4194304 = 0 \\ a + 64 = 0 \end{cases}$$

15. Or you may divide the first of those two quantities by 4, then instead of that first quantity the Quotient is to be equated to a Square, and so this following Duplicate equality ariseth,

$$\text{viz. } \begin{cases} aa - 6144a + 1048576 = 0 \\ a + 64 = 0 \end{cases}$$

16. But because in the Duplicate equality last express'd, the Square numbers 1048576 and 64 are unequal, I divide the greater of them by the less, and by the Quotient 16384 I multiply $a + 64$, and then the Product $16384a + 1048576$ is to be equated to a Square instead of $a + 64$; so at length this Duplicate equality remains to be resolved

$$\text{viz. } \begin{cases} aa - 6144a + 1048576 = 0 \\ 16384a + 1048576 = 0 \end{cases}$$

17. Now in order to resolve that Duplicate equality, first, supposing the former of the two quantities to be equated to be the greater, their difference is $aa - 22528a$, then according to the method used in divers preceding Questions of this Book, two such numbers are to be found out that the Product of their multiplication may make the said difference $aa - 22528a$, and that as well in the half of the sum as in the half of the difference of the said two numbers there may be found 1024, which is the side of the Square 1048576. But two such numbers are 11a and $\frac{1}{11}a - 2048$; the half-difference of these is $\frac{10}{11}a + 1024$, and the Square of $\frac{10}{11}a + 1024$ being equated to $16384a + 1048576$ will give $a = \frac{22528}{121}$, to wit, the Area of the Triangle sought.

Since then the value of a is found to be $\frac{22528}{121}$, and s was before assumed to be 64; according to these values of a and s , the twelfth step will discover the two values of e to be $\frac{2}{11}$ and $\frac{176}{11}$; by either of which if you resolve the positions in the fourth and fifth steps, you will find the sides about the right-angle to be $\frac{20}{11}$ and $\frac{440}{11}$; therefore the Hypotenusal is $\frac{442}{11}$, and the sum of all the three sides is the Cube 64, to which if you add the Area $\frac{22528}{121}$ it makes the Square $\frac{121}{121}$, whose side is $\frac{11}{11}$.

QUEST. 112. (Quest. 25. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the Square of the Hypotenusal may be compos'd of some Square and its side; and that the Square of the said Hypotenusal being divided by one of the sides about the right-angle may give a number compos'd of a Cube and its side.

RESOLUTION.

- For one of the sides about the right-angle put a
- And for the other, aa
- The sum of their Squares is equal to the Square of the Hypotenusal, $aaaa + aa$ to wit,
- Now 'tis evident that $aaaa + aa$ is compos'd of the Square $aaaa$ and its side aa ; but if the said $aaaa + aa$ be divided by a , (which was put for one of the sides about the right-angle,) it gives the Quotient $aaa + a$, which is compos'd of a Cube with its side, so that it remains only to equate $aaaa + aa$ to a Square, that is, to find out a right-angled Triangle that one of the sides about the right-angle may be equal to the Square of the other of the same sides: But the preceding 95th Question shews how to find out such a Triangle; take if you please that in the first Example of the said Question, to wit, $\frac{4}{9}$, $\frac{16}{9}$, $\frac{20}{9}$, which will solve this 112th Question; for the Square of the Hypotenusal $\frac{400}{81}$, viz. $\frac{400}{81}$, is compos'd of the Square $\frac{16}{81}$ and its side $\frac{4}{9}$. Moreover, if the

if the said $\frac{400}{81}$ be divided by $\frac{4}{9}$, (one of the sides about the right-angle,) it gives the Quotient $\frac{100}{9}$, which is compos'd of the Cube $\frac{8}{9}$ and its side $\frac{2}{3}$.

A prospect of all Diophantus's kinds of Duplicate Equality, shewing also at first sight in which of the preceding Questions they are resolved.

- I. The first kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of an unknown Root or number a , with some absolute or known number, and the numbers prefix to the Root a are equal to one another, as in the five following Examples.

$$1. \begin{cases} \text{If } a + 192 = 0 \\ \text{And } a + 128 = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 8. of this Book 3.}$$

$$2. \begin{cases} \text{If } 104 + 54 = 0 \\ \text{And } 104 + 6 = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 8.}$$

$$3. \begin{cases} \text{If } 192 - a = 0 \\ \text{And } 64 - a = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 14. See the like in Quest. 50.}$$

$$4. \begin{cases} \text{If } a - 27 = 0 \\ \text{And } a - 15 = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 15.}$$

$$5. \begin{cases} \text{If } a + 12 = 0 \\ \text{And } a - 8 = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 16.}$$

- II. The second kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of an unknown Root a , and of one and the same known Square number. Also, when the numbers prefix to a in both Quantities are unequal, and the absolute numbers are unequal Squares, as in the six following Examples,

$$1. \begin{cases} \text{If } 4 + a = 0 \\ \text{And } 4 - a = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 17.}$$

$$2. \begin{cases} \text{If } 9 + a = 0 \\ \text{And } 4 - a = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 18.}$$

$$3. \begin{cases} \text{If } 8a + 4 = 0 \\ \text{And } 6a + 4 = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Quest. 33.}$$

$$4. \begin{cases} \text{If } 4 - 2a = 0 \\ \text{And } 4 - 3a = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Observat. 1. Quest. 33.}$$

$$5. \begin{cases} \text{If } 4 + 2a = 0 \\ \text{And } 4 - 3a = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Observat. 1. Quest. 33.}$$

$$6. \begin{cases} \text{If } 10a + 9 = 0 \\ \text{And } 5a + 4 = 0 \\ \text{What is the number } a? \end{cases} \text{ Resolved in Observat. 2. Quest. 33.}$$

- III. The third kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of some number of a , and an absolute number not a Square, but the numbers prefix to a have such proportion to one another as a square number to a square number, as in the three following Examples.

x. If

$$1. \left\{ \begin{array}{l} \text{If } \dots a + 12 = \square \\ \text{And } \dots 4a + 12 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 29.}$$

$$2. \left\{ \begin{array}{l} \text{If } \dots \frac{13}{10}a + \frac{2}{10} = \square \\ \text{And } \dots \frac{13}{10}a + \frac{2}{10} = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 35.}$$

$$3. \left\{ \begin{array}{l} \text{If } \dots \frac{41}{5} - \frac{1}{5}a = \square \\ \text{And } \dots \frac{41}{5} - \frac{1}{5}a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 50.}$$

IV. The fourth kind of Duplicate Equality is, when two Quantities to be equated to Squares are diversly compos'd of aa and a , or of aa , a and absolute numbers; in which cases, to the end the Duplicate Equality propos'd may be explicable by rational numbers, these two things are requisite to be found therein; viz. First, either the numbers prefix to aa , or the absolute numbers must be rational Squares: Secondly, the difference of the Quantities propos'd must be either some sole number of a , or compos'd of some number of a and an absolute number, or of some numbers of aa and a ; as in the following Examples.

$$1. \left\{ \begin{array}{l} \text{If } \dots aa - a = \square \\ \text{And } \dots aa - a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 19.}$$

$$2. \left\{ \begin{array}{l} \text{If } \dots 9aa - 2a = \square \\ \text{And } \dots 9aa - 2a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 20.}$$

$$3. \left\{ \begin{array}{l} \text{If } \dots 4aa + 5a = \square \\ \text{And } \dots 9aa + 5a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 21. See the like in } \text{Quest. 22.}$$

$$4. \left\{ \begin{array}{l} \text{If } \dots 4aa + 3a + 1 = \square \\ \text{And } \dots 4aa - a + 1 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 20.}$$

$$5. \left\{ \begin{array}{l} \text{If } \dots 4aa - 15a = \square \\ \text{And } \dots 4aa - a - 4 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 31.}$$

$$6. \left\{ \begin{array}{l} \text{If } \dots 4aa - a + 1 = \square \\ \text{And } \dots 3a + 1 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 32.}$$

$$7. \left\{ \begin{array}{l} \text{If } \dots aa + d = \square \\ \text{And } \dots ba + d = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 55.}$$

$$8. \left\{ \begin{array}{l} \text{If } \dots aa + 2pa + hh = \square \\ \text{And } \dots \frac{a}{p} + 1 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 59.}$$

$$9. \left\{ \begin{array}{l} \text{If } \dots 3aa - 6a + 4 = \square \\ \text{And } \dots 4aa + 4 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 94. See the like in } \text{Quest. 93.}$$

$$10. \left\{ \begin{array}{l} \text{If } \dots ee + 1 = \square \\ \text{And } \dots e + 1 = \square \\ \text{What is the number } e? \end{array} \right\} \text{ Resolved in } \text{Quest. 99.}$$

$$11. \left\{ \begin{array}{l} \text{If } aa - 6144a + 1048576 = \square \\ \text{And } \dots a + 64 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in } \text{Quest. 111.}$$



A brief Exposition upon Monsieur Fermat's Analytical Invention, inserted in the late Edition of Monsieur Bachet's Comment upon Diophantus, printed at Toloze, Anno 1670.

JAC. de BILLY, who collected the said Invention out of Fermat's Letters, divides it into three Parts. The first is an Improvement of one of Diophantus's kinds of Duplicate Equality, (to wit, the fourth and last kind in my preceding Prospect, but the sixth with Bachet in his Comment upon Quest. 24. Book VI. Diophant.) whereas Diophantus's method of resolving that kind of Duplicate Equality, finds out one value or two at the most of the Root sought, and sometimes the value found out is Negative, viz. less than nothing, Fermat's method finds out innumerable affirmative values.

The second part shews how to resolve two kinds of Triplicate Equality, by reducing them to a Duplicate Equality of the same kind with that above mentioned.

The third part shews how to equate a Quantity compos'd of five Terms to a Square; likewise a Quantity of four Terms to a Square or Cube, and for the most part to find out innumerable affirmative values of the Root sought.

These three parts I shall explain in order, and according to my usual method put a (instead of N by Diophantus) for a Root or number unknown, aa for the Square of that Root, aaa for the Cube, &c.

Concerning the First part of Fermat's Analytical Invention.

Fermat's Rule to find out innumerable affirmative values of the Root sought, in a Duplicate equality of the kind before mentioned, is this,

First, by the vulgar method of Diophantus, (explain'd in divers preceding Questions of this Book,) find out one value of the sought Root, (a) it matters not whether it be affirmative or negative; then to a joyn that value with its own sign, whether it be $+$ or $-$, and take the sum for a new Root instead of a ; then according to the said new Root let a new Duplicate equality be deduced from the first, and find out the value of a in the new Duplicate equality by the vulgar method. That done, connect this latter value to the first before found, with their own signs, and it will give a second value of the Root sought in the first Duplicate equality. In like manner by the help of this second value you may find out a third, and from the third a fourth, and so infinitely.

Note. After one value of the Root sought is found out in the vulgar way, there will always certainly arise a known Square in each of the two new Algebraic quantities to be equated to Squares, in the second, third, or any following Duplicate equality deduced from the first as the Rule doth direct, the reason whereof will be evident to him that diligently examines the Operation. But when those two known Squares are unequal, the latter must be reduced to the greater, (in such manner as before hath been shewn in Quest. 21. of this Book,) to the end there may be one and the same known square number in each of the two quantities to be equated, and then the difference of the said quantities will be composed of some number of aa and some number of a ; which kind of difference is absolutely necessary in the use of Fermat's Rule before express'd. All which will be made manifest by the following Questions 113, 114, 115, 116.

QUEST. 113.

To find a number, that if to the Product of its Square multiplied by 33 you add unity, and from the sum subtract eight times the number sought, the remainder may be a Square. Also, that eight times the number sought, together with unity, may make a Square. Let

Y 2

- Let a be put for the number sought, and then the Question may be stated thus, viz.
1. If $\dots 32aa - 8a + 1 = \square$ } what is the number a ?
 2. And $\dots 8a + 1 = \square$ }

RESOLUTION.

3. To resolve that Duplicate equality, first, (according to the vulgar method before explained in *Quest.* 32. of this Book,) I take the difference of the two quantities to be equated, and because in this Example, either of them may be taken for the greater, I suppose the first to be the greater, so their difference is $32aa - 16a$; then I seek two such quantities that the Product of their multiplication may make $32aa - 16a$, and that as well in half the sum as in half the difference of the same two quantities, there may be found the absolute number 2, (to wit, the double of the Square Root of 1, the known Square in each of the two quantities to be equated,) so I find $4a - 2$ and $8a$, whose Product is $32aa - 16a$, and their difference is $4a + 2$; the half of this difference is $2a + 1$, whose Square $4aa + 4a + 1$ being equated to $8a + 1$, (which was supposed to be the lesser of the two quantities in the Duplicate equality,) will give, after due Reduction, $a = 1$, according to which value, the first of the two quantities propos'd to be equated, to wit, $32aa - 8a + 1$, makes the Square 25, and the other quantity $8a + 1$ makes the Square 9; wherefore the Question is solved, and the said value of a , to wit, 1, is the only value either affirmative or negative that can be found out by the common method. But *Fermat's Rule*, by the help of the said first value finds out a second, and from the second a third, &c. As, for example, to find out another number, or value of a besides 1 to solve the Question, let $a + 1$ be taken for a new Root instead of a , ($+1$, because the first Root was found $+1$, but if it had been found -1 , then $a - 1$ ought to be put for the new Root;) then let a new Duplicate equality be deduced from that before resolved in this manner, viz.
4. Instead of $32aa$ take 32 times the Square of the new Root $a + 1$, that is, $32aa + 64a + 32$
5. To that Product add unity, and it makes this sum, $32aa + 64a + 33$
6. From that sum subtract eight times $a + 1$, (instead of $8a$ in the first Duplicate equality,) that is, $\dots + 8a + 8$
7. And the remainder must be equal to a Square, viz. $32aa + 56a + 25 = \square$
8. Again, instead of $+8a$ the later quantity of the first Duplicate equality, take eight times $a + 1$, that is, $\dots + 8a + 8$
9. And to that Product add unity, so this sum must be equal to a Square, viz. $\dots + 8a + 9 = \square$
10. the seventh and ninth steps give a new Duplicate equality, viz. $32aa + 56a + 25 = \square$
 $\dots + 8a + 9 = \square$
11. But because in this Duplicate equality the Squares 25 and 9 are unequal, and *Fermat's Rule* takes place only in a Duplicate equality which hath one and the same known square number in each quantity to be equated, let 25 be divided by 9, and by the Quotient $\frac{25}{9}$ multiply $8a + 9$, so this quantity comes forth (instead of $8a + 9$) to be equated to a Square, viz. $\dots + \frac{25}{9}a + \frac{25}{9} = \square$
12. Thus at length, from the Duplicate equality before express'd in the first and second steps, another is deduced, and qualified as *Fermat's Rule* requires, viz. $32aa + 56a + 25 = \square$
 $\dots + \frac{25}{9}a + \frac{25}{9} = \square$
13. Then by resolving this last Duplicate equality in like manner as the first, you will find $a = -\frac{1}{1697809}$
14. To which negative value of a if you add 1, because $a + 1$ was taken for the new Root, you will find $a + 1 = -\frac{1}{1697809}$ for a second Root or value of a in the Duplicate equality in the first and second steps, viz.

I say $\frac{1}{1697809}$, besides 1 first found, will solve the Question; for, if you multiply the Square of $\frac{1}{1697809}$ by 32, and add 1 to the Product, and from this sum subtract the Product of the said $\frac{1}{1697809}$ into 8, there will remain a Square whose side is $\frac{1}{1697809}$. Moreover, if the Product of the said $\frac{1}{1697809}$ into 8, be increased with 1, it makes a Square, whose side is $\frac{1}{1697809}$.

In

In like manner, by taking or substituting $a + \frac{1}{1697809}$ for a new Root instead of a , and proceeding as before, you may find out a third number, and from the third a fourth, &c. to solve the Question propos'd, but the operation in finding out Answers by this method is so excessively laborious, that for the most part he that hath found out two Answers, will hardly have patience to find out a third.

QUEST. 114.

To find a number, that if to the double of its Square you add unity, and from that sum subtract four times the number sought, the remainder may be a Square. Also, that if from unity you subtract the double of the number sought, the remainder may be a Square.

- Let a be put for the number sought, and then the Question may be stated thus, viz.
1. If $\dots 2aa - 4a + 1 = \square$ } what is the number a ?
 2. And $\dots -2a + 1 = \square$ }

RESOLUTION.

3. If that Duplicate equality be resolved by the vulgar method, the only value of a will be found -4 ; but this being less than nothing, I search out an affirmative value of a by *Fermat's Rule*, thus,
4. Instead of a I take for a new Root $a - 4$
5. Then instead of $2aa$ I take the double of the Square of the new Root $a - 4$, viz. $2aa - 16a + 32$
6. To which double Square I add unity, (as the Question requires,) and it makes this sum, $2aa - 16a + 33$
7. Then from that sum I subtract four times the new Root $a - 4$, (instead of $4a$ in the Duplicate equality in the first step,) viz. I subtract $\dots + 4a - 16$
8. So this remainder must be equal to a Square, viz. $2aa - 20a + 49 = \square$
9. Again, if from unity, the double of the new Root $a - 4$ be subtracted, the remainder must be equal to a Square, viz. $\dots - 2a + 9 = \square$
10. So in the eighth and ninth steps there is a new Duplicate equality deduced from that in the first and second steps; but because in the new Duplicate equality the known square numbers 49 and 9 are unequal, I divide 49 by 9, and by the Quotient $\frac{49}{9}$ I multiply $-2a + 9$ in the ninth step, and it gives to be equated to a Square $2aa - 20a + 49 = \square$
 $\dots - \frac{49}{9}a + 49 = \square$
11. The eighth and tenth steps give this new Duplicate equality to be resolved, in which there is the same known square number (to wit, 49, as *Fermat's Rule* requires, viz. $2aa - 20a + 49 = \square$
 $\dots - \frac{49}{9}a + 49 = \square$
12. Now to resolve the last preceding Duplicate equality, according to the vulgar method, I take the difference of the two quantities to be equated, which, by supposing the first quantity to be the greater, is $2aa - \frac{49}{9}a$
13. Then I search out two such quantities, that being mutually multiplied may make the said difference, and that as well in half the sum as in half the difference of the said two quantities there may be found 7, (the Square Root of 49,) so I find the two quantities to be $\frac{11}{9}a - 14$ and $\frac{2}{9}a$
14. Half the difference of those two quantities is $\frac{11}{18}a - 7$
15. Then the Square of the said half-difference being equated to $-\frac{2}{9}a + 49$ (in the eleventh step) will give $a = \frac{1611149}{39130049}$
16. From which value of a I subtract 4, because the new Root was assumed to be $a - 4$, and the remainder is

I say $\frac{1611149}{39130049}$ shall be a value of the Root a in the Duplicate equality in the first and second steps, and therefore will solve the Question propos'd; for if $a = \frac{1611149}{39130049}$, then $2aa - 4a + 1$ makes a Square whose side is $\frac{1611149}{39130049}$; also, $1 - 2a$ makes a Square whose side is $\frac{1611149}{39130049}$. Thus although a number less than nothing, -4 was found out for the first value of the Root a , yet by the help thereof an affirmative Root or number is found out to solve the Question propos'd; and from that second Root you may find out a third.

QUEST. 115.

QUEST. 117.

To find a number, that if it be multiplied by three given numbers severally, suppose by 1, 2, 5, and to every one of the Products one and the same given Square, suppose 1, be added, the three summs may be Squares.

RESOLUTION.

1. For the number sought put a
2. Then the Question requires that every one of these three summs may make a Square, viz. $a+1 = \square$
 $2a+1 = \square$
 $5a+1 = \square$
3. Now to resolve that Triplicate equality, first form a Square from any number of $a+1$ the side of the given Square 1, as from $a+1$, whose Square is $aa+2a+1$; then divide $aa+2a$ (the two first terms of that Square) by any one of the three numbers prefix to a in the said Triplicate equality, as by 1, which is tacitly prefix to a in the first quantity $a+1$, and take the Quotient $aa+2a$ for a new Root instead of a , (which was put for the number sought,) whereby the first part of the Question is solved indefinitely; for if $aa+2a$ be put for the number sought, then unity added to it makes a Square, to wit, $aa+2a+1$. Then multiply the said $aa+2a$ by 2 which is prefix to a in the second quantity $2a+1$, and it produceth $2aa+4a$, to which add the given Square 1, and it makes $2aa+4a+1$ to be equated to a Square. Again, multiply $aa+2a$ by 5 which is prefix to a in the third quantity $5a+1$, and it makes $5aa+10a$; to this add unity and it makes $5aa+10a+1$, to be equated to a Square; hence the following Duplicate equality aritheth,

$$\text{viz. } \begin{cases} 2aa+4a+1 = \square \\ 5aa+10a+1 = \square \end{cases}$$

4. This Duplicate equality being resolved by the vulgar way, will give $a=-6$, by which value of a if $aa+2a$ be expounded, it makes 24; (for the Square of -6 is $+36$, to which if you add -12 the double of -6 , it makes $+24$;) wherefore 24 is the number sought, and will solve the Question: For if unity be added first to 24, then to 48 the double of 24, and lastly, to 120 the quintuple of 24, the three summs are Squares, to wit, 25; 49; 121.

If you desire another number besides 24 to solve the Question proposed, you may assume $a=-6$ for a new Root of the Duplicate equality last resolved, and thence (by the method before explained in the first Part) find out a second number to solve that Duplicate equality, and consequently the Question.

Note. When in a Triplicate equality of the first kind before defined, the greatest of the three numbers of a is equal to the sum of the other two, then in such case that Triplicate equality, although it may be possible in it self, is inexplicable by the method of resolving the preceding Quest. 117. As, for example, if these three quantities be proposed to be equated to as many Squares, viz. $5a+1$; $16a+1$; $21a+1$; where the greatest number of a is equal to the sum of the other two, (for $21=16+5$;) and the value of the Root a is 3, according to which, those three quantities being expounded will give these three Squares, 16, 49, 64; it will be in vain to seek out any Answer to that Triplicate equality by Fermat's Rule, for it will produce an absurd or fruitless Equation, as you will find upon trial.

QUEST. 118.

To find a number, that if it be multiplied by three given numbers, suppose by 3, 4, 5, and the Products be severally subtracted from 1 a given Square, the three remainders may be Squares.

RESOLUTION.

1. For the number sought put a
2. Then the Question requires that these three Remainders may be Squares, viz. $1-3a = \square$
 $1-4a = \square$
 $1-5a = \square$
3. This Triplicate equality may be resolved like that in the foregoing Quest. 117. For first, I form a Square from 1—any number of a , as from $1-3a$, whose Square is $1-6a+9a^2$; then I divide $9aa-6a$ (the two first terms of that Square,) by 3

which

Quest. 119.

Fermat's Analytical Invention.

which is prefix to a in the first quantity $1-3a$, and the Quotient is $3aa-2a$, this with its contrary signs is $-3aa+2a$, which I put for a new Root instead of a (the number sought,) whereby the first part of the Question is satisfied indefinitely; for if the triple of $-3aa+2a$ be subtracted from the given Square 1, the remainder is a Square, to wit, $9aa-6a+1$. Then I multiply the said new Root $-3aa+2a$ by 4 and 5 severally, (which are prefix to a in the second and third quantities of the Triplicate equality in the second step,) and subtracting the Products $-12aa+8a$ and $-15aa+10a$ severally from the given Square 1, the remainders $12aa-8a+1$ and $15aa-10a+1$ are to be equated to Squares, so it remains only to solve this following Duplicate equality,

$$\text{viz. } \begin{cases} 12aa-8a+1 = \square \\ 15aa-10a+1 = \square \end{cases}$$

4. This Duplicate equality being solved in the ordinary way, gives $a=\frac{1}{12}$, by which value of a , the new Root $-3aa+2a$ being expounded will give $\frac{1}{12}$ for the value of a in the Triplicate equality in the second step. Wherefore $\frac{1}{12}$ will solve the Question; for if its triple, quadruple and quintuple be severally subtracted from unity, the three remainders are Squares, to wit, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$.

QUEST. 119.

To find a number, as also four other numbers in Geometrical proportion continued, that if from these Proportionals severally the first number be subtracted, the three remainders may be Squares.

RESOLUTION.

1. For the first number sought put $a-1$
2. Then multiply a into any four known numbers continually proportional, as into 1, 2, 4, 8, and assume the Products to be the four Proportionals sought, viz. $a, 2a, 4a, 8a$
3. Then subtract the number in the first step from those four Proportionals severally, and every one of the remainders must make a Square; but the first remainder is manifestly the Square 1, it remains therefore to resolve this Triplicate equality, viz. $a+1 = \square$
 $3a+1 = \square$
 $7a+1 = \square$

Now to resolve that Triplicate equality you may take $aa+2a$ for a new Root instead of a , whereby the first part of the Triplicate equality will be solved indefinitely, for if $aa+2a$ be increased with 1 it makes a Square, to wit, $aa+2a+1$; then the two other parts of the said Triplicate equality (by the like Operation as was used in Quest. 117.) will be converted into this following Duplicate equality,

$$\text{viz. } \begin{cases} 3aa+6a+1 = \square \\ 7aa+14a+1 = \square \end{cases}$$

4. Which Duplicate equality being resolved in the vulgar way gives $a=-12$, whence $aa+2a$ (the new Root) will be found 120; (for the Square of -12 is $+144$, to which if you add -24 , that is, $2a$, it makes 120.) Therefore the first number sought by the Question is 119, (that is, $a-1$;) and the four numbers required to be in continual proportion are 120, 240, 480, 960; (which answer to $a, 2a, 4a, 8a$, in the second step;) for if 119 be subtracted from those four Proportionals severally, the remainders are the Squares 1, 121, 361, 841.

QUEST. 120.

Three square numbers Geometrically proportional being given, to find a number, that if it be added to those Proportionals severally, the three summs may be Squares.

RESOLUTION.

1. Let the three given Squares in continual proportion be $1, 4, 16$
2. For the number sought put a
3. Then the Question requires $a+1 = \square$
 $a+4 = \square$
 $a+16 = \square$

X

4. Now

QUEST. 123. (Probl. 9. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, that if it be multiplied by three given numbers in Arithmetical Progression, and the Products be subtracted from the Square of the number sought, the three remainders may be Squares. But the following Resolution presupposeth, (for the reason before given in the *Note* at the end of *Quest.* 117.) that the greatest of the three numbers given is not equal to the sum of the other two. Let therefore the given numbers be 3, 4, 5.

RESOLUTION.

1. For the number sought put a
2. Then (according to the import of the Question) this Triplicate equality must be resolved, viz. $\begin{cases} aa - 3a = \square \\ aa - 4a = \square \\ aa - 5a = \square \end{cases}$
3. But that (according to the Rule in *Quest.* 121.) will be reduced to this, viz. $\begin{cases} 1 - 3a = \square \\ 1 - 4a = \square \\ 1 - 5a = \square \end{cases}$

In which Triplicate equality last express the value of the Root a was by *Quest.* 118: found $\frac{1}{12}$, but because in the reduction of the Triplicate equality in the second step, $\frac{1}{12}$ was taken for a new Root instead of a , we must divide 1 by the said $\frac{1}{12}$, so there ariseth $\frac{12}{1}$ for the number sought by the Question; for if $\frac{12}{1}$ be multiplied by 3, 4, 5 severally, and the Products be severally subtracted from the Square of $\frac{12}{1}$, the three remainders will be the Squares of $\frac{22}{1}$, $\frac{24}{1}$ and $\frac{25}{1}$.

Note 1. Sometimes, when four, five or more quantities are to be equated to Squares, they may be resolved by the method before explain'd: As, for example,

$$\text{If this Quadruplicate equality be propos'd to be resolved, } \begin{cases} 20a + 64 = \square \\ 12a + 16 = \square \\ 8a + 4 = \square \\ 2a + 1 = \square \end{cases}$$

$$\text{You may (by the method in the preceding Quest. 120.) reduce that four-fold equality to this, viz. } \begin{cases} 20a + 64 = \square \\ 48a + 64 = \square \\ 128a + 64 = \square \\ 128a + 64 = \square \end{cases}$$

Which last Quadruplicate equality is in effect but a Triplicate equality, for there are two Terms which happen to be the same, to wit, $128a + 64$; and therefore you may resolve that Triplicate equality by the method before delivered.

Note 2. Sometimes also by the preceding method of resolving a Triplicate equality of the first kind, you may resolve one of *Diophantus's* kinds of Duplicate equality, (to wit, that explain'd in the preceding *Quest.* 33.) more easily than by his method, as will appear by the following *Quest.* 124, 125, 126.

QUEST. 124. (The same with the foregoing *Quest.* 33.)

To find a number less than 2, and such, that if it be multiplied severally by two numbers given, suppose by 6 and 8, and to each of the Products there be added the same given Square 4, the two summs may be Squares.

RESOLUTION.

1. For the number sought put a
2. Then the Question requires this Duplicate equality to be resolved, $\begin{cases} 6a + 4 = \square \\ 8a + 4 = \square \end{cases}$
3. To which end you may proceed thus; First, (according to the method of resolving a Triplicate equality before delivered,) form a Square from $a - 2$, (2 being the side of the given Square 4,) so that Square will be $aa + 4a + 4$; then take $\frac{1}{2}$ part of $aa + 4a$, ($\frac{1}{2}$ part, because 6 is prefixt to a in the first part of the given Duplicate equality,) and it is $\frac{1}{2}aa + 2a$, which is to be assumed for a new Root instead of a . Whence 'tis evident, that if the given Square 4 be added to six times $\frac{1}{2}aa + 2a$, it makes a Square, to wit, $aa + 4a + 4$, whereby the first part of the Question is solved indefinitely. Then multiply the new Root $\frac{1}{2}aa + 2a$ by 8, and to the Product add 4,

Quest. 125.

so there comes forth $\frac{1}{2}aa + 2a + 4$ to be equated to a Square, the side whereof must be so feigned that $\frac{1}{2}aa + 2a$ (the new Root) may be less than 2; but if $\frac{1}{2}aa + 2a$ be less than 2, it will follow that a is less than 2. Wherefore let $\frac{1}{2}aa + 2a + 4$ be equated to a Square, so, as that a may be less than 2; to which end the side of the said Square may be feigned $2 - \frac{1}{2}a$, any number of a greater than $3\frac{1}{2}$; let therefore the said side be feigned $4a - 2$, and then the Square of $4a - 2$ being equated to $\frac{1}{2}aa + 2a + 4$, will give $a = \frac{16}{5}$, by which, if the new Root $\frac{1}{2}aa + 2a$ be resolved, it makes $\frac{16}{5}$ for the number sought by the Question proposed: For first, it is less than 2; secondly, six times $\frac{16}{5}$ together with 4 makes a Square, to wit, $\frac{144}{5} + 20$, whose side is $\frac{14}{5}$; and lastly, eight times $\frac{16}{5}$ together with 4 makes the Square $\frac{144}{5} + 20$, whose side is $\frac{14}{5}$.

Again, if this Duplicate equality were proposed, $\begin{cases} 4 - 6a = \square \\ 4 - 8a = \square \end{cases}$

You may put $\frac{1}{2}a - \frac{1}{2}aa$ for a new Root instead of a , and then proceed as before.

Again, if this Duplicate equality were proposed, $\begin{cases} 4 - 6a = \square \\ 4 - 8a = \square \end{cases}$

You may take $\frac{1}{2}aa - \frac{1}{2}a$ for a new Root instead of a , and then proceed as before.

QUEST. 125. (Probl. 10. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, as also three other numbers in Geometrical proportion, that if from the Proportionals severally the first number be subtracted, the three remainders may be Squares.

RESOLUTION.

1. For the first number sought put a
2. And for the three continual Proportionals sought put $a, 2a, 4a$
3. From which if you subtract the first number sought, the remainders are $\begin{cases} 1 - a, 1, 1 + 2a \end{cases}$
4. Among which remainders the mean is a Square, wherefore $\begin{cases} 1 - a = \square \\ 1 + 2a = \square \end{cases}$ it remains to equate each of the two extremes to a Square, viz.

Now to resolve that Duplicate equality, you may take $2aa + 2a$ for a new Root instead of a , whence $1 - 2a$ will be converted into the Square $4aa - 4a + 1$, and $1 + 2a$ into $1 - 2a - 2aa$, which must be equated to a Square, but with this Caution, That the new Root $2aa + 2a$ may be less than 1, and that $4aa - 4a + 1$ may exceed 1, and consequently that the value of a may fall between $\frac{1}{2}$ and $\frac{1}{3}$; to which end, the side of the said Square may be feigned $1 - 2a$, whole Square $4aa - 4a + 1$ being equated to $1 - 2a - 2aa$, gives $a = \frac{1}{5}$; therefore the new Root $2aa + 2a$ is $\frac{2}{5}$, and the first number sought, which was represented by $2a - 1$, is consequently $\frac{3}{5}$, and the three desired Proportionals are $\frac{3}{5}, \frac{6}{5}, \frac{12}{5}$; which will solve the Question, as may easily be proved.

QUEST. 126.

To find two numbers, such, that if their sum be increased and lessened, as well by their difference as the difference of their Squares, the summs and remainders may be Squares.

RESOLUTION.

1. If unity be divided into any two parts their difference is equal to the difference of their Squares, (as hath been shewn in *Quest.* 53. of this Book,) therefore for the two numbers sought put $\begin{cases} \frac{1}{2} + a \\ \frac{1}{2} - a \end{cases}$
2. Whence their difference, as also the difference of their Squares is $2a$
3. It remains therefore that the sum of the two numbers in the first step, to wit, unity, being increased and lessened by $2a$, the sum and remainder may be Squares; hence this Duplicate equality ariseth, viz. $\begin{cases} 1 + 2a = \square \\ 1 - 2a = \square \end{cases}$
4. Now to resolve that Duplicate equality, you may take $\frac{1}{2}aa - a$ for a new Root instead of a , whence $1 - 2a$ will be converted into the Square $aa + 2a + 1$, and $1 + 2a$ into $1 - 2a - aa$, which must be equated to a Square, but with this Caution, That the new Root $\frac{1}{2}aa - a$ may be less than $\frac{1}{2}$, and consequently a less than $\frac{1}{2}$, that

that is, less than $\frac{1}{2}aa$; but to cause that effect, the side of the said Square may be feigned to be $1 - \text{any number of } a \text{ greater than } a$, as $1 - 3a$, the Square whereof being equated to the said $1 - 2a - aa$ will give $a = \frac{2}{3}$, therefore the new Root $\frac{1}{2}aa + a$ is $\frac{1}{2}$, according to which, the Positions in the first step being expounded, $\frac{1}{2}aa + a$ is $\frac{1}{2}$ and $\frac{1}{2}a$ for the numbers sought: For if to their sum, which is 1 , you will give $\frac{2}{3}$ and $\frac{1}{3}$ for the numbers sought: For if to their sum, which is 1 , you add and subtract their difference $\frac{2}{3}$, the sum and remainder are Squares, to wit, $\frac{25}{9}$ and $\frac{1}{9}$; and these will also come forth when the difference of the Squares of the two numbers $\frac{4}{9}$ and $\frac{1}{9}$ is added to and subtracted from their sum 1 , because the difference of the two numbers is equal to the difference of their Squares.

Concerning the third Part of Fermat's Analytical Invention.

I. The Scope of this third Part is chiefly to shew how to equate a quantity compos'd of five Terms, viz. of some numbers of $aaaa$, aaa , aa , a with an absolute (or known) number, to a Square; as also to equate a quantity compos'd of four Terms to a Square or Cube; and that in such manner, that for the most part innumerable values of the unknown Root a may be found out.

II. In equating a quantity compos'd of five Terms to a Square, one of these two things is absolutely necessary, viz. either the first Term must be a Biquadrate, or else the last Term, to wit, the absolute number, must be a rational Square. Likewise, in equating a quantity consisting of four Terms to a Square, one of the extremes must be a Square. And lastly, in equating a quantity of four Terms to a Cube, one of the extremes must be a Cube.

III. When a quantity compos'd of five Terms is given to be equated to a Square, and the first Term is a Biquadrate, but the absolute number, that is, the last Term, hath not a rational Square Root, then the side of the Square must be feigned so, as that in the Square it self there may be found the same numbers of $aaaa$, aaa and aa as are in the quantity given to be equated, to the end that those three first Terms may by Reduction destroy one another, and consequently an Equation remain between some number of a and an absolute number, whence the value of the Root a , if it hath any possible value, may be expressible by some rational number either affirmative or negative, but how to feign the said side so as to cause that effect, the following Proposition and Canon will shew; where to evidence the certainty thereof, I shall assume b , c , d to stand for Coefficients or known numbers prefixt to aaa , aa and a , and n for the absolute number, (or last Term,) which in this case is supposed to have no rational Square Root.

P R O P.

1. Let this quantity be given to be equated to a Square, viz. $aaaa + baaa + caa + da + n = \square$

C A N O N.

2. When $\frac{1}{2}c$ exceeds $\frac{1}{2}bb$, then let the side of the Square sought be feigned $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{2}bb$
 3. But if $\frac{1}{2}bb$ exceeds $\frac{1}{2}c$, then let the side of the Square be feigned $\frac{1}{2}bb - \frac{1}{2}c + \frac{1}{2}ba - aa$

Examples in Numbers.

4. Let this quantity be given to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 2a + 7 = \square$
 5. Here, because $\frac{1}{2}c$ exceeds $\frac{1}{2}bb$, viz. half 6 exceeds $\frac{1}{2}$ of the Square of 4, the first part of the Canon gives this feigned side of a Square, viz. $aa + 2a + 1$
 6. Then by equating the Square of the said side $aa + 2a + 1$ to the given quantity $aaaa + 4aaa + 6aa + 2a + 7$, the value of a , after due Reduction, will be found 3, by which if the given quantity be resolv'd, it makes the Square 256, whose side is 16.
 7. Again, let this quantity be given to be equated to a Square, $aaaa + 4aaa + 2aa - 6a + 11 = \square$
 8. Here, because $\frac{1}{2}bb$ exceeds $\frac{1}{2}c$, viz. $\frac{1}{2}$ of the Square of 4 exceeds the half of 2, the latter part of the Canon gives this feigned side of a Square, viz. $1 - 2a - aa$

9. Then

9. Then by equating the Square of the said feigned side $1 - 2a - aa$ to the given quantity $aaaa + 4aaa + 2aa - 6a + 11$, the value of the Root a will be found 3, by which if the given quantity be resolv'd it makes the Square 1156, whose side is 34.

IV. When a quantity compos'd of five Terms is to be equated to a Square, and the first Term is not a perfect Biquadrate, but the last Term, that is, the absolute number is a Square, then the side of a Square must be feigned so, as that in the Square it self there may be found the same numbers of aa , a and absolute number, as are in the quantity given to be equated; to the end that those three last Terms by due Reduction may vanish out of each part, and an Equation remain between some numbers of $aaaa$ and aaa , whence the value of the Root a , if it hath any possible value, may be expressible by some rational number either affirmative or negative. But how to feign the said side so as to cause such an effect, the following Proposition and Canon will shew; where to evidence the certainty thereof, I put b , c , d to stand for the Coefficients or known numbers prefixt to aaa , aa , a ; also, rr (whose side is r) for the last Term, which in this Case is a rational square number, and f , which is prefixt to $aaaa$, stands for a number not a Square.

P R O P.

1. Let this quantity be given to be equated to a Square, viz. $faaaa + baaa + caa + da + rr = \square$

C A N O N.

2. When $4crr$ exceeds dd , then let the side of the Square sought be feigned $\frac{4crr - dd}{8rr}aa + \frac{d}{2r}a + r$
 But if dd exceeds $4crr$, then let the side of the Square be feigned $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rr}aa$

Examples in Numbers.

3. Let this quantity be given to be equated to a Square, viz. $10aaaa + 4aaa + 19aa + 6a + 9 = \square$
 4. Here, because $4crr$ exceeds dd , viz. four times $19 \times 9 \times 9$ exceeds the Square of 6, the first part of the Canon gives this feigned side of a Square, viz. $3aa + 3a + 3$
 5. Then by equating the Square of the said side $3aa + 3a + 3$ to the given quantity $10aaaa + 4aaa + 19aa + 6a + 9$, the value of the Root a will be found 2, according to which the said given quantity being expounded makes the Square 289, whose side is 17.
 6. Again, let this quantity be given to be equated to a Square, viz. $2aaaa + 3aaa + 3aa + 6a + 1 = \square$
 7. Here, because dd exceeds $4crr$, viz. the Square of 6 exceeds four times $3 \times 1 \times 1$, the latter part of the Canon gives this feigned side of a Square, viz. $1 + 3a - 3aa$
 8. Then by equating the Square of the said side $1 + 3a - 3aa$ to the given quantity $2aaaa + 3aaa + 3aa + 6a + 1$, the value of the Root a will be found 3; according to which, the said given quantity being expounded makes the Square 289.

V. When a Quantity compos'd of five Terms given to be equated to a Square is such, that the first Term is a Biquadrate, and the last Term (that is, the absolute number) hath not a rational Square Root, then the side of a Square to be equated to such Quantity may be varied six several ways, (including into this number the two last preceding Canons,) by every one of which the value of the Root a may oftentimes be found out, and express'd by a rational number either affirmative or negative. To evidence this, I shall (as before) put b , c , d to stand for the Coefficients or known numbers prefixt to aaa , aa and a ; and rr (whose side is r) for the rational square number which is the last Term. As, for example, Let this Quantity be propos'd to be equated to a Square, viz. $aaaa + baaa + caa + da + rr = \square$

1. Then to the end that $aaaa + da + rr$ may be found in a Square to be equated to the quantity $aa + \frac{d}{2r}a + r$ propos'd, the side of that Square may be feigned

2. Or,

2. Or, to cause the same effect, the side of the said Square may be feigned $r + \frac{d}{2r}a - aa$
3. Again, that $caa + da + rr$ may be found in a Square to be equated to the quantity proposed, the side of such Square, when $4crr$ exceeds dd , may (agreeable to the Canon in *Self. 4.*) be feigned $\frac{acrr - dd}{4crr}aa + \frac{d}{2r}a + r$
4. But to cause the same effect, if dd exceeds $4crr$, then let the side of the Square be feigned $r + \frac{d}{2r}a - \frac{dd - 4crr}{8crr}aa$
5. Again, that $aaaa + baaa + rr$ may be found in a Square to be equated to the quantity proposed, let the side of the Square be feigned $aa + \frac{1}{2}ba + r$
6. Or, to cause the same effect, the side of the said Square may be feigned $aa + \frac{1}{2}ba - r$
7. Again, that $aaaa + baaa + caa$ may be found in a Square to be equated to the quantity proposed, the side of that Square, when $4c$ exceeds $\frac{1}{2}bb$, may (agreeable to the Canon in *Self. 3.*) be feigned $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{2}bb$
8. But, to cause the same effect, if $\frac{1}{2}bb$ exceeds $4c$, let the side of the said Square be feigned $\frac{1}{2}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers of the preceding sides of Squares express'd by Letters.

Let this quantity be proposed to be equated to a Square, viz. $aaaa + 4aaa + 10aa + 20a + 1 = \square$

1. Then supposing b, c, d, r , to stand for 4, 10, 20, 1, the first of the preceding literal sides will give $aa + 10a + 1$

The Square of which side $aa + 10a + 1$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, there will arise, after due Reduction, $16aa = -92a$, whence by dividing each part by $16a$, there comes forth $-\frac{23}{4}$ for the value of the Root a , according to which, the proposed quantity being expounded will give the Square $\frac{14425}{16}$, whose side is $\frac{119}{4}$.

2. Again, the third literal side gives $1 + 10a - aa$

The Square of which side $1 + 10a - aa$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{11}{3}$, according to which, the same quantity being resolved, makes the Square of $\frac{211}{9}$.

3. Again, because in the quantity proposed, dd exceeds $4crr$, the fourth literal side gives $1 + 10a - 45aa$

The Square of which side $1 + 10a - 45aa$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{21}{13}$, according to which, the same quantity being expounded makes the Square of the side $\frac{23228}{169}$.

4. Again, the fifth literal side gives $aa + 2a + 1$

The Square of which side $aa + 2a + 1$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -4$, according to which, the same quantity being expounded makes the Square 81.

5. Again, the sixth literal side gives $aa + 2a - 1$

The Square of which side $aa + 2a - 1$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -3$, according to which the said quantity being resolved makes the Square 4.

6. Lastly, because $\frac{1}{2}c$ exceeds $\frac{1}{2}bb$, the seventh literal side gives $aa + 2a + 3$

The Square of which side $aa + 2a + 3$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = 1$, according to which the same quantity being resolved makes the Square 36.

V.I. Some:

V.I. Sometimes, when a quantity compos'd of five Terms is equated to a Square formed from one of the eight literal sides before express'd in *Self. V.* no value of the Root a either affirmative or negative can thence be discovered. As, for example,

If this quantity be propos'd to be equated to a Square, viz. $aaaa - 18aaa - 12caa - 351a + 1519 = \square$

The Canon in *Self. III.* (or the seventh literal side in *Self. VI.*) will give this feigned side of a Square, viz. $aa + 9a - 19\frac{1}{2}$

The Square of which side $aa + 9a - 19\frac{1}{2}$ is $aaaa - 18aaa - 12caa - 351a + 1\frac{1}{4}$

Which Square being equated to the quantity proposed will give this fruitless and absurd Equation, viz. $1138\frac{1}{2} = 0$

VII. When negative Terms are intermingled with affirmative, in a quantity compos'd of five Terms given to be equated to a Square, the side of the Square, (when such an Equation is possible,) shall be one of the eight literal sides before express'd in *Self. V.* finding that one, and sometimes two of its signs $+$ must be changed into $-$. As, for example,

If $aaaa - 8aaa + 28aa - 40a + 4$ be given to be equated to a Square, it may variously done, in regard the extremes $aaaa$ and 4 are Squares. First then, I imagine all the Terms of the proposed quantity to be affirmative, so it will be $aaaa + 8aaa + 28aa + 40a + 4$; now to feign the side of a Square, that $aaaa + 8aaa + 4$ may by due Reduction vanish out of each part, the fifth literal side in *Self. V.* being resolved into numbers will give $aa + 4a + 2$ for the feigned side; but here two of its signs $+$ must be changed into $-$, that in its Square there may be found $aaaa - 8aaa + 4$ to destroy $aaa - 8aaa + 4$ in the quantity given to be equated, to which end, the said side $aa + 4a + 2$ must be converted into $aa - 4a - 2$, and then the Square of this side being equated to the proposed quantity $aaaa - 8aaa + 28aa - 40a + 4$ will give $a = \frac{1}{2}$.

Again, to feign the side of a Square to be equated to the same given quantity $aaaa - 8aaa + 28aa - 40a + 4$, so, as to destroy the first, second and last Terms in each part, the first of the literal sides in *Self. V.* being resolved into numbers, gives $aa + 10a + 2$ for the feigned side, if all the Terms of the given quantity were affirmative; but that $aaaa - 8aaa + 4$ may vanish out of each part, the said side $aa + 10a + 2$ must be changed into $aa - 10a + 2$, and then the Square of this side being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = \frac{1}{2}$.

Again, to feign the side of a Square to be equated to the same given quantity $aaaa - 8aaa + 28aa - 40a + 4$, in such manner that $aaaa - 40a + 4$ may vanish out of each part, the first of the eight literal sides in *Self. V.* being resolved into numbers, gives $aa + 10a + 2$ for the feigned side, if all the Terms of the given quantity were affirmative; but that $aaaa - 40a + 4$ may be expounded out of each part, the said side $aa + 10a + 2$ must be changed into $2 - 10a - aa$, and then the Square of this side being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = -\frac{1}{2}$. Other sides might likewise be feigned, as is evident by the foregoing *Self. V.* and to him that is a little exercis'd in this Method it will not be difficult to change the Signs in any possible case.

VIII. When a quantity consisting of five Terms is to be equated to a Square, and one or more values of the Root a are found out, either affirmative or negative, by the Rules before given, you may from every one of those Primitive Roots or values, find out other values of the Root a , even as many as you please, which latter, *Fermat* calls Derivative Roots of the first, second, third, &c. degree. As, for example, To find a Derivative Root of the first degree out of the quantity $aaaa + 4aaa + 10aa + 20a + 1$, before proposed in *Self. V.* to be equated to a Square, take one of its Primitive Roots there found out, to wit, -3 , and connect it to a , so it makes $a - 3$, then instead of a take $a - 3$ for a new Root, according to which the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa - 8aaa - 28aa - 40a + 4$, as here you see;

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aaaa	aaaa	- 12aaa	+ 5aaa	- 108a	+ 81
4aaa		+ 4aaa	- 36aa	+ 108a	- 108
10aa			+ 1caa	- 60a	+ 90
20a				+ 20a	- 60
1					+ 1
Summ	aaaa	- 8aaa	+ 28aa	- 40a	+ 4

This summ must be equated to a Square, whose side (as before hath been shewn in *Self. 7.*) may be feigned $aa - 4a - 2$, the Square whereof being equated to the said summ will give $a = \frac{1}{2}$; but because the new Root was put $a - 3$, out of $\frac{1}{2}$ subtract 3, and there will remain $\frac{5}{2}$ for a second value of the Root a in the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, which second value may be called a Derivative Root of the first degree.

In like manner by the help of the said second value $\frac{1}{2}$ you may find out a third, by joining $\frac{1}{2}$ to a , and taking $a + \frac{1}{2}$ for a new Root, according to which, the given quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa + 6aaa + \frac{1}{2}aa + \frac{1}{2}a + \frac{1}{2}$ to be equated to a Square, the side whereof may be feigned $aa + 3a + \frac{1}{2}$, (agreeable to the fifth literal side in *Self. 5.*) the Square whereof being equated to the given quantity, there will thence arise $a = -11$, therefore the new Root $a + \frac{1}{2}$ gives $a = -\frac{11}{2}$ for a third value of the Root a in the given quantity, that is, a Derivative Root of the second degree.

Nor will the Operation be otherwise to find out a fourth value, or derivative Root of the third degree, by putting for a new Root $a - \frac{1}{2}$, for according to this, every member of the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$ being resolved, there will come forth $aaaa - 38aaa + 122\frac{1}{2}aa - 222\frac{1}{2}a + 122\frac{1}{2}$, this summ is to be equated to a Square, whose side may be feigned $aa - 19a - \frac{1}{2}$, and the Square of this side being equated to the said summ will give $a = \frac{1}{2}$, from which if you subtract $\frac{1}{2}$, (because the new Root was put $a - \frac{1}{2}$), there will remain $\frac{1}{2}$ for a fourth value of the Root a , that is, a derivative Root of the third degree, out of the quantity first proposed to be equated to a Square.

Lastly, as by the help of one of the primitive Roots of the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$ other Roots have been derived, so by the help of any one of the rest of the primitive Roots of the same quantity, found out in the Examples of *Self. 5.* you may proceed to find out other derivative Roots, but sometimes you will meet with fruitless Equations.

IX. A quantity compos'd of four Terms may be equated to a Square, when either the absolute number, that is, the last Term is a Square, or the first Term a Biquadrate.

First, let $20aaa + 5aa + 40a + 16$ be given to be equated to a Square. Feign the side so, that $40a + 16$ may vanish out of each part, to which purpose, let the side be $5a + 4$, (5 being the Quotient that ariseth by dividing 40 in the $4a$, by 8 the double of the side of the given Square 16;) then by equating the Square of $5a + 4$ to the given quantity $20aaa + 5aa + 40a + 16$ you will find $a = 1$, according to which, that quantity being resolved makes the Square 81. Now to find a second value of the Root a , you may put for a new Root $a + 1$, according to which, the given quantity $20aaa + 5aa + 40a + 16$ will be converted into $20aaa + 65aa + 10a + 81$ to be equated to a Square, the side whereof, (that $10a + 81$ may vanish out of each part,) may be feigned $\frac{1}{2}a + 9$, whence after due Reduction, there will arise $a = -\frac{1}{2}$; therefore $a + 1$ (which was put for the new Root) gives $1 - \frac{1}{2}$, that is, $\frac{1}{2}$ for a second value of the Root a , (or a derivative Root of the first degree,) and by putting $a + \frac{1}{2}$ for a new Root you may find out a third value, and so infinitely.

Secondly, an Example where the first Term is $aaaa$ may be this, *viz.* Let $aaaa + 4aaa - 3aa + 2a$ be given to be equated to a Square. That $aaaa + 4aaa$ may vanish out of each part, feign the side of a Square, to be $aa + 2a$, (2 in the $2a$ being equated to 4 prefix to aaa in the given quantity,) then the Square of $aa + 2a$ being equated to $aaaa + 4aaa - 3aa + 2a$ will give $a = \frac{1}{2}$; and to find out a second value of a you may put $a + \frac{1}{2}$ for a new Root, and proceed as in former Examples.

Thirdly,

Thirdly, although some intermediate Term be omitted in a quantity compos'd of four Terms, such quantity may be equated to a Square: As, to equate $5aaa + 16aaa + 24aa + 16$ to a Square, you may feign its side to be $3aa + 4$, (3 in the $3aa$ being the Quotient that ariseth by dividing 24 which is prefix to aa in the given quantity, by 8 the double of the side of the given Square 16,) and thence the value of a will be found 4; then you may put $a + 4$ for a new Root to find out a second value of a . In like manner, if $aaaa + 60aa + 80a + 500$ be proposed to be equated to a Square, you may feign its side to be $aa + 30$, (30 being the half of 60 which is prefix to aa in the proposed quantity,) whence you will find $a = 5$, and for a derivative Root you may put $a + 5$.

X. A Quantity compos'd of four Terms may be equated to a Cube, when either the absolute number, (that is, the last Term,) or the first Term is a perfect Cube.

First, let $2aaa + aa + 3a + 1$ be given to be equated to a Cube. That the two last Terms may vanish out of each part, feign the side of a Cube to be $a + 1$, (1 being the side of the given Cube 1, and a being the Quotient that ariseth by dividing $3a$ in the given quantity by 3 the triple Square of 1; the cubick Root of the given Cube 1,) then the Cube of $a + 1$, that is, $aaa + 3aa + 3a + 1$ being equated to the given quantity $2aaa + aa + 3a + 1$, will give $a = 2$; then to find out a second value of a you may put $a + 2$ for a new Root.

Secondly, but if the first Term be a rational Cube, as, if $8aaa + 24aa + 2a + 48$ be given to be equated to a Cube; that the first and second Terms may vanish out of each part, feign the side of a Cube to be $2a + 2$, (2a being the side of the Cube $8aaa$, and 2 being the Quotient that ariseth by dividing 24 which is prefix to aa , by 12 and 2 being the cubick Root of 8 in $8aaa$;) then the Cube of $2a + 2$ being equated to the given quantity $8aaa + 24aa + 2a + 48$, will give $a = \frac{1}{2}$; whence you may find out derivative Roots as before.

XI. If the first Term of a Quantity compos'd of four Terms given to be equated to a Cube be a rational Cube, and the last Term, to wit, the absolute number be also a Cube, then that given Quantity may be equated to a Cube in a threefold manner.

As, for example, if $aaa + 2aa + 4a + 1$ be proposed to be equated to a Cube; first, that the first and last Terms may vanish out of each part, feign the side of the Cube to be $a + 1$, (which is compos'd of the cubick Roots of aaa and 1; then the Cube of $a + 1$ being equated to the quantity proposed will give $a = 1$. Secondly, that of $a + 1$ being equated to the quantity proposed, you may feign the side to be the first and second Terms may vanish out of each part, you may feign the side to be $a + \frac{1}{2}$, (a being the side of the Cube aaa , and $\frac{1}{2}$ being the Quotient that ariseth by dividing 2 which is prefix to aa , by 3 the triple Square of the cubick Root of 1 which is prefix to aaa ;) whence $a = -\frac{1}{2}$. And lastly, that the third and fourth Terms may vanish, you may feign the side to be $\frac{1}{2}a + 1$, (1 being the side of the given Cube 1, and $\frac{1}{2}a$ being the Quotient that ariseth by dividing $4a$ by 3 the triple Square of the cubick Root of the given Cube 1,) whence $a = \frac{2}{3}$; and by the help of those three primitive Roots you may find out derivatives, in like manner as before.

XII. Sometimes when a Quantity compos'd of four Terms, whereof one or both the extremes are Cubes, is to be equated to a Cube, no value of the Root a either affirmative or negative can be found out by any of the Rules before delivered.

As, if $4aaa + 3aa + 3a + 1$ be given to be equated to a Cube, its side can only be feigned $a + 1$, the Cube whereof being equated to the given quantity will give $3aaa = 0$; and therefore the given quantity cannot be equated to a Cube. Also, if $aaa + 2aa + 3a + 1$ be to be equated to a Cube, there can but one primitive Root be found out, although there be a threefold way of feigning the side of the Cube according to *Self. XI.* which primitive Root will be discovered from the feigned side $a + \frac{1}{2}$, but neither of the other two ways will prove effectual.

I shall now add a few Questions to illustrate the foregoing Third Part of *Fermat's Invention*, and so conclude this Book.

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QUEST. 127:

QUEST. 127.

(The same with the foregoing Quest. 105. but resolved after another manner.)

To find a right-angled Triangle, that the Area being subtracted as well from the Hypothenusal as from one of the sides about the right-angle, each remainder may be a Square.

RESOLUTION.

- Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle in numbers. Divide those three sides severally by a , and put the Quotients for the three sides of the Triangle sought, viz. $\frac{h}{a}, \frac{b}{a}, \frac{p}{a}$.
- Then by subtracting the Area $\frac{bp}{aa}$, as well from $\frac{b}{a}$ (one of the sides about the right-angle, as from the Hypothenusal $\frac{h}{a}$, each remainder must be equal to a Square, and by multiplying each remainder by the Denominator aa , this Duplicate equality aritheth, viz. $ba - \frac{1}{2}bp = \square$
 $ha - \frac{1}{2}bp = \square$
- Let the first of those two quantities be equated to some Square, viz. suppose $ba - \frac{1}{2}bp = bb$
 $a = b + \frac{1}{2}p$
- Whence, after due Reduction,
- Therefore by multiplying b into $b + \frac{1}{2}p$, (instead of a), the latter of the two quantities in the second step, will be converted into this quantity, which must be equated to a Square, viz. $bb + \frac{1}{2}bp - \frac{1}{2}bp = \square$
- Now since h, b, p were put for the Hypothenusal, Base and Perpendicular of a right-angled Triangle, the quantity in the fifth step shews that a right-angled Triangle must be found out such, that if the Hypothenusal be multiplied into the sum of one of the sides about the right-angle and half the other side, and the Product be lessened by the Area, the remainder must be a Square: But such a right-angled Triangle, by Fermat's method, (before explained,) may be found out thus, viz.
- Form a right-angled Triangle from two numbers taken at pleasure, as from $a+1$ and 1, $aa + 2a + 2 =$ Hypoth.
 $aa + 2a =$ Base,
 $+ 2a + 2 =$ Perpend.
- The Product of the Hypothenusal into the sum of the Base and half the Perpendicular is $aaaa + 5aaa + 9aa + 8a + 2$
- The Area is $\dots + aaa + 3aa + 2a$
- Which subtracted from the said Product leaves this quantity to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 6a + 2 = \square$
- Feign the side of that Square, (according to the 7th literal side in the foregoing Sect. 5. Part 3.) $aa + 2a + 1$
- Then the Square of the said side $aa + 2a + 1$ being equated to the quantity in the tenth step will give $a = -\frac{1}{2}$, therefore $a+1$ and 1 the numbers forming the Triangle shall be $\frac{1}{2}$ and 1, or (in Integers in the same Reason) 1 and 2, by which, if a right-angled Triangle be formed, one of the sides about the right-angle will be less than nothing, to wit, -3 ; (for the Square of 2 is to be subtracted from the Square of 1, because 1 and 2 answer to $a+1$ and 1 the numbers that formed the Triangle in the seventh step;) to cause therefore all the sides to be affirmative, the work must be renewed in manner following, viz.
- Let a right-angled Triangle be formed from $a+1$ and 2, so the three sides will be these, $aa + 2a + 5 =$ Hypoth.
 $aa + 2a - 3 =$ Base,
 $+ 4a + 4 =$ Perpend.
- The Product of the Hypothenusal into the sum of the Base and half the Perpendicular is $aaaa + 6aaa + 12aa + 18a - 5$
- The Area is $\dots + 2aaa + 6aa - 2a - 6$
- Which being subtracted from the said Product, leaves this quantity to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 20a + 1 = \square$

17. The

- The side of that Square may be variously feigned, according to the preceding Sect. 5.) let it be the second literal side in that Sect. viz. $1+10a-aa$
- Then the Square of the said side $1+10a-aa$ being equated to the quantity in the sixteenth step will give $a = \frac{2}{3}$, therefore $a+1$ and 2 shall be $\frac{5}{3}$ and 2, that is, in Integers in the same Reason, 29 and 12, by which if you form a right-angled Triangle, the three sides will be 985, 697, 696, that is, b, b, p ; then according to the Positions in the first step, divide every one of those sides by 1045, that is, by $b + \frac{1}{2}p = 4$, (as appears by the fourth step,) so the Quotients $\frac{985}{1045}, \frac{697}{1045}, \frac{696}{1045}$ shall be the sides of a right-angled Triangle to solve the Question: For if the Area be subtracted from the Hypothenusal $\frac{985}{1045}$ and the Base $\frac{697}{1045}$ severally, the remainders will be the Squares of these sides $\frac{88}{1045}$ and $\frac{60}{1045}$.

And because the quantity in the sixteenth step is capable of being equated to innumerable Squares, (according to the Method before explained,) the Question is also capable of innumerable Answers; but in larger numbers than those, that may be found out by the foregoing Quest. 105. which is the same with this.

QUEST. 128. (Probl. 1. in cap. 1. part. 1. Dioph. redivivi.)

To find a right-angled Triangle, that if the double of its Area be subtracted from every one of the three sides, the remainders may be Squares.

RESOLUTION.

- Let b, p, h represent the Base, Perpendicular, and Hypothenusal of a right-angled Triangle. Divide those sides severally by a , $\frac{b}{a}, \frac{p}{a}, \frac{h}{a}$ and assume the Quotients to be the three sides of the Triangle sought, viz. $\frac{b}{a}, \frac{p}{a}, \frac{h}{a}$.
- Then by subtracting the double Area $\frac{bp}{aa}$ from every one of those three sides, the remainders must be Squares, and multiplying the remainders severally by the Denominator aa , this Triplicate equality aritheth to be resolved, viz. $ba - bp = \square$
 $pa - bp = \square$
 $ha - bp = \square$
- Now in order to resolve that Duplicate equality, let the first of its three quantities be equated to some Square, viz. suppose $ba - bp = bb$
 $a = b + p$
- Whence, after due Reduction to find out the value of a , you will discover
- Then by multiplying $b + p$, instead of a , into p , the second of the three quantities in the second step (to wit, $pa - bp$), will be converted into this quantity, which is manifestly a Square, viz. $bp + pp - bp = \square = pp$
- And by multiplying $b + p$, instead of a , into h , the third quantity in the second step will be converted into this quantity to be equated to a Square, viz. $hb + hp - bp = \square$
- Thus the Triplicate equality in the second step is reduced to a Duplicate equality in the fifth and sixth steps, and because the first of the two quantities in that Duplicate equality happens to be a Square, to wit, pp , it remains only to equate $hb + hp - bp$ (in the sixth step,) to a Square, which discovers the Scope of our search must be this, viz. to find a right-angled Triangle, such, that if the Hypothenusal be multiplied by the sum of the sides about the right-angle, and the Product be lessened by the double of the Area, the remainder must be a Square: But such a right-angled Triangle may be found out thus, viz.
- Form a right-angled Triangle from two numbers taken at pleasure, as from $a+1$ and 2, $aa + 2a + 5 =$ Hyp.
 $aa + 2a - 3 =$ Base,
 $+ 4a + 4 =$ Perp.
- The Product of the Hypothenusal into the sum of the Base and Perpendicular is $aaaa + 8aaa + 18aa + 32a + 5$
- The double Area is $\dots + 4aaa + 12aa - 4a$
- Which subtracted from the said Product, leaves this quantity to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 36a + 17 = \square$
- Feign

12. Feign the side of that Square according to the Canon in the preceding *Self. 3. Part 3.* and it will be $aa + 2a + 1$
13. Then the Square of the said side $aa + 2a + 1$ being equated to the quantity in the eleventh step, will give $a = -\frac{1}{2}$; therefore $a + 1$ and 2, the numbers forming the Triangle in the eighth step, shall be $\frac{1}{2}$ and 2, or, (in Integers in the same Reason,) 1 and 4, by which if a right-angled Triangle be formed, one of the sides about the right-angle will be less than nothing, to wit, $-\frac{1}{2}$; (for the Square of 4 is to be subtracted from the Square of 1, because 1 and 4 answer to $a - 1$ and 2, the numbers that formed the Triangle in the eighth step, where $a - 1$ was supposed to exceed 2.) To cause therefore all the sides to be affirmative, the work must be renewed thus, viz.
14. Form a right-angled Triangle from $a + 1$ and 4, so the three sides will be these, viz.
- $$\begin{array}{rcl} aa + 2a + 17 & = & \text{Hyp.} \\ aa + 2a + 15 & = & \text{Base,} \\ + 8a + 8 & = & \text{Perp.} \end{array}$$
15. The Product of the Hypothenusal into the sum of the Base and Perpendicular is $aaaa + 12aaa + 30aa + 156a - 119$
16. The double Area is $+ 8aaa - 24aa - 104a - 120$
17. Which subtracted from the said Product, leaves this quantity to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 260a + 1 = \square$
18. The side of that Square may be variously feigned, (according to the preceding *Self. V.*) let it be the second literal side in that *Self. viz.* $1 + 130a - aa$
19. The Square of the said side $1 + 130a - aa$ being equated to the quantity in the seventeenth step, will give $a = \frac{4223}{20590417}$, therefore $a + 1$ and 4 shall be $\frac{4223}{20590417}$ and 4, or, (in Integers in the same Reason,) 4223 and 4223; from which, a right-angled Triangle being formed, the three sides will be 18465217, 18325825, 2264592; that is, h, b, p : Then according to the Positions in the first step, divide those three sides severally by 20590417, that is, by $b + p = a$, (as is evident by the fourth step;) so the Quotients $\frac{18465217}{20590417}$, $\frac{18325825}{20590417}$, $\frac{2264592}{20590417}$ shall be the sides of a right-angled Triangle to solve the Question, as may easily be proved.

Note 1. Although the Question be truly solved, yet 'tis evident that it was by chance that the Triplicate equality in the second step came to be reduced to a single equality; for if the quantity to be equated to a Square in the fifth step, had not happened to have been a Square, there would have been an inexplicable Duplicate equality.

Note 2. It is easie to perceive by the second, third and fourth steps, that instead of bp the double Area, the Product of bp multiplied by any square number may be given in the Question: As, if it were required to find out a right-angled Triangle, that $4bp$, that is, eight times the Area being subtracted from every one of the three sides may leave Squares, you need only to multiply the Denominator 20590417 of the three sides before found by 4, without altering the Numerators; or, if $9bp$, that is, eighteen times the Area, were prescribed, then to multiply the Denominator by 9.

QUEST. 129. (Probl. 1. in cap. 2. part. 1. *Dioph. redivivi.*)

To find a right-angled Triangle, that the Product of the Hypothenusal into one of the sides about the right-angle, being subtracted from every one of the three sides, may leave Squares.

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, and for the three sides of the Triangle sought put $\frac{b}{a}, \frac{b}{a}, \frac{p}{a}$
2. Then from $\frac{bp}{aa}$, (the Product of the Hypothenusal into the Perpendicular,) subtract every one of the three sides, and the remainders must be Squares; therefore also those remainders multiplied into the Denominator aa must make Squares; hence this Triplicate equality ariseth, viz.
- $$\begin{array}{rcl} ha - bp & = & \square \\ ba - bp & = & \square \\ pa - bp & = & \square \end{array}$$

3. Now

3. Now in order to resolve that Triplicate equality, let the first of its three quantities be equated to some Square, viz. suppose $ha - bp = hb$
4. Whence, after due Reduction to find out the value of a , you will discover $a = b + p$
5. Then by multiplying $b + p$, instead of a , into b , the second of the three quantities in the second step will be reduced to this to be equated to a Square, viz. $bb + bp - bp = \square$
6. Likewise by multiplying $b + p$, instead of a , into p , the third quantity in the second step will be reduced to this quantity, which is manifestly a Square, viz. $bp + pp - bp = \square = pp$
7. Thus the Triplicate equality in the second step is reduced to a Duplicate equality in the fifth and sixth steps; and because the latter quantity in that Duplicate equality happens to be a Square, to wit, pp , it remains only to equate the former, that is, $bb + bp - bp$, to a Square; which shews that a right-angled Triangle must be found, such, that if the sum of the Hypothenusal and Perpendicular be multiplied by the Base, and from the Product you subtract the Product of the Hypothenusal into the Perpendicular, the remainder may be a Square: But such a right-angled Triangle may be found out thus, viz.
8. Form a right-angled Triangle from $a + 2$ and 1, being numbers taken at pleasure, so the three sides will be these, viz.
- $$\begin{array}{rcl} aa + 4a + 5 & = & \text{Hyp.} \\ aa + 4a + 3 & = & \text{Base,} \\ + 2a + 4 & = & \text{Perp.} \end{array}$$
9. The sum of the Hypothenusal and Perpendicular being multiplied by the Base, produceth $aaaa + 10aaa + 36aa + 54a + 27$
10. The Product of the Hypothenusal and Perpendicular is $2aaa + 12aa + 26a + 20$
11. Which latter Product being subtracted from the former, leaves to be equated to a Square, $aaaa + 8aaa + 24aa + 28a + 7 = \square$
12. Feign the side of that Square according to the Canon in the preceding *Self. III. Part 3.* and it will be $aa + 4a + 4$
13. Then the Square of the said side being equated to the quantity in the eleventh step will give $a = -\frac{1}{2}$, and therefore $a + 2$ and 1, which were the numbers forming the Triangle in the eighth step, shall be $-\frac{1}{2}$ and 1, but one of these being negative, the work must be renewed; and now a right-angled Triangle may be confidently formed from $a - 1$ and 4, which Triangle being used like the former in the ninth, tenth and eleventh steps, at length there will remain $aaaa - 4aaa + 6aa - 260a - 1$ to be equated to a Square, the side whereof may be variously feigned, let it be $1 - 130a + aa$, then the Square of this side being equated to the said $aaaa - 4aaa + 6aa - 260a + 1$, will give $a = 66$; therefore $a - 1$ and 4 the numbers forming the Triangle shall be 65 and 4, by which if you form a right-angled Triangle, the three sides will be found 4241, 4209, 520; that is, h, b, p . Then according to the Positions in the first step, divide those three numbers severally by 4761; that is, by $b + p = a$, as appears by the fourth step, and the Quotients $\frac{4241}{4761}$, $\frac{4209}{4761}$, $\frac{520}{4761}$ shall be the sides of a right-angled Triangle to solve the Question, as may easily be proved.

QUEST. 130. (Probl. 39. in cap. 1. part. 1. *Dioph. redivivi.*)

To find a right-angled Triangle, whose Area subtracted from one of the sides about the right-angle, may leave a given number, suppose 2, (or n .)

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, then multiply those sides severally by a , and put the Products for the sides of the Triangle sought, viz. $ha; ba; pa$
2. The Area of which Triangle being subtracted from one of its sides about the right-angle, suppose from ba , must leave a remainder equal to the given number 2, (or n ;) therefore $ba - \frac{1}{2}bpa = n = 2$
3. Which Equation divided by $\frac{1}{2}bp$, gives $\frac{b}{\frac{1}{2}bp}a - aa = \frac{n}{\frac{1}{2}bp}$
4. Now

Examples, shewing more at large the Signification of the foregoing Characters.

- $a + b$. . . Signifies the sum of the right lines or numbers represented by a and b .
 $a - b$. . . The excess by which the right line or number a exceeds the right line or number b ; or, it imports that the latter quantity b is subtracted, or to be subtracted from the former Quantity a .
 $a \times b$, or ab . . . The Rectangle or Product made by the multiplication of the right line or number a , by the right line or number b .
 aa . . . The Square of the right line or number signified by a .
 $\frac{aa}{c}$. . . The right line arising by the Application of the Square of the right line a , to the right line c ; or, the Quotient arising by the Division of the Square of the number a , by the number c .
 $\frac{ab}{c}$. . . The right line or number arising by the Application or Division of the Rectangle or Product of a into b , by c .
 $a : b :: c : d$, viz. As a is to b ; so c to d .
 a, b, c . . . viz. As a is to b ; so b to c .
 \sqrt{ab} . . . The Square Root of the Product of a into b ; or, the side of a Square equal to the Rectangle ab .
 $\sqrt{aa + bb}$. . . The Square Root universal of $aa + bb$; or, the side of a Square equal to the sum of the Squares aa and bb .
 $\sqrt{aa - bb}$. . . The Square Root universal of $aa - bb$; or, the side of a Square equal to the excess of the Square aa above the Square bb .
 $a = b$. . . The line or number a is equal to the line or number b .
 $a = 5c$. . . The line or number a is equal to five times the line or number c .
 $a = \frac{1}{2}d$. . . The line or number a is equal to half the line or number d .
 $a > f$. . . The line or number a is greater than the line or number f .
 $a < g$. . . The line or number a is less than the line or number g .
 $AB \parallel CD$. . . The line AB is parallel to the line CD .
 $\angle ABC$. . . The angle ABC . Observe here, that when an angle is express'd by three letters, the middle letter stands at the angular point.
 $\angle A$. . . The angle A .
 $\angle ABC$ is \perp . . . The angle ABC is a right-angle.
 $AB \perp BC$. . . The right-line AB is perpendicular to the right-line BC .
 $ABCD$ is \circ . . . $ABCD$ is a Circle. Observe here, that the first letter towards the left hand is usually set at the Center.
 $\square AD$. . . Signifies either the Square AD when the letters A and D stand at the opposite angles of the Square; or else, the Square of the right-line AD , when A and D stand at the ends of the side of the Square.
 $\square A$. . . The Square of the right-line A .
 $\square AB, BC$. . . The long Square, or Rectangle, made of the right-lines AB and BC .
 $\square AB, C$. . . The Rectangle of the right-lines AB and C .
 $\square A, B$. . . The Rectangle of the right-lines A and B .
 $\triangle ABC$. . . The Triangle ABC .

Explication of Abbreviations and Citations.

- Probl. . . Problem.
 Suppos. . . Suppositions.
 Req. . . It is (or, let it be) required.
 Prepar. . . Preparation.
 Constr. . . Construction.
 Req. demonstr. . . It is (or, let it be) required to Demonstrate.
 Conclus. . . Conclusion.
 Coroll. . . Corollary.
 Annot. . . Annotation.
 Explicat. . . Explication.
 Per Prop. 11. . . By the 11th Proposition of the fifth Book of Euclid's Elements.
 Elem. 5. . .

Pr

Per Defin. 29. } By the 29th Definition of the first Book of Euclid's Elements.
 Elem. 1. }

Per Ax. 1. } By the first Axiom of the second Chapter of this fourth Book.
 Chap. 2. }

Note. In the handling of every Proposition, whether it be a Theorem or Problem, proceed from the beginning to the end by Steps numbred in the Margin by 1, 2, 3, 4, 5, &c. that by referring to preceding Steps, the title of the following may be apparent: so when 'tis said, [Therefore out of, or, From 3^o. and 4^o.] it imports, that the thing asserted or infer'd is manifest by the third and fourth Steps of the Proposition in hand. If any other Abbreviations occur, their meaning will be obvious to every intelligent Reader.

CHAP. II.

The explication of Axioms, or common notions, upon which the force of Inferences or Conclusions, about the Equality, Majority and Minority of Quantities compared to one another, doth chiefly depend.

Axiom 1.

1. If each of two Quantities be equal to a third, those two are equal between themselves.

Explicat.

If $AB = EF$, A ——— B E ——— F
 And $CD = EF$,
 Then $AB = CD$; per Ax. 1. C ——— D

That is to say,

If AB be equal to EF , and CD be equal to EF , then AB is equal to CD , by the first Axiom of Chap. 2.

Axiom 2.

2. Quantities which are equal to equal Quantities, are also equal between themselves.

Explicat.

If $\begin{cases} C = D, \\ A = C, \\ B = D, \end{cases}$ A ——— C ———
 B ——— D ———
 Then $A = B$; per Ax. 2.

Axiom 3.

3. That which is greater or less than one of two equal Quantities, is also greater or less than the other.

Explicat.

If $B = C$, B ———
 And $A < B$, A ——— C ———
 Then $A < C$; per Ax. 3.

That is to say,

If B be equal to C , and A be greater than B , then A is greater than C , by Ax. 3.

Axiom 4.

4. If one of two equal Quantities be greater or less than a third, the other of those two shall be also greater or less than the same third.

Explicat.

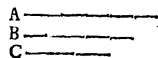
If $A = B$, A ——— C ———
 And $A < C$, B ———
 Then $B < C$; per Ax. 4. Z 2 $Axiom$ 5;

Axiom 5.

5. That which is greater than the greater of two Quantities, is also greater than the lesser; and that which is less than the lesser of two Quantities, is also less than the greater.

Explicat.

If $B < C$,
And $A < B$,
Then $A < C$; per Ax. 5.

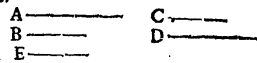


Axiom 6.

6. The exchanging of equal Quantities doth not alter equality.

Explicat.

If $A + B = C + D$,
And $E = B$,
Then $A + E = C + D$; per Ax. 6.

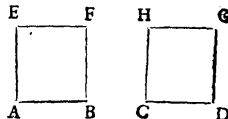


Axiom 7.

7. Interpretation doth not change equality.

Explicat.

Suppos. $\left\{ \begin{array}{l} \square AF = \square CG, \\ AF \text{ is } \square AB, \\ CG \text{ is } \square CD, \\ \square AB = \square CD; \text{ per Ax. 7.} \end{array} \right.$



That is to say,

The Square AF is equal to the Square CG, by supposition.

AF is the Square of the side AB, by supposition.

CG is the Square of the side CD, by supposition.

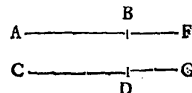
The Square of the side AB is equal to the Square of the side CD, by the seventh Axiom of Chap. 2.

Axiom 8.

8. If to equal Quantities you add equal Quantities, or one and the same Quantity, the wholes shall be equal.

Explicat.

If $AB = CD$,
And $BF = DG$,
Then $AB + BF = CD + DG$,
That is, $AF = CG$; per Ax. 8.

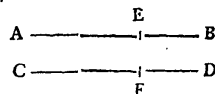


Axiom 9.

9. If from equal Quantities you take away equal Quantities, or one and the same Quantity, the Quantities remaining shall be equal to one another.

Explicat.

If $AB = CD$,
And $AE = CF$,
Then $AB - AE = CD - CF$,
That is, $EB = FD$; per Ax. 9.

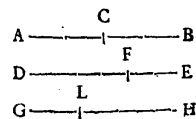


Axiom 10.

10. If from a whole the half be taken away, half will remain; and if more than half be taken away, less than half will remain; but if one third part be taken away, two thirds will remain, &c.

Explicat.

If $AC = \frac{1}{2} AB$,
Then $CB = \frac{1}{2} AB$, per Ax. 10.
If $DF = \frac{1}{2} DE$,
Then $FE = \frac{1}{2} DE$, per Ax. 10.
If $GL = \frac{1}{3} GH$,
Then $LH = \frac{2}{3} GH$, per Ax. 10.



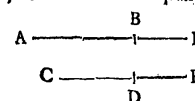
Axiom 11.

Axiom 11.

11. If to unequal quantities equal quantities be added, the wholes are unequal.

Explicat.

If $AB < CD$,
And $BE = DF$,
Then $AE < CF$; per Ax. 11.

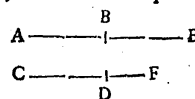


Axiom 12.

12. If to equal quantities you add unequal quantities, the wholes are unequal.

Explicat.

If $AB = CD$,
And $BE < DF$,
Then $AE < CF$; per Ax. 12.

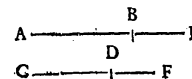


Axiom 13.

13. If to unequal quantities unequal quantities be added, the greater to the greater; and the less to the less, the wholes are unequal, to wit, the former the greater, and the latter the lesser.

Explicat.

If $AB < CD$,
And $BE < DF$,
Then $AE < CF$; per Ax. 13.

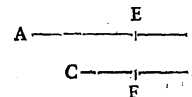


Axiom 14.

14. If from unequal quantities equal quantities or one and the same quantity be taken away, the remainders will be unequal.

Explicat.

If $AB < CD$,
And $EB = FD$,
Then $AE < CF$; per Ax. 14.

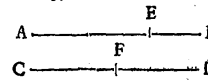


Axiom 15.

15. If from equal quantities unequal quantities be taken away, the remainders are unequal.

Explicat.

If $AB = CD$,
And $AE < CF$,
Then $EB > FD$; per Ax. 15.

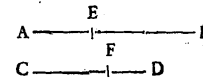


Axiom 16.

16. If from unequal quantities unequal quantities be taken away, from the greater the lesser, and from the lesser the greater, the remainders are unequal; to wit, the former the greater, and the latter the lesser.

Explicat.

If $AB < CD$,
And $CF < AE$,
Then $EB < FD$; per Ax. 16.

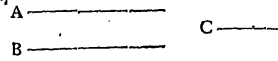


Axiom 17.

17. Quantities which are the doubles of one and the same quantity, or of equal quantities, are equal between themselves. Conceive the same of triples, quadruples, &c.

Explicat.

If $A = 2C$,
And $B = 2C$,
Then $A = B$; per Ax. 17.

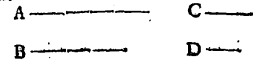


Axiom 18.

18. The double of the greater of two quantities is greater than the double of the lesser.

Explicat.

If $\left\{ \begin{array}{l} C < D, \\ A = 2C, \\ B = 2D, \end{array} \right.$
Then $A < B$; per Ax. 18.



Axiom 19.

Definit. II.

2. *Inverse Reason* is the taking of the Consequent as the Antecedent, to compare it to the Antecedent as if it were the Consequent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right. \\ \text{Then inversely, } \dots \dots \dots \left\{ \begin{array}{l} b : a :: d : c \\ 4 : 6 :: 8 : 12 \end{array} \right. \end{array}$$

Per Coroll.
prop. 4.
Elem. 5.

Definit. III.

3. *Composition of Reason* is the taking of the Antecedent and Consequent both as one, to compare it to the same Consequent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 12 :: 4 : 8 \end{array} \right. \\ \text{Then by Composition, } \dots \dots \dots \left\{ \begin{array}{l} a+b : b :: c+d : d \\ 18 : 12 :: 12 : 8 \end{array} \right. \end{array}$$

Per prop.
18. Elem. 5.

Annot. 1.

4. *Composition of Reason converse* is the taking of the Antecedent and Consequent both as one, to compare it to the same Antecedent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right. \\ \text{Then by Compos. converse, } \dots \dots \dots \left\{ \begin{array}{l} a+b : a :: c+d : c \\ 10 : 6 :: 20 : 12 \end{array} \right. \end{array}$$

Per Schol.
1. Clavii
in prop. 18.
Elem. 5.

Annot. 2.

5. *Composition of Reason contrary* is the comparing of the Antecedent to the Antecedent and Consequent taken both as one.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right. \\ \text{Then by Compos. contrary, } \dots \dots \dots \left\{ \begin{array}{l} a : a+b :: c : c+d \\ 6 : 10 :: 12 : 20 \end{array} \right. \end{array}$$

Per Schol.
2. Clavii
in prop. 18.
Elem. 5.

Annot. 3.

6. *Composition of Reason inversely contrary* is the comparing of the Consequent to the Antecedent and Consequent taken both as one.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right. \\ \text{Then by Composition inversely contrary, } \dots \dots \dots \left\{ \begin{array}{l} b : a+b :: d : c+d \\ 4 : 10 :: 8 : 20 \end{array} \right. \end{array}$$

Per Schol.
2. Heringon.
in prop. 18.
Elem. 5.

Definit. IV.

7. *Division of Reason* is the comparing of the excess whereby the Antecedent exceeds the Consequent, to the same Consequent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right. \\ \text{Then by Division, } \dots \dots \dots \left\{ \begin{array}{l} a-b : b :: c-d : d \\ 2 : 4 :: 4 : 8 \end{array} \right. \end{array}$$

Per prop.
17. Elem.
5.

But in this way of arguing by Division of Reason, 'tis manifest that the Antecedent must necessarily be greater than the Consequent.

Annot. 1.

Annot. 1.

8. *Division of Reason converse* is the comparing of the Consequent to the excess whereby the Antecedent exceeds the Consequent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 9 : 4 :: 18 : 8 \end{array} \right. \\ \text{Then by Division converse, } \dots \dots \dots \left\{ \begin{array}{l} b : a-b :: d : c-d \\ 4 : 5 :: 8 : 10 \end{array} \right. \end{array}$$

Per Schol.
1. Clavii
in prop. 17.
Elem. 5.

Annot. 2.

9. *Division of Reason contrary* is the comparing of the Antecedent, to the excess whereby the Consequent exceeds the Antecedent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 4 : 6 :: 8 : 12 \end{array} \right. \\ \text{Then by Division contrary, } \dots \dots \dots \left\{ \begin{array}{l} a : b-a :: c : d-c \\ 4 : 2 :: 8 : 4 \end{array} \right. \end{array}$$

Per Schol.
2. Clavii
in prop. 17.
Elem. 5.

But here 'tis manifest that the Consequent must be greater than the Antecedent.

Annot. 3.

10. *Division of Reason inversely contrary* is the comparing of the excess whereby the Consequent exceeds the Antecedent, to the same Antecedent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 4 : 6 :: 8 : 12 \end{array} \right. \\ \text{Then by Division inversely contrary, } \dots \dots \dots \left\{ \begin{array}{l} b-a : a :: d-c : c \\ 2 : 4 :: 4 : 8 \end{array} \right. \end{array}$$

Per Schol.
2. Heringon.
in prop. 17.
Elem. 5.

Definit. V.

11. *Conversion of Reason* is the comparing of the Antecedent to the excess by which the Antecedent exceeds the Consequent.

Explicat.

$$\begin{array}{l} \text{If } \dots \dots \dots \left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right. \\ \text{Then by converse Reason, } \dots \dots \dots \left\{ \begin{array}{l} a : a-b :: c : c-d \\ 6 : 2 :: 12 : 4 \end{array} \right. \end{array}$$

Per Coroll.
prop. 19.
Elem. 5.

Definit. VI.

12. *Reason of equality* is, when more than two quantities in one Rank, and as many in another are such, that if two to two be compared, they are in the same Reason; and it also happens, that as the first is to the last in the first rank of Quantities, so is the first to the last in the latter rank. Or otherwise, 'tis a comparison of the extremes to one another, the mean quantities being taken away.

But there are two ways of arguing by Reason of equality, to wit, one when the Proportion is Ordinate, the other when the Proportion is Inordinate or Disturbed; both which are explain'd in the two following Definitions.

Definit. VII.

13. *Ordinate proportion* is, when in the first rank of quantities, as the Antecedent is to the Consequent; so in the latter rank is the Antecedent to the Consequent: and when in the first rank as the Consequent is to some other, so in the latter rank is the Consequent to some other.

A 2

Explicat.

Explicat.

If to these quantities propounded, $\begin{cases} A,4 & B,6 & C,12 & D,8 \\ E,10 & F,15 & G,30 & H,20 \end{cases}$

These Analogies do happen, $\begin{cases} A,4 & B,6 & :: & E,10 & F,15 \\ C,12 & D,8 & :: & G,30 & H,20 \end{cases}$

Per prop.
22. Elem.
5.

Then by Reason of equality, $\begin{cases} A,4 & D,8 & :: & E,10 & H,20 \end{cases}$

That is to say, when the proportion in both ranks of quantities propounded is ordinate, (according to *Defin. 7.*) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C, D , so shall the first be to the last in the second rank of quantities E, F, G, H .

Definit. VIII.

14. *Inordinate proportion* is, when three quantities standing in one rank and three in another do afford these Analogies, viz. as the first quantity in the first rank is to the second in the same rank; so is the second quantity in the second rank to the third in the same rank: and as the second quantity in the first rank, is to the third in the same rank; so is the first quantity in the second rank to the second in the same rank.

Explicat.

If to these quantities propounded, $\begin{cases} A,4 & B,6 & C,3 \\ D,20 & E,10 & F,15 \end{cases}$

These Analogies do happen; $\begin{cases} A,4 & B,6 & :: & E,10 & F,15 \\ B,6 & C,3 & :: & D,20 & F,15 \end{cases}$

Per prop.
23. Elem.
5.

Then by reason of Equality, $\begin{cases} A,4 & C,3 & :: & D,20 & F,15 \end{cases}$

That is to say, when the proportion in both ranks of quantities propounded is inordinate, (according to *Defin. 8.*) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C , so shall the first be to the last in the latter rank of quantities, D, E, F .

CHAP. IV.

Various fundamental Theorems frequently used in Mathematical Resolution and Composition.

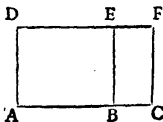
Theorem I.

A Rectangle (or right-angled Parallelogram) comprehended under any right-line and the difference of any two right-lines, is equal to the difference of two Rectangles comprehended under the first line and each of the two latter.

Suppos.

1. AD is a right-line,
2. AC and BC are right-lines,
3. ABC is a right-line,
4. $AB = AC - BC$.

5. . . . *Req. demonstr.* . . . $\square AD, AB = \square AD, AC - \square AD, BC$



Preparat.

6. Make $\square AF$ to be contain'd under AD and AC,
7. Make $BE \perp AC$.

Dem.

Demonstration.

8. By Constr. in 6° , and 7° . (and *per prop. 1. Elem. 2.*) $\square AE + \square BF = \square AF$
9. Therefore, by subtracting $\square BF$ from each part of that Equation, $\square AE = \square AF - \square BF$
10. That is, (per *Ax. 7. Chap. 2.*) $\square AD, AB = \square AD, AC - \square AD, BC$ Which was to be demonstrated.

Illustration Algebraical.

Let three right-lines be represented by $\begin{cases} a & \dots & 3 \\ b & \dots & 6 \\ c & \dots & 2 \\ b-c & \dots & 2 \end{cases}$

Suppose also
Then if the first line be multiplied by the difference of the second and third, that is, $a \times b - c$, the Product will be $ab - ac$ $18 - 6 = 12$

Which Product is manifestly the difference between the Product of the first line a into the second b , and the Product of the first line a into the third c , according to the tenour of *Theorem 1.*

Theorem II.

If a right-line be cut into any two parts, the Square described upon the whole line is equal to the Squares described upon the parts, and to twice the Rectangle comprehended under the parts.

Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. $AC + CB = AB$.

4. . . . *Req. demonstr.* . . . $\square AB = \square AC + \square CB + 2\square AC, CB$.

Prepar.

5. Upon AB describe the $\square AD$, (per *prop. 46. Elem. 1.*)
6. Draw the Diameter EB
7. Draw $CF \parallel AE$ (or BD), and cutting EB in G, (per *prop. 31. Elem. 1.*)
8. By the point G, draw $HGI \parallel AB$, (or ED .)

Demonstration.

9. By Constr. in 5° , $\square AD$ is $\square AB$.
10. Therefore, (per *29. Defin. 1. Elem.*) $\angle A, \angle AED, \angle D, \angle DBA$ are \perp .
11. And out of $7^\circ, 8^\circ$, and 10° , (per *29. prop. 1. Elem.*) $\angle EHG, \angle EFG, \angle HGF$ are \perp .
12. And out of 5° , (per *29. defin. 1. Elem.*) $\angle E = \angle B = \angle D = \angle E$.
13. Therefore out of 10° , and 12° , (per *prop. 5, & 32. Elem. 1.*) $\angle AEB$ is $\frac{1}{2} \perp$.
14. Likewise, out of 10° , and 12° , $\angle DEB$ is $\frac{1}{2} \perp$.
15. Likewise out of 11° , and 13° , $\angle HGE$ is $\frac{1}{2} \perp$.
16. Likewise out of 11° , and 14° , $\angle FGE$ is $\frac{1}{2} \perp$.
17. Therefore out of 15° , and 15° , (per *prop. 6. Elem. 1.*) $HE = HG$.
18. Likewise out of 14° , and 16° , $EF = FG$.
19. And from 7° , and 8° , (per *prop. 34. Elem. 1.*) $EF = HG$.
20. Wherefore out of $11^\circ, 17^\circ, 18^\circ, 19^\circ$, (per *29. defin. 1. Elem.*) HF is $\square HG$, or $\square AC$.
21. And in the same respect, CI is $\square CB$.
22. Therefore from 21° , $CG = CB$.
23. And from 22° , (per *36. prop. 1. Elem.*) $\square AG$, (or $\square AC, CG$), $= \square AC, CB$.

A a 2

24. But

24. But (per 43. prop. 1. Elem.) $\square AG = \square GD$.
 25. Therefore out of 25° , and 24° $\square GD = \square AC, CB$.
 (per Ax. 1. Chap. 2.)
 26. But (per Ax. 28. Chap. 2.) $\square AD = \square HF + \square CI + \square AG + \square GD$.
 27. Wherefore, out of $5^\circ, 20^\circ, 21^\circ$ $\square AB = \square AC + \square CB + 2 \square AC, CB$.
 $24^\circ, 25^\circ, 26^\circ$; (per Ax. 7. Chap. 2.)
 Which was to be dem.

Coroll. 1.

Hence it is manifest that the Parallelograms which are about the Diameter of a Square, are also Squares themselves.

Coroll. 2.

It also appears that the Diameter of any Square divides its angles into two equal parts.

Illustration Algebraical.

Suppose a right-line to be cut into two parts, to wit, a and b . . . 6 and 2
 Then the sum of the parts is . . . $a + b$. . . 8
 Which sum multiplied by it self produceth the Square $aa + 2ba + bb$ 64
 of the whole line, to wit, . . .

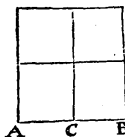
Which Product or Square doth manifestly consist of the Squares of the parts a and b , and twice the Product (or Rectangle) of the same parts; according to the tenour of the preceding Theor. 2.

Theorem III.

A Square described upon any right-line is equal to four times the Square of the half of the same line; and consequently, a quarter of the former Square is equal to the latter.

Suppos.

1. AB is a right-line;
 2. AC = CB, therefore
 3. AB = AC + CB = 2AC or 2CB.



4. . . . Reg. demonstr. . . . $\square AB = 4 \square AC$, (or $4 \square CB$;) Also,
 $\frac{1}{4} \square AB = \square AC$, or $\square CB$.

Demonstration.

5. By supposition, . . . AC = CB.
 6. Therefore, (per Sch. of prop. 46. Elem. 1.) $\square AC = \square CB$.
 7. And out of 5° , (per prop. 36. Elem. 1.) $\square AC = \square AC, CB$.
 8. And out of 7° , (per Ax. 17. Chap. 2.) $2 \square AC = 2 \square AC, CB$.
 9. And out of 6° , (per Ax. 8. Chap. 2.) $2 \square AC = \square AC + \square CB$.
 10. And out of 8° , and 9° , (per Ax. 8.) $4 \square AC = \square AC + \square CB + 2 \square AC, CB$.
 11. But per Theor. 2. of this Chap. $\square AB = \square AC + \square CB + 2 \square AC, CB$.
 12. Wherefore out of 10° , and 11° ,
 (per Ax. 1.) $\square AB = 4 \square AC$, (or $4 \square CB$.)

- Conclus. $\left\{ \begin{array}{l} 13. \text{ And out of } 12^\circ, \text{ (per Ax. 21.)} \\ \text{Chap. 2.)} \end{array} \right. \frac{1}{4} \square AB = \square AC, \text{ (or } \square CB.)$
 Which was to be dem.

Illustration Algebraical.

Let a right-line be represented by . . . $2a$ | 10
 The half thereof is . . . a | 5
 The Square of the whole line $2a$ is . . . $4aa$ | 100
 The Square of the half, to wit, of a is . . . aa | 25
 The first of those Squares is evidently equal to four times the latter, and consequently a quarter of the former is equal to the latter; as is affirmed by Theor. 3.

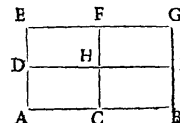
Theor. IV.

Theorem IV.

A Rectangle (or long Square) comprehended under any two unequal right-lines is equal to four times the Rectangle comprehended under the half of each of those lines; and consequently a quarter of the first Rectangle is equal to the latter.

Suppos.

1. AB and AE are right-lines,
 2. AC = CB = $\frac{1}{2}$ AB,
 3. AD = DE = $\frac{1}{2}$ AE.



4. . . . Reg. demonstr. . . . $\square AB, AE = 4 \square AC, AD$. Also,
 $\frac{1}{4} \square AB, AE = \square AC, AD$.

Prepar.

5. Make $\square AG$ to be contain'd under AB and AE.
 6. Draw CF || AE, (or BG.) Likewise, DI || EG, (or AB.)

Demonstration.

7. By Constr. in 5° , . . . AG is $\square AB, AE$.
 8. Therefore (per 30. defin. 1. Elem.) $\angle A, \angle E, \angle G, \angle B$ are \perp .
 9. And because by Constr. CF || AE || BG,
 in 6° , DI || EG || AB.
 10. Likewise by Constr. in 6° ,
 11. Therefore out of $8^\circ, 9^\circ$, and 10° ; (per prop. 29. Elem. 1.) AH, DF, HG, CI are \square .
 12. And because by Suppos. AC = CB. Also AD = DE.
 in 2° and 3° , . . .
 13. Therefore out of 11° and 12° , (per prop. 36. Elem. 1.) $\square AH = \square DF = \square HG = \square CI$.
 14. And from 13° , (per Ax. 8. Chap. 2.) $\square AH + \square DF + \square HG + \square CI = 4 \square AH$.
 15. But (per Ax. 28. Chap. 2.) $\square AH + \square DF + \square HG + \square CI = \square AG$.
 16. Therefore out of 14° and 15° , (per Ax. 1.) $\square AG = 4 \square AH$.
 17. That is, (per Ax. 7. Chap. 2.) $\square AB, AE = 4 \square AC, AD$.
 18. And consequently from 17° , (per Ax. 21.) $\frac{1}{4} \square AB, AE = \square AC, AD$.
 Which was to be dem.

Conclus.

Illustration Algebraical.

Let a right-line be represented by . . . $2a$ | 6
 And another right-line by . . . $2b$ | 4
 The half of the former line is . . . a | 3
 And the half of the latter is . . . b | 2
 The Product or Rectangle of the two whole lines is . . . $4ab$ | 24
 The Product of the half of each line is . . . ab | 6

The first of those Products is evidently equal to four times the latter, and consequently a quarter of the former is equal to the latter, according to the tenour of Theor. 4.

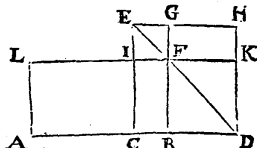
Theor. V.

Theorem V.

If a right-line be cut into any two unequal parts, the Square of the difference of the parts is equal to the Squares of the parts, less by twice the Rectangle (or long Square) comprehended under the parts: Also, the Square of half the difference of the said parts is equal to a quarter of each of the Squares of the parts, less by half the Rectangle of the parts.

Suppos.

1. AB is a right-line.
2. AC and CB are parts of AB.
3. $AC < CB$.



Prepar.

4. Produce AB to D, so, that $AC = CD$, then 'tis manifest that BD is the difference of the parts AC and CB, for CD (or AC) $- CB = BD$.
5. Upon CD describe the $\square CH$; (per prop. 46. Elem. 1.)
6. Draw the Diameter ED.
7. By the point B draw $BG \parallel CE$ (or DH), and cutting ED in F.
8. By the point F draw $LFK \parallel AD$, or EH .
9. By the point A draw $AL \parallel CE$.

10. . . Req. demonstr. . . $\square BD = \square AC + \square CB - 2\square AC, CB$. Also, $\square \frac{1}{2}BD = \frac{1}{4}\square AC + \frac{1}{4}\square CB - \frac{1}{2}\square AC, CB$.

Demonstration.

11. By Constr. in 4° and 5° . CH is $\square CD$ or $\square AC$.

12. Therefore (per Coroll. 1.) BK is $\square BD$.

13. Likewise IG is $\square EG$ or $\square CB$.

14. And (per prop. 43. Elem. 1.) $\square CF = \square FH$.

15. Therefore (per Ax. 8. Chap. 2.) by adding $\square IG$ to each part of the Equation in 14° . $\square CG = \square IH$.

16. Again, (by Constr. in 4° and 5° . $CE = AC$.

17. And 'tis evident that $CB = CB$.

18. Therefore from 16° and 17° (per prop. 35. Elem. 1.) $\square CE, CB$ (or $\square CG$) $= \square AC, CB$.

19. Therefore out of 15° and 18° (per Ax. 6. & 8. Ch. 2.) $\square CG + \square IH = 2\square AC, CB$.

20. And because (per Ax. 28. & 8. Chap. 2.) $\square BK + \square CG + \square IH = \square CH + \square IG$.

21. Therefore out of 19° and 20° (per Ax. 6. Ch. 2.) $\square BK + 2\square AC, CB = \square CH + \square IG$.

22. Therefore from 21° (per Ax. 9. Chap. 2.) $\square BK = \square CH + \square IG - 2\square AC, CB$.

23. Therefore out of 22° , 12° , 11° , 13° ; (per Ax. 7. Chap. 2.) $\square BD = \square AC + \square CB - 2\square AC, CB$.

Conclus. 1. Which was to be dem.

24. Moreover, out of 23° (per Ax. 21. Chap. 2.) $\frac{1}{4}\square BD = \frac{1}{4}\square AC + \frac{1}{4}\square CB - \frac{1}{2}\square AC, CB$.

25. And per The. 3. of this Ch. $\frac{1}{4}\square BD = \square \frac{1}{2}BD$.

26. Therefore out of 24° and 25° (per Ax. 1.) $\square \frac{1}{2}BD = \frac{1}{4}\square AC + \frac{1}{4}\square CB - \frac{1}{2}\square AC, CB$.

Conclus. 2. Which was also to be dem.

Illustra-

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, to wit, } a and b | 16 and 10
 Suppose also } $a = b$ |
 Then the difference of the parts is } $a - b$ | 6
 And half the difference of the parts is } $\frac{1}{2}a - \frac{1}{2}b$ | 3
 The Square of the whole difference is } $aa + bb - 2ab$ | 36
 The Square of the half difference is } $\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ab$ | 9

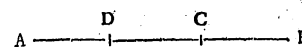
Which two Squares do manifestly prove the certainty of what is affirmed in the foregoing Theor. 5.

Theorem VI.

If a right-line be cut into any two unequal parts, the Square of the whole line together with the Square of the difference of the parts is equal to twice the Squares of the parts, and consequently half the Square of the whole line together with half the Square of the difference of the parts is equal to the sum of the Squares of the parts.

Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. $AC < CB$.



Prepar.

4. From CA cut off CD = CB, thence it follows that AD (= AC - CB) is the difference of the parts AC and CB.

5. . . . Req. demonstr. . . $\square AB + \square AD = 2\square AC + 2\square CB$. Also, $\frac{1}{2}\square AB + \frac{1}{2}\square AD = \square AC + \square CB$.

Demonstration.

6. By Theor. 2. of this Chap. $\square AB = \square AC + \square CB + 2\square AC, CB$.

7. And by Theor. 5. of this Chap. $\square AD = \square AC + \square CB - 2\square AC, CB$.

8. Therefore out of 6° and 7° , (per Ax. 8. Chap. 2.) $\square AB + \square AD = 2\square AC + 2\square CB$.

9. And consequently, (per Ax. 21. Chap. 2.) $\frac{1}{2}\square AB + \frac{1}{2}\square AD = \square AC + \square CB$.

Which was to be dem.

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, to wit, } a and b | 6 & 4
 Suppose also } $a = b$ |
 Then the sum of the parts is } $a + b$ | 10
 And the difference of the parts is } $a - b$ | 2
 The Square of the whole line, that is, the Square of the sum of the parts, is } $aa + bb + 2ab$ | 100
 The Square of the difference of the parts is } $aa + bb - 2ab$ | 4
 The sum of those Squares is } $2aa + 2bb$ | 104

Which sum, (according to the tenour of the preceding Theor. 6.) is manifestly equal to twice the sum of the Squares of the parts.

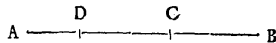
Theorem VII.

If a right-line be cut into any two unequal parts, the Square of the whole line is equal to four times the Rectangle (or long Square) comprehended under the parts, together with the Square of the difference of the parts: Also, the Square of half the said right-line, (or of half the sum of the parts,) is equal to the Rectangle of the parts together with a quarter of the Square of the difference of the parts.

Suppos.

Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. $AC \sqsubset CB$.



Prepar.

4. From CA cut off $CD = CB$, whence $AD (= AC - CB)$ is the difference of the parts A C and C B.

5. . . . Reg. demonstr. . . . $\left\{ \begin{array}{l} \square AB = 4 \square AC, CB + \square AD. \text{ Also,} \\ \square \frac{1}{2} AB = \square AC, CB + \frac{1}{4} \square AD. \end{array} \right.$

Demonstration.

6. By Theor. 5. of this Chapt. . . . $\square AD = \square AC + \square CB - 2 \square AC, CB$.
7. Therefore by adding $4 \square AC, CB$ to each part of that Equation this ariseth, (per ax. 8. ch. 2.) . . . $4 \square AC, CB + \square AD = \square AC + \square CB + 2 \square AC, CB$.
8. But per Theor. 2. of this Chapt. . . . $\square AB = \square AC + \square CB + 2 \square AC, CB$.
9. Therefore from 7° and 8° , . . . $\square AB = 4 \square AC, CB + \square AD$.
(per Ax. 1. Chap. 2.) . . . Which was to be dem.
10. Moreover, from 9° , (per Ax. 21. Chap. 2.) . . . $\frac{1}{4} \square AB = \square AC, CB + \frac{1}{4} \square AD$.
11. And by Theor. 3. of this Chapt. . . . $\frac{1}{4} \square AB = \square \frac{1}{2} AB$.
12. Therefore from 10° and 11° , . . . $\square \frac{1}{2} AB = \square AC, CB + \frac{1}{4} \square AD$.
(per Ax. 1.) . . . Which was also to be dem.

Illustration Algebraical.

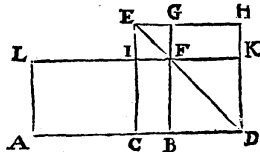
- | | |
|--|---------|
| Suppose a right-line to be cut into two parts, to wit, a and b . . . | 6 and 4 |
| Suppose also . . . $a \sqsubset b$ | |
| Then the sum of the parts is . . . $a + b$. . . | 10 |
| And the difference of the parts is . . . $a - b$. . . | 2 |
| The Square of the sum of the parts, that is, the Square of the whole line is . . . $aa + bb + 2ba$ | 100 |
| And the Square of the difference of the parts is . . . $aa + bb - 2ba$ | 4 |
| To which Square of the difference if you add the quadruple Product of the parts, to wit, . . . $4ba$ | 96 |
- The sum (according to the tenour of Theor. 7.) makes the Square of the whole line, to wit, $aa + bb + 2ba$.

Theorem VIII.

If a right-line be cut into any two unequal parts, the Rectangle (or long Square) comprehended under the whole line and the difference of the parts is equal to the difference of the Squares of the parts. Also, the Rectangle under half the said right-line, (that is, half the sum of the parts) and half the difference of the parts, is equal to a quarter of the difference of the Squares of the parts.

Suppos.

1. AB is a right-line.
2. AC and CB are parts of AB.
3. $AC \sqsubset CB$.



Prepar.

4. Produce AB to D, so, that $AC = CD$, thence it follows that BD is the difference of the parts AC and CB; for CD (or AC) $- CB = BD$.

5. Upon

5. Upon CD describe the $\square CH$.
6. Draw the Diameter ED.
7. By the point B draw $BG \parallel CE$ (or DH), and cutting ED in F.
8. By the point F draw $LFG \parallel AD$, or EH .
9. By the point A draw $AL \parallel CE$.

10. . . . Reg. demonstr. . . . $\left\{ \begin{array}{l} \square AB, BD = \square AC - \square CB. \text{ Also,} \\ \square \frac{1}{2} AB, \frac{1}{2} BD = \frac{1}{4} \square AC - \frac{1}{4} \square CB. \end{array} \right.$

Demonstration.

11. By Constr. in 5° , . . . CH is $\square CD$.
12. Therefore out of 6° and 11° , . . . $IG = \square IF$, and $BK = \square BD$.
(per Cor. 1. The. 2. of this Chapt.)
13. And from 12° , (per 29. defin. 1. Elem.) . . . $BF = BD$.
14. By Constr. in 7° and 8° , . . . CF is \square .
15. Therefore from 14° , (per 34. prop. 1. Elem.) . . . $BF = CI$, and $IF = CB$.
16. And from 13° and 15° , (per Ax. 1.) . . . $BD = CI$.
17. By Constr. in 4° and 5° , . . . $DH = CA$.
18. Therefore out of 16° and 17° , . . . $\square BD, DH = \square CI, CA$.
(per 36. prop. 1. Elem.) . . . $\square BH = \square AI$.
19. That is, (per Ax. 7. Chap. 2.) . . . $Gnomon, ICDHG = AF = \square AB, BD (BF)$.
20. Therefore by adding $\square CF$ to each part of the Equation in 19° , . . . $Gnomon, ICDHG = \square CH - \square IG$.
21. But 'tis manifest (per Ax. 9. Chap. 2.) that . . . $\square AB, BD = \square CH - \square IG$.
(per Ax. 1.)
22. And became from $4^\circ, 5^\circ, 11^\circ$, and 15° , . . . $\square CD$ (or $\square AC$) $- \square CB$ (or $\square IG$) $= \square CH - \square IG$.
23. Therefore from 22° and 23° , . . . $\square AB, BD = \square AC - \square CB$.
(per Ax. 1.) . . . Which was to be dem.
24. Moreover, from 24° , (per Ax. 21. Chap. 2.) . . . $\frac{1}{4} \square AB, BD = \frac{1}{4} \square AC - \frac{1}{4} \square CB$.
25. And by Theor. 4. of this Chapt. . . . $\frac{1}{4} \square AB, BD = \square \frac{1}{2} AB, \frac{1}{2} BD$.
26. Therefore out of 25° and 26° , . . . $\square \frac{1}{2} AB, \frac{1}{2} BD = \frac{1}{4} \square AC - \frac{1}{4} \square CB$.
(per Ax. 1.) . . . Which was also to be dem.

Conclus. 1.

Conclus. 2.

Illustration Algebraical.

- | | |
|--|---------|
| Suppose a right-line to be cut into two parts, to wit, a and b . . . | 6 and 4 |
| Suppose also . . . $a \sqsubset b$ | |
| Then the sum of the parts is . . . $a + b$. . . | 10 |
| And the difference of the parts is . . . $a - b$. . . | 2 |
| Therefore the Rectangle (or Product) of the sum and difference of the parts is . . . $aa - bb$ | 20 |
- Which Rectangle (according to the tenour of Theor. 8.) is manifestly equal to the difference of the Squares of the parts a and b . And by multiplying $\frac{1}{2}a + \frac{1}{2}b$ into $\frac{1}{2}a - \frac{1}{2}b$, the latter part of the said Theorem will be also manifest.

Theorem IX.

If a right-line be cut into any two unequal parts, the greater part shall be equal to half the whole line, together with half the difference of the parts: And, the lesser part shall be equal to half the whole line less by half the difference of the parts.

B b

Suppos.

Suppos.

1. AB is a right-line;
2. AE and EB are parts of AB;
3. $AE \sqsubset EB$.

Prepar.

4. From AB cut off $AD = EB$, thence it follows that DE is the difference of the parts AE and EB; for $DE = AE - AD$ (EB.)
5. Divide DE into two equal parts in C; therefore $DC = CE = \frac{1}{2} DE$.
6. \therefore Reg. demonstr. $\left\{ \begin{array}{l} AE = \frac{1}{2} AB + \frac{1}{2} DE. \\ EB = \frac{1}{2} AB - \frac{1}{2} DE. \end{array} \right.$

Demonstration.

7. Because by Constr. in 4° , $\left\{ \begin{array}{l} AD = EB. \\ DC = CE = \frac{1}{2} DE. \end{array} \right.$
8. And by Constr. in 5° , $\left\{ \begin{array}{l} AC = CB = \frac{1}{2} AB. \\ AE = \frac{1}{2} AB + \frac{1}{2} DE. \end{array} \right.$
9. Therefore the sum of the Equations in 7° and 8° , gives (per Ax. 8. Chap. 2.) $\left\{ \begin{array}{l} AE = \frac{1}{2} AB + \frac{1}{2} DE. \\ EB = \frac{1}{2} AB - \frac{1}{2} DE. \end{array} \right.$ Which was to be Dem.
10. And the sum of the Equations in 8° and 9° gives
11. And the Equation in 8° subtracted from the Equation in 9° gives

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, $\left\{ \begin{array}{l} a \text{ and } b \\ a \sqsubset b \end{array} \right\}$ 6 and 4
to wit, $\left\{ \begin{array}{l} a \sqsubset b \\ \frac{1}{2} a + \frac{1}{2} b \\ \frac{1}{2} a - \frac{1}{2} b \end{array} \right\}$ 5
Suppose also
Then half the whole line, that is, half the sum of the parts is $\frac{1}{2} a + \frac{1}{2} b$
And half the difference of the parts is $\frac{1}{2} a - \frac{1}{2} b$

Now (according to the import of Theor. 9.) the sum of the said half sum and half difference doth manifestly make a the greater part: And the excess of the said half sum above the said difference is manifestly equal to b the lesser part.

CHAP. V.

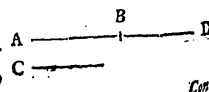
A Collection of Canonical Geometrical Effections, frequently used in the Construction of Plane Problems; more especially of those whose Solutions are found out by the Algebraical Art.

AS all Arithmetical Operations are compris'd under five kinds, to wit, Addition, Subtraction, Multiplication, Division and the Extraction of Roots, so all those Geometrical Constructions which are formed according to Canons deduced from the Algebraical Resolutions of Problems, do principally depend upon the like kinds of Operations, or Effections; but how these Geometrical Effections, (or the Arithmetick of Geometry) may be perform'd, so far as is necessary to the Construction of Plane Problems, to wit, such as may be solved by drawing only right-lines and describing the Circumferences of Circles, I shall shew in this Chapter, the Contents whereof are extracted out of the first six Books of Euclid's Elements, wherein I presuppose the Reader to be competently versed.

Problem I.

To add a given right-line to a right-line given.

Let AB and C be two right-lines given to be added together, viz. let it be required to find out a right-line which shall be equal to both the given right-lines taken together as one right-line,



Con.

Construction.

By prop. 2. Elem. 1. produce (or continue) the given line AB to D, that BD may be equal to C, so is AD the right-line sought; for by Construction $AD = AB + BD$, and $BD = C$, therefore (per Ax. 6. Chap. 2.) $AD = AB + C$, as was required.

Probl. II.

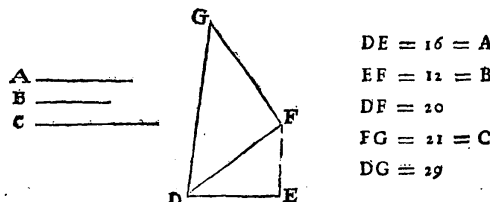
Two or more Squares being given, to find a Square equal to them all.

Suppos.

1. A, B, C are the sides of three Squares given.

Req. to find

2. DG a right-line, such, that $\square DG = \square A + \square B + \square C$.



Constr.

3. Make $DE = A$.
4. Make $EF \perp DE$.
5. Make $EF = B$.
6. Draw DF .
7. Make $FG \perp DF$.
8. Make $FG = C$.
9. Draw DG , which shall be the side of the Square required.
10. . . . Reg. demonstr. $\square GD = \square A + \square B + \square C$

Demonstration.

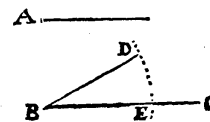
11. Because by Constr. in 7° and 4° , $\angle DFG = \angle DEF$.
12. Therefore, (per prop. 47. Elem. 1.) $\square DG = \square FG + \square FD$.
13. Likewise $\square FD = \square DE + \square EF$.
14. Therefore from 12° and 13° , $\square DG = \square DE + \square EF + \square FG = \square A + \square B + \square C$. (per Ax. 6. Chap. 2.) Which was to be done.

After the same manner of Construction, as many Squares as one will may be added into one. But if Planes of any other kind, as Long-Squares, Rhombs, Rhomboids, Triangles, &c. be given to be added, they must first be transformed into Squares, which may be done by Prop. 14. Elem. 2. or by various ways delivered in the practical Geometry of divers Mathematicians, and then they may be added together as before.

Probl. III.

To subtract or cut off a right-line given from a greater right-line given.

The subtracting or cutting off one right-line from another, to wit, a lesser from a greater, is perform'd by Prop. 3. Elem. 1. For, if two unequal right-lines be given, suppose BC the greater, and A the lesser, then by describing a Circle from B as a Center, with the distance or Semidiameter BD equal to the lesser line A, the right-line BE = BD or A will be cut off from the greater line BC, and consequently, EC is the excess whereby BC exceeds A or BE.



Bb 2

Probl. IV.

Probl. IV.

Two unequal Squares being given, to find a Square equal to the excess whereby the greater exceeds the less.

Suppos.

1. C and AB are the sides of two Squares given.
2. $C > AB$.

Req. to find

3. BE a right-line, such, that $\square BE = \square C - \square AB$.

Construction.

4. Upon the point B, (one of the ends of the given line AB,) erect a Perpendicular, and draw it forth at length, as BE.
5. From A as a Center, at the distance of the given line C, describe the arch DE, to cut the Perpendicular BE, suppose in E; for by supposition the line C is greater than AB, and therefore a Circle described upon the Center A, at the distance of C shall necessarily cut the Perpendicular BE produced infinitely.
6. I say BE shall be the side of the Square required.
7. . . . Req. demonstr. . . . $\square BE = \square C - \square AB$.

Demonstration.

8. By Constr. in 4° and 5° , . . . $\angle ABE$ is \perp , and $C = AE$.
9. Therefore (per prop. 47. Elem. 1.) . . . $\square AB + \square BE = \square AE = \square C$.
10. Therefore (per Ax. 9. Chap. 2.) . . . $\square BE = \square C - \square AB$. Which was to be done.

Another Construction of Probl. 4:

Suppos.

11. AB and D are the sides of two Squares given.
12. $AB < D$.

Req. to find

13. EB a right-line, such, that $\square EB = \square AB - \square D$.

Construction.

14. Upon the given line AB as a Diameter, describe the Semicircle CAEB, and inscribe $AE = D$, which is possible to be done, for by supposition $AB > D$. Lastly, draw EB which shall be the side of the Square required.
15. . . . Req. demonstr. . . . $\square EB = \square AB - \square D$.

Demonstration.

16. By Constr. in 14° , and per prop. 31. . . . $\angle AEB$ is \perp , and $AE = D$.
17. Therefore, per prop. 47. Elem. 1. . . . $\square AE$ (or $\square D$) + $\square EB = \square AB$.
18. Therefore per Ax. 9. Chap. 2. . . . $\square EB = \square AB - \square D$. Which was to be done.

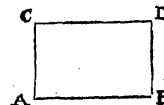
Note. If many Squares be given to be subtracted from a Square given, those to be subtracted must first be added together, by the preceding Probl. 2. of this Chap. and then subtraction may be made by either of the two foregoing Constructions of Probl. 4. But if Planes which are not Squares be to be subtracted, they must first be reduced to Squares, by Prop. 14. Elem. 2.

Probl. V.

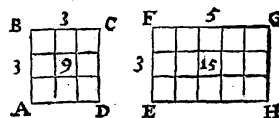
Probl. V.

Concerning Geometrical Multiplication.

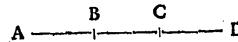
1. A right-line is said to be multiplied by a right-line, when a right-angled Parallelogram, whether it be a Square or a Long-square, is comprehended under one of the said right-lines as a length, and the other as a breadth. As, if the right-line AC be conceived to be moved along the line AB, so, as that AC always makes a right-angle with the line AB, until the point C be come to the point D, and the point A to the point B, then the right-angled Parallelogram ACDB is described by such moving of the line AC, and imports the same thing with the Product of the multiplication of the line AB by the line AC. Which Product, or right-angled Parallelogram, is also usually called a Rectangle.



2. A Rectangle is also implied by the Product of the multiplication of any two numbers, for the Area of a Rectangle is equal to the Product made by the multiplication of the number expressing the measure of one of the sides about the right-angle, by the number expressing the measure of the other side about the same angle. As in the Rectangle or long-Square EG, if its length EH or FG be 5 feet, and the breadth EF or HG, 3 feet, then the Product of the multiplication of 5 by 3, to wit, 15, imports the Area, or number of square feet contain'd in the said Rectangle. Likewise, if AB or AD the side of the Square AC be 3 feet, the Area is 9 square feet. All which evidently appears by the Diagrams here in view.

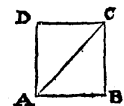


3. If a right-line be to be multiplied by a whole number, it may be done by Addition; (by Probl. 1. of this Chap.) As, if the line AB be to be multiplied by 3, it implies only the producing or continuing of the said line to such a point D, that the whole line AD may be equal to the triple of the given line AB.



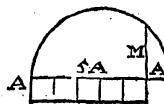
4. But if a right-line AD be to be multiplied by a Fraction, or (which is of the same import,) if it be required to cut off some segment, as $\frac{2}{3}$ parts from AD; first, (per Schol. of Prop. 10. Elem. 6.) divide the line AD into three equal parts, which suppose to be AB, BC, CD, then the segment AC which is compos'd of two of those three parts is manifestly $\frac{2}{3}$ of the line AD.

5. If a Square be to be multiplied by a whole number, the side of the Square sought may be found out by addition of Squares, as before in Probl. 2. So if the Square of the right-line AB, or BC be to be doubled, or multiplied by 2, it implies only the finding of the right-line AC; (by Probl. 2. of this Chap.) whose Square is equal to the double of the Square of AB or BC. For if $AB = BC$, and $BC \perp AB$, then (per prop. 47. Elem. 1.) $\square AC = \square AB + \square BC = 2 \square AB$ or $2 \square BC$. The same thing also may be done by the way delivered in the following Sect. 6.



$$\begin{aligned} AB &= 3 \\ BC &= 3 \\ AC &= \sqrt{18} \end{aligned}$$

6. If a Square be to be multiplied by a whole number, or by any Fraction whatever, whether it be a proper or improper Fraction, that is, if it be required to find the side of a Square that shall be equal to any prescribed Multiple, or to any part or parts of a given Square, it may be done thus: Let the right-line A be the side of a Square given, and let it be required to find a Square which shall contain five times the given Square whose



$$\begin{aligned} A &= 2 \\ 5A &= 10 \\ 5A &= 20 \\ M &= \sqrt{20} \end{aligned}$$

side

side is A. To effect this, find (per Probl. 9. of this Chap.) a mean Proportional between the given side A, and a right-line equal to five times A, which mean suppose to be the right-line M : I say the line M is the Square required; which I prove thus,

Req. demonstr. $\square M = 5 \square A$.

Demonstration.

By Constr. these are Proportionals, viz. . . . $A : M :: M : 5 A$
 Therefore, (per prop. 17. Elem. 6.) . . . $\square A, 5 A = \square M$
 But (per prop. 1. Elem. 2. respect being had to the }
 last preceding Diagram,) . . . $\square A, 5 A = 5 \square A$
 Therefore from the two preceding Equations, }
 (per Ax. 1.) . . . $\square M = 5 \square A$.

Which was to be done.

In like manner a mean Proportional between A and $\frac{2}{3} A$ shall be the side of a Square equal to $\frac{2}{3} \square A$. Also a mean Proportional between $3 \frac{1}{2} A$ (or $\frac{7}{2} A$) and A shall be the side of a Square equal to $\frac{7}{2} \square A$, or $3 \frac{1}{2} \square A$. The same thing may be effected by Probl. 11. of this Chap. the proportion of the Square given to the Square sought being first exprest by two right-lines, by the help of the foregoing Sect. 3. or 4. of this Probl. 5.

Probl. VI.

Concerning Geometrical Division, or Application.

That Geometrical Effection which answers to Division in Arithmetick is called Application, the Scope whereof when 'tis exercis'd about the Construction of Plane Problems is only this, viz. A Rectangle, (or right-angled Parallelogram) being given, as also a right-line, to find out another right-line, such, that a Rectangle contain'd under the line found out, and the line given shall be equal to the Rectangle first given, which Effection (or Construction) is called the Application of a given Rectangle to a right-line given, and the right-line arising out of the Application is called the Parabola, or the Geometrical Quotient, which is found out by the Rule of Three in right-lines by the following 7th or 8th Problems of this Chapter: For as the line given is to either of the sides about the right-angle of the given Rectangle, so is the other side about the same angle, to the line sought, to wit, the Geometrical Quotient.

This will be made manifest by the two following Examples, in the first whereof the Rectangle given is a Square, in the latter a long-Square.

Suppos. 1.

1. A is the side of a Square given,
2. BC is a right-line given,

Req. to find

3. BD, a right-line, such, that $\square BD, BC = \square A$.

Construction.

4. By Probl. 7. of this Chap. let it be made as BC }
 to A, so A to a third Proportional, which sup- } $BC : A :: A : BD$
 pose to be the line BD, therefore, . . . }
5. I say BD is the right-line sought by the Probl. propounded; it remains then to prove that a Rectangle contain'd under the right lines BD and BC is equal to the Square of line A.

Prepar.

6. Let a Rectangle be made of the lines BD and BC, as $\square CD$, (per prop. 16. Elem. 1.) Then
7. . . . Req. demonstr. $\square CD = \square A$.

Demonstration.

8. By Constr. in 4th, . . . $BC : A :: A : BD$.
9. Therefore, (per prop. 17. Elem. 6.) . . . $\square A = \square BC, BD$.
10. But by Constr. in 6th, . . . $\square CD = \square BC, BD$.
11. Therefore (per Ax. 1. Chap. 2.) . . . $\square CD = \square A$.

Which was to be done.

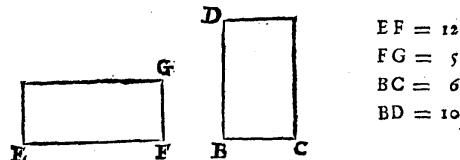
Suppos. 2.

Suppos. 2.

12. EG is a \square whose sides EF, FG are given severally.
13. BC is a right-line given.

Req. to find

14. BD a right-line, such, that $\square BC, BD = \square EG$.



Construction.

15. By Probl. 8. of this Chap. let it be made }
 as BC to EF, so FG to a fourth Proportio- } $BC : EF :: FG : BD$
 nal, which suppose to be the line BD, therefore, }
16. I say BD is the right-line sought; it remains therefore to prove that a Rectangle contain'd under BC and BD is equal to the given Rectangle EG.

Prepar.

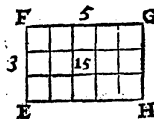
17. Let a Rectangle be made of the lines BC and BD, as $\square CD$, (per prop. 16. Elem. 1.) Then
18. . . . Req. demonstr. $\square CD = \square EG$.

Demonstration.

19. By Constr. in 15th, . . . $BC : EF :: FG : BD$.
20. Therefore, (per prop. 16. Elem. 6.) . . . $\square EF, FG = \square BC, BD$.
21. But by Constr. in 17th, . . . $\square CD = \square BC, BD$.
22. Therefore from the two last preceding Equations, (per Ax. 1. Chap. 2.) . . . $\square CD = \square EF, FG$, or $\square EG$.

Which was to be done.

23. From the premises 'tis evident that Geometrical Application answers to Division in Arithmetick, for the Rectangle applied is correspondent to the Dividend, and the right-line to which the Rectangle is applied answers to the Divisor, and the right-line arising out of the Application, the Quotient: Therefore, if the Area of a Rectangle and either of its sides be given, the other side is also given; for if the Area be divided by the given side, the Quotient is the other side. So if FG, or EH, one of the sides of the Rectangle EG, be 5, and FE, or GH, the other side, 3, the Area 15 divided by 5, (to wit, FG,) gives 3 for FE. Likewise the Area 15 divided by 3, (to wit, FE,) gives 5 for FG, or EH.



Probl. VII.

Unto two right-lines given to find a third Proportional.

Suppos.

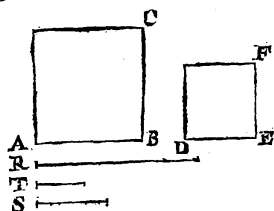
1. A and B are two right-lines given

Req. to find

2. HE a right-line, such, that $A : B :: B : HE$.

Req. to find

4. S a right-line, such, that . . . $\square AC . \square DF :: R . S$.



$$\begin{aligned} AB &= 6 \\ \square AB &= 36 \\ DE &= 4 \\ \square DE &= 16 \\ R &= 9 \\ T &= 2\frac{1}{2} \\ S &= 4 \end{aligned}$$

Constrution.

5. By *Probl. 7.* of this *Chapt.* let it be made as AB to DE, so DE to a third proportional, which suppose to be the line T, therefore $AB : DE :: DE : T$.
6. Again, (by the foregoing *Probl. 8.*) let it be made as AB to T, so R to a fourth proportional, which suppose to be the line S, therefore $AB : T :: R : S$.

I say the line S is that required by the *Problem*, therefore

7. . . . *Req. demonstr.* . . . $\square AC . \square DF :: R : S$.

Demonstration.

8. Because by *Constrution* in 5°, . . . $AB : DE :: DE : T$.
9. Therefore, (per *Coroll. Prop. 20. Elem. 6.*) $\square AB : \square DE :: AB : T$.
10. And because by *Constr.* in 6°, . . . $AB : T :: R : S$.
11. Therefore from 9° and 10°, (per *Prop. 11. Elem. 5.*) $\square AB . \square DE :: R . S$.
12. That is, (per *Ax. 7. Chap. 2.*) $\square AC . \square DF :: R . S$.
- Which was to be done.

Probl. XI.

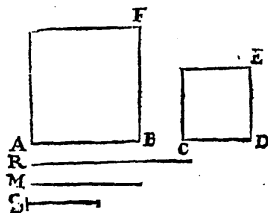
A Square being given, to find out another Square greater or less than that given, according to a Proportion given.

Suppos.

1. AF is a Square given, whose side is AB.
2. R and S are two right-lines expressing the given Proportion.

Req. to make

3. CE a Square, such, that . . . $R : S :: \square AF . \square CE$.



$$\begin{aligned} AB &= 6 \\ \square AB &= 36 \\ R &= 9 \\ S &= 4 \\ M &= 6 \\ CD &= 4 \\ \square CD &= 16 \end{aligned}$$

Constrution.

4. By *Probl. 9.* of this *Chapt.* find a mean proportional between the given lines R and S, which mean suppose to be the right-line M, therefore $R : M :: M : S$.
5. Again, (by *Probl. 8.* of this *Chapt.*) let it be made as R to M, so AB (the side of the given Square,) to a fourth proportional, which suppose to be the right-line CD, therefore $R : M :: AB : CD$.
6. Upon the line CD describe a Square, as CE, which shall be that required by the *Problem*, therefore
7. . . . *Req. demonstr.* . . . $R : S :: \square AF : \square CE$.

Demonstration.

8. Because by *Constr.* in 4°, . . . $R : M :: M : S$.
9. Therefore, (per *Coroll. prop. 20. Elem. 6.*) $\square R . \square M :: R . S$.
10. And because by *Constr.* in 5°, . . . $R : M :: AB : CD$.
11. Therefore, (per *prop. 22. Elem. 6.*) $\square R . \square M :: \square AB . \square CD$.
12. Therefore out of 9° and 11°, (per *prop. 11. Elem. 5.*) $R : S :: \square AB . \square CD$.
13. That is, (per *Ax. 7. Chap. 2.*) $R : S :: \square AF . \square CE$.
- Which was to be done.

Probl. XII.

The mean of three proportional right-lines being given; as also the difference of the extremes, to find the extremes.

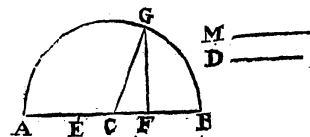
Suppos.

1. M = the mean of three proportionals is given.
2. D = the difference of the extremes is given.

Req. to find

3. AF and FB two right-lines, such, that $AF - FB = D$; Also, that
4. $AF . M :: M . FB$.

$$\begin{aligned} M &= 12 & AF &= 18 \\ D &= 10 & FB &= 8 \\ EF &= 10 & AE &= 8 \\ FG &= 12 & CA &= 13 \\ EC &= 5 & CG &= 13 \\ CF &= 5 & CB &= 13 \end{aligned}$$



Constrution.

5. Make $EF = D$ (the given difference.)
6. Upon the point F erect $FG \perp FE$.
7. Make $FG = M$ (the given mean.)
8. Divide EF into two equal parts in C.
9. Draw the right-line CG.
10. From C as a Center, at the distance of CG describe the Semicircle CAGB.
11. Draw the Diameter AECFB.
12. I say AF and FB are the extreme proportionals required.
13. . . . *Req. demonstr.* . . . $\square AF - FB = D$.

C.c 2

Demon.

Demonstration.

14. Because by *Construction* in 16° , CAGB is a Semicircle whose Center is C; therefore, $CA = CB = CG$.
 (per *defn.* 15. *Elem.* 1.) $CE = CF$.
 15. And because by *Constr.* in 8° , $EA = FB$.
 16. Therefore the Equation in 15° being subtracted from that in 14° , gives (as is evident by the Diagram) $AF - EA = EF$.
 17. It is also evident by the Diagram that $AF - FB = EF$.
 18. Therefore out of 16° and 17° , (per *Ax.* 6. Chap. 2.) $D = EF$.
 19. But by *Constr.* in 5° , $AF - FB = D$.
 20. Therefore from 18° and 19° , (per *Ax.* 1.) Which was to be dem.
 21. Again, because (per *Coroll.* *Probl.* 9. of this Chap.) $AF \cdot FG :: FG \cdot FB$.
 22. And by *Constr.* in 7° , $M = FG$.
 23. Therefore from 21° and 22° , by exchanging equal right-lines, $AF \cdot M :: M \cdot FB$.
 Which was also to be dem.

Therefore the *Problem* propounded is satisfied; and out of the premises the following *Theorem* is deducible, for the solving of the same *Probl.* 12. Arithmetically.

Theorem.

24. If half the difference of the extremes of three Proportionals, be added to the side (or square Root) of that Square which is equal to the Square of half the difference of the extremes together with the Square of the mean, the sum shall be the greater extreme: But if the said half difference be subtracted from the said side, the remainder shall be the lesser extreme. Hence,

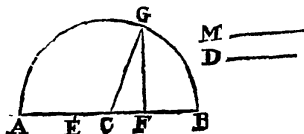
The Arithmetical solution of *Probl.* 12.

Suppos.

25. AF, FG, FB are \div , viz. $AF : FG :: FG : FB$.
 26. $AF \sqsubset FB$.
 27. $FG = 12$.
 28. $EF = 10 = AF - FB$. } Given.

Req. to find

29. AF and FB in numbers.



Operation Arithmetical.

30. By *Suppos.* in 28° , $EF = 10$.
 31. Therefore $\frac{1}{2}EF = CF = 5 = CE$.
 32. And consequently, $\square CF = 25$.
 33. Also from 27° , $\square FG = 144$.
 34. The sum of the Equations in 32° and 33° , gives (per *prop.* 47. *Elem.* 1.) $\square CG = 169$.
 35. The square Root of each part of the Equation in 34° gives $CG = 13 = CA$.
 36. Therefore the sum of the Equations in 31° and 35° gives $AF = 18$. } Sought.
 37. And by subtracting the Equation in 31° from that in 35° , $EA = FB = 8$.

The Proof.

38. It is manifest that these are Proportionals, for the Product of the extremes is equal to the Square of the mean, viz. $18 \cdot 12 :: 12 \cdot 8$.
 $AF \cdot FG :: FG \cdot FB$.
 39. Also,

39. Also the mean proportional is 12, and the difference of the extremes is 10, as was prescribed.

Coroll. 1.

40. From the premises it may be infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Equation so constituted, that the Square of an unknown right-line less by a Rectangle contain'd under that unknown line and some known right-line, is equal to a known Plane; the said unknown line shall be given by the preceding Geometrical *Construction* of *Probl.* 12. But here is to be noted, that if the said known Plane be not a Square, it must first be reduced to a Square, by *Probl.* 9. of this Chap. or by *prop.* 14. *Elem.* 1.

41. As, in this Equation $aa - da = mm$;
 If we suppose aa to represent an unknown Square whose side is a , also da a Rectangle contain'd under the said unknown side a and some known right-line represented by d , and that the said unknown Square aa less by the said Rectangle da is equal to some known Square, as mm , whose side is m ; then the said unknown side or right-line a may be found out by the Geometrical *Construction* of the preceding *Probl.* 12.
 42. For (by *prop.* 14. *Elem.* 6.) the Equation above propos'd in 41° may be resolved into these Proportionals,

$$a - d \cdot m :: m \cdot a;$$

Of which three Proportionals the mean m is known by supposition, as also d the difference of the extremes a and $a - d$, (for a exceeds $a - d$ by d), therefore by the *Construction* of the foregoing *Probl.* 12. the extremes shall be given severally, the greater whereof is the line a sought.

43. Moreover, if in the Equation above propounded, to wit, $aa - da = mm$, we suppose $m = 12$, and $d = 10$, then the quantity of the line a shall be also given in number, for by the first part of the preceding *Theorem* in 24° ,

$$a = \frac{1}{2}d + \sqrt{\frac{1}{4}dd + mm} = 18.$$

Coroll. 2.

44. It also follows from the preceding *Construction* of *Probl.* 12. that if by the Algebraical Resolution of a Geometrical Problem, an Equation be found out, such, that the Square of a right-line sought, together with a Rectangle contain'd under that unknown line and some known right-line, is equal to a known Plane; the said unknown line shall be given by the Geometrical *Construction* of the said *Probl.* 12. But if (as before hath been said) the said known Plane be not a Square, it must first be reduced to a Square.

45. As, in this Equation, $aa + da = mm$;
 If we suppose aa to represent an unknown Square whose side is a , also da a Rectangle contain'd under the said unknown side a , and some known right-line represented by d , and that the said unknown Square aa together with the said Rectangle da is equal to some known Square, as mm , whose side is m , then the said unknown side, or right-line a , may be found out by the Geometrical *Construction* of the preceding *Probl.* 12.

46. For the Equation above propounded in 45° may be resolved into these Proportionals, viz.

$$a + d \cdot m :: m \cdot a;$$

Of which three Proportionals the mean m is known by supposition, as also d the difference of the extremes $a + d$ and a , (for $a + d$ exceeds a by d), therefore by the *Construction* of the foregoing *Probl.* 12. the extremes shall be given severally, the lesser whereof is the line a sought.

47. Lastly, if in the Equation above propounded in 45° , to wit, $aa + da = mm$, we suppose $m = 12$ and $d = 10$, then the quantity of the line a shall be also given in number; for by the latter part of the preceding *Theorem* in 24° ,

$$a = \sqrt{\frac{1}{4}dd + mm} - \frac{1}{2}d = 8.$$

Probl. XIII.

The mean of three proportional right-lines being given, as also the sum of the extremes, to find the extremes. But the given mean must not exceed half the given sum of the extremes.

Suppos.

1. M = the mean of three Proportionals is given.

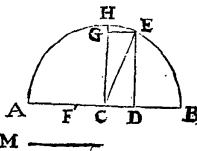
2. AB = the sum of the extremes is given.

Also M not $\leq \frac{1}{2} AB$.

Req. to find

3. AD and DB two right-lines, such, that $AD + DB = AB$. Also, that

4. $AD \cdot M :: M \cdot DB$.



Construction.

5. Divide AB into two equal parts in C .
 6. From C as a Center, with the distance CA , or CB , describe the Semicircle $CAHB$.
 7. Upon the Center C erect $CH \perp AB$.
 8. From CH cut off $CG = M$, (the given mean Proportional,) which is possible to be done, for by the Determination annex'd to the Problem propounded, M not $\leq CH$, or $\frac{1}{2} AB$.
 9. By the point G draw $GE \parallel AB$, which GE will either touch the Semicircle in H , when $M = CH = \frac{1}{2} AB$, or else cut the Semicircle, when $M > CH$, (or $\frac{1}{2} AB$), suppose then in this Example that $M > CH$, and consequently that GE cuts the Circumference, as in E .
 10. From the point E let fall $ED \perp AB$, then shall AD and DB be the extreme Proportionals required, for first their sum, in regard ADB is a right-line, (to wit, the Diameter of the Semicircle $CAHB$), is equal to AB the given sum; it remains to prove that as AD is to M , so M to DB , therefore

11. . . . *Req. demonstr.* $AD \cdot M :: M \cdot DB$.

Demonstration.

12. Because by *Construction* in 5° and 6° , . . . $CAEB$ is a Semicircle.
 13. And by *Constr.* in 10° , . . . $DE \perp AB$.
 14. Therefore from 12° and 13° , (per *Coroll.* in 12°) $AD \cdot DE :: DE \cdot DB$.
 15. And because by *Constr.* in 7° , 9° , 10° , . . . DG is \square .
 16. Therefore (per *prop.* 14. *Elem.* 1.) . . . $DE = CG$.
 17. But by *Constr.* in 8° , . . . $M = CG$.
 18. Therefore out of 16° and 17° , (per *Ax.* 1.) $DE = M$.
 19. Wherefore from 14° and 18° , (by taking M instead of DE), . . . $AD \cdot M :: M \cdot DB$.
 Which was to be done.

20. The reason of the Determination annex'd to the Problem, to wit, that the line prescribed for the mean must not exceed half the line given for the sum of the extremes, will be evident by this that follows. First, if the right-line M be less than CH , or $\frac{1}{2} AB$, and at the distance of M a line be drawn parallel to the Diameter AB , as the parallel GE , it will necessarily cut the Semicircle, as in E , in which case the extreme Proportionals, to wit, the segments of the Diameter which are made by the falling of the Perpendicular ED , will always be unequal. Secondly, if the line M be equal to CH , or $\frac{1}{2} AB$, and at the distance of M or CH a line be drawn parallel to the Diameter AB , such parallel will touch the Semicircle in H , and consequently HC which is perpendicular to AB will be a mean between AC and CB , in which case, the three Proportionals AC , CH and CB are equal to one another, for each of them is the Semidiameter of the Semicircle. Lastly, if the line M be greater than CH , or $\frac{1}{2} AB$, then 'tis eassie to perceive, that a right-line drawn parallel to the Diameter AB at the distance of such line M cannot possibly either touch or cut the Semicircle, but will lye altogether without the same, and consequently such line M cannot be a mean

$M = 12$
 $AB = 26$
 $AC = 13$
 $CE = 13$
 $CB = 13$
 $AD = 18$
 $DB = 8$
 $DE = 12$
 $CD = 5$
 $CF = 5$

proportional between any two segments of the Diameter; for a mean proportional line between two extremes is a right-line within a Circle, drawn perpendicularly to the Diameter, and extended only unto (not beyond) the Circumference: And therefore that there may be a possibility of solving this 13th Problem, the line given for the mean Proportional must not be longer than half the line given for the sum of the extremes; which is the Determination annex'd to the Problem.

The premises being well understood, it will not be difficult to apprehend how the following Theorem is thence deducible, for the solving of Probl. 13. Arithmetically.

Theorem.

21. If half the sum of the extremes of three Proportionals be increased with the side (or square Root) of that Square which is equal to the excess whereby the Square of the said half sum exceeds the Square of the mean; the said half sum so increased shall be equal to the greater extreme: But if the said half sum be lessened by the side (or square Root) aforesaid, the remainder shall be the lesser extreme. Hence,

The Arithmetical solution of Probl. 13.

Suppos.

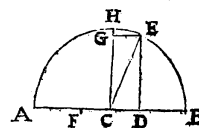
22. AD, DE, DB are $\div \div$, viz. $AD \cdot DE :: DE \cdot DB$.

23. $DE = 12$ } Given.

24. $AB = 26 = AD + DB$. }

Req. to find

25. AD and DB in numbers.



Operation Arithmetical. M ———

26. By *Suppos.* in 24° , } $AB = 26$.
 27. Therefore, } $\frac{1}{2} AB = CB = 13 = CE = CA$.
 28. And consequently, } $\square CE = 169$.
 29. And from 23° , } $\square DE = 144$.
 30. Therefore by subtracting the Equation in 29° ,
 from that in 28° , there will remain (per *prop.*) . . . $\square CD = 25$.
 47. *Elem.* 1.)
 31. And consequently, by extracting the square } $CD = 5$.
 Root out of each part of the last Equation,
 32. Therefore by adding together the Equations } $AD = 18$. } Sought.
 in 29° and 31° , }
 33. And by subtracting the Equation in 31° from } $DB = 8$. }
 that in 29° , }

Coroll.

34. From the premises it may be infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Equation so constituted, that a Rectangle contain'd under some known right-line and a right-line sought, less by the Square of the same line sought, is equal to a known Plane; then the said right-line sought shall be given by the preceding Geometrical Construction of Probl. 13. But here is to be noted, that if the said known Plane be not a Square, it must first be reduced to a Square, by Probl. 9. of this Chap. or by Prop. 14. *Elem.* 2.
 35. As, if this Equation be proposed, $sa - aa = mm$;
 We may suppose a in that Equation to represent a right-line unknown, aa the Square of that line, s a right line known, and that sa the Rectangle contain'd under those lines, less by the said unknown Square aa , is equal to some known Square, as mm , whose side is m ; then shall the said unknown right line a be given by the preceding Construction of Probl. 13.
 36. For the Equation before propos'd in 35° , may be resolved into these Proportionals, viz.

$$s - a \cdot m :: m \cdot a$$

Of which three Proportionals the mean m is known by supposition, as also s the sum of

of the extremes $s - a$ and a ; therefore, by the foregoing Construction of *Probl. 12*, the extremes shall be given severally, either of which may be taken for the line a sought.

37. For (viewing the Diagram belonging to *Probl. 13*.)
 let us suppose that $AD \cdot DE :: DE \cdot DB$.
 38. Then putting $m = DE$.
 39. Also $s = AB$.
 40. And $a = AD$.
 41. It follows that $s - a = DB = AB - AD$.
 42. And $a \cdot m :: m \cdot s - a$.
 43. Wherefore, by comparing the Rectangle of the extremes to the Square of the mean, this Equation is produced, $sa - aa = mm$.

Which is the same with the Equation before propos'd in 35°.

44. Again, the same Equation will arise if we put $a = DB$, the Suppositions in 37°, 38° and 39° remaining unalter'd; for,
 45. Then it will follow that $s - a = AD$.
 46. And $s - a \cdot m :: m \cdot a$.
 47. That is, $AD \cdot DE :: DE \cdot DB$.
 48. Therefore from 46°, $sa - aa = mm$.

Which is the same with the Equation before produced in 43°.

49. Whence 'tis manifest that the right line a sought in all Quadratick Equations of the same form with that before propos'd in 35°, may be either of two right lines, represented in the Diagram by AD and DB , for which cause such Equation is called *Ambiguous*.
 50. Lastly, if in the Equation above propounded, to wit, $sa - aa = mm$, we suppose $m = 12$, and $s = 26$, then the quantity of the line a shall be also given in number; for by the preceding Theorem in 21°,

$$a = \frac{1}{2}s \pm \sqrt{\frac{1}{4}s^2 - mm} = 18;$$

$$\text{Or, } a = \frac{1}{2}s - \sqrt{\frac{1}{4}s^2 - mm} = 8.$$

Probl. XIV.

To divide a right-line given into two parts, such, that another right-line may be a mean Proportional between the parts. But the line given for the mean must not exceed half the line given to be divided.

Suppos.

1. AB is a right line given to be cut into two parts.
 2. AC is a right line given to be a mean Proportional between the parts.
 3. AC not $\leq \frac{1}{2}AB$.

Req. to find

4. AF and FB such parts of AB that $AF \cdot FB = AC^2$; Also, that
 5. $AF \cdot AC :: AC \cdot FB$.

Construction.

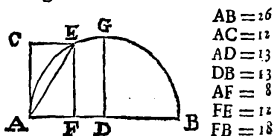
6. Upon A one of the ends of AB , erect $AC \perp AB$.
 7. Divide AB into two equal parts in D .
 8. From D as a Center, with the distance DA , or DB , describe the Semicircle $DAEB$.
 9. By the point C draw $CE \parallel AB$, so shall CE (by what hath been said in 20° of *Probl. 13*.) either touch or cut the Semicircle $DAEB$, for by *Suppos.* AC not $\leq \frac{1}{2}AB$, (or DG .) suppose then in this Example that $AC > DG$, or DA , and consequently that CE cuts the Semicircle as in E .
 10. From the point E let fall $EF \perp AB$, then shall AF and FB be the desired parts of AB , between which parts, AC is a mean Proportional, as I shall in the next place demonstrate.

11. *Reg. demonstr.* $AF \cdot AC :: AC \cdot FB$.

Preparat.

12. Draw the right lines AE and EB .

Demon-



Demonstration.

13. Because by *Constr.* in 8°, $DAEB$ is a Semicircle, } $\angle AEB$ is \perp .
 therefore (per prop. 31. *Elem.* 3.) } $EF \perp AB$.
 14. And because by *Constr.* in 10°, } $AF \cdot FE :: FE \cdot FB$.
 15. Therefore from 13° and 14°, (per *Coroll. prop.* 8. } CF is \square .
Elem. 6.) } $AC = FE$.
 16. And because by *Constr.* in 6°, 9° and 10°, . . . } $AC = FE$.
 17. Therefore, (per prop. 34. *Elem.* 1.) } $AF \cdot AC :: AC \cdot FB$.
 18. Therefore from 17° and 15°, by taking AC in- }
 stead of FE , }
 Which was to be done.

Note. This Problem may be solved Arithmetically in the same manner as the preceding *Probl. 13*.

Probl. XV.

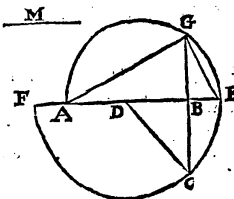
The mean of three Squares in continual proportion being given, as also a Square equal to the difference of the extremes, to find out the extremes.

Or thus;

The Base of a right-angled Triangle being given, as also a mean proportional between the Hypothenufal and Perpendicular, to find the Triangle.

Suppos.

1. AB = the Base of a right-angled Triangle is given.
 2. M = a mean proportional between the Hypothenufal and Perpendicular is given.



Req. to find out the Triangle.

3. Making the given Base AB the first of three Proportionals, and the given Mean M the second, find out a third, (per *Probl. 7*. of this *Chapt.*) which suppose to be BC , therefore,
 4. Upon B one of the ends of the given Base AB , make $BC \perp AB$, whence $\angle ABC$ is \perp .
 5. Divide AB into two equal parts in D .
 6. Draw the right-line DC .
 7. From D as a Center, at the distance of DC , describe the Semicircle $DFCE$ having $FADBE$ for its Diameter.
 8. Upon AE as a Diameter describe the Semicircle AGE .
 9. Continue CB to the Circumference in G .
 10. Draw the right lines AG and EG .
 11. I say ABG is the right-angled Triangle required; now we must shew that it will satisfy the Problem propounded. First then, by *Supposition* AB is equal to the given Base, but that the angle ABG is a right-angle, and that the given line M is a mean proportional between the Hypothenufal AG and the Perpendicular BG , the following *Demonstration* will make manifest.

12. *Reg. demonstr.* $\angle ABG$ is \perp ; Also,
 $AG \cdot M :: M \cdot BG$;

Demonstration.

13. Because by *Constr.* in the fourth step, } $\angle ABC$ is \perp .
 14. And by *Constr.* in 9°, } CBG is a right-line;
 15. Therefore, (per *Coroll. Prop.* 13. *Elem.* 1.) } $\angle ABG$ is \perp . Which was to be Dem.
 16. Again, because by *Constr.* in 7°, . . . } $DF = DE = DC$.
 17. And by *Constr.* in 5°, } $DA = DB$.
 18. Therefore by subtracting the Equation } $AF = BE$.
 in 17° from that in 16°, } Dd

19. And

19. And by adding AB to each part of the last preceding Equation, . . . FB = AE.
 20. Again, out of 4° and 7° (per Coroll. Probl. 9. of this Chap. . . . FB . BC :: BC . BE.
 21. Therefore from 20°, (per prop. 17. Elem. 6.) . . . □FB, BE = □BC.
 22. And from 19°, (per prop. 1. Elem. 6.) by taking BE as a common altitude, . . . □FB, BE = □AE, BE.
 23. Therefore out of 21° and 22° (per Ax. 1. Chap. 2.) . . . □BC = □AE, BE.
 24. Again, because by Constr. in 8°, . . . AGE is a Semicircle.
 25. Therefore, (per prop. 31. Elem. 3.) . . . <AGE is ∟.
 26. And because by what hath been proved in 15°, . . . GB ⊥ BA.
 27. Therefore from 25° and 26°, (per Coroll. 2. prop. 8. Elem. 6.) . . . AE . GE :: GE . BE.
 28. Therefore out of 27°, (per prop. 17. Elem. 6.) . . . □GE = □AE, BE.
 29. But it hath been proved in 23°, that . . . □BC = □AE, BE.
 30. Therefore from 28° and 29°, (per Ax. 1.) . . . □BC = □GE.
 31. And consequently, . . . BC = GE.
 32. Again, out of 25° and 26°, (per prop. 8. Elem. 6.) . . . △ABG and △GBE are like.
 33. Therefore from 32°, (per prop. 4. Elem. 6.) . . . AB . AG :: BG . GE.
 34. And from 33°, (per prop. 16. Elem. 6.) . . . □AG, BG = □AB, GE.
 35. And out of 31°, by taking AB for a common altitude, . . . □AB, BC = □AB, GE.
 36. Therefore from 34° and 35°, (per Ax. 1.) . . . □AB, BC = □AG, BG.
 37. But by Constr. in 5°, and per prop. 17. Elem. 6. . . . □AB, BC = □M.
 38. And consequently from 36° and 37°, (per Ax. 1.) . . . □AG, BG = □M.
 39. Therefore out of 38°, (per prop. 14. Elem. 6.) . . . AG . M :: M . BG.
 Which was to be dem. And therefore the Problem is satisfied.

Coroll. 1.

40. From the premises it may be infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in continual proportion, such, that the greater extreme is a Square unknown, the mean a known Square, and the lesser extreme the excess whereby that unknown Square exceeds some known Square, then the side of the said unknown Square shall be given by the Geometrical Construction of the preceding Probl. 15.

As, for example, let there be propos'd the following Analogy, where aa represents a Square unknown, and a its side; also mm and bb two known Squares, whose sides are m and b ;

$$aa - bb . mm :: mm . aa.$$

41. Then, (because, by prop. 22. Elem. 6. the sides of proportional Squares are proportionals also,) from the Analogy propos'd this ariseth, viz.

$$\sqrt{aa - bb} : m :: m : a.$$

42. Which three Proportionals last express'd being well consider'd, it will be manifest that the greater extreme, to wit, a , may be esteem'd the Hypotenusal of a right-angled Triangle whose Base is b , and Perpendicular $\sqrt{aa - bb}$: the lesser extreme: Now in that right-angled Triangle the Base b is given by supposition, as also m a right line which is a mean proportional between the said Hypotenusal and Perpendicular, and therefore the Hypotenusal and Perpendicular shall be given severally by the preceding Construction of Probl. 15; which Hypotenusal shall be the line represented by a in the Analogy propos'd.

43. It is also manifest, that if the Terms of the Analogy propos'd for an Example in 40, be suppos'd to represent numbers, then by comparing the Rectangle of the the extremes to the Rectangle of the means, this following Biquadratic Equation will thence be produced, viz.

$$aaaa - bb aa = mmmm;$$

Where,

where, if we suppose $bb = 6$, also $mm = \sqrt{27}$, and a to stand for some number unknown, that Biquadratic Equation may be express'd thus, viz.

$$aaaa - 6aa = 27.$$

44. In which last Equation, the unknown number a may be found out, either by an Arithmetical Operation deducible from the preceding Geometrical Construction of Probl. 15, or else after the said Equation is resolv'd into these three continual Proportionals,

$$aa - 6 . \sqrt{27} :: \sqrt{27} . aa,$$

The greater extreme aa may first be made known after the manner of the Arithmetical Solution of the foregoing Probl. 12. and then the Square Root of that number found out for the value of aa shall be the number a sought. All which will be manifest by the following Operation and Diagram.

Suppos.

45. AF, FG, FB :: viz. AF . FG :: FG . FB.

46. AF ⊥ FB.

47. □FG = 27.

48. AF - FB = 6 = EF. } given.

Req. to find out

49. AF the greater extreme, signified by aa in the Analogy before express'd in 44°.

Operation Arithmetical.

50. By Supposition in 48°, . . . EF = 6.

51. Therefore . . . EF = CF = 3 = CE.

52. And consequently, . . . CF = 9.

53. By Suppos. in 47°, . . . □FG = 27.

54. The sum of the two last Equations gives . . . □CG = 36.

(per prop. 47. Elem. 1.)

55. The Square Root of the last Equation gives . . . CG = 6 = CA.

56. The sum of the Equations in 51° and 55° gives . . . AF = 9 = aa .

57. Wherefore, . . . $\sqrt{9} = 3 = a$.

58. Moreover, out of the premises ariseth the following Canon, for the Arithmetical Resolution of all Biquadratic Equations falling under the same form with that before express'd in 43°, and here-under repeated, where a represents a number sought, but b and m two numbers given; viz.

$$If aaaa - bb aa = mmmm.$$

Then $a = \sqrt{\frac{1}{2}bb + \sqrt{\frac{1}{4}bbbb + mmmm}}$

Coroll. 2.

59. From the premises also it may be infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in continual proportion, such, that the lesser extreme is a Square unknown, the mean a known Square, and the greater extreme is composed of the said unknown Square and some known Square; then the side of the said unknown Square shall be given by the Geometrical Construction of the foregoing Probl. 15. As, for example,

If this Analogy be propos'd, where aa represents a Square unknown, and a its side; also mm and bb two known Squares, whose sides are m and b ;

60. Then (per prop. 22. Elem. 6.) these also shall be Proportionals,

61. Which three last preceding Proportionals being well examined, it will be manifest that the greater extreme, to wit, $\sqrt{aa + bb}$ may be esteem'd the Hypotenusal of a right-angled Triangle whose Base is b , and Perpendicular a , (the lesser extreme.)

Now in that right-angled Triangle the Base b is given by supposition, as also m a right line which is a mean proportional between the said Hypotenusal and Perpendicular, and therefore the Hypotenusal and Perpendicular shall be given severally by the preceding Construction of Probl. 15. which Perpendicular shall be the line represented by a in the Analogy propos'd in 59°.

D d 2

62. It



62. It is also manifest, that if the Terms of the Analogy in 59° be supposed to represent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this following Biquadratic Equation will thence be produced, viz.

$$aaaa + bbba = mmmm.$$

- Where, if we suppose $bb = 6$, also $mm = \sqrt{135}$, and a to represent some number unknown; that Biquadratic Equation may be expressed thus, 63. In which last Equation the unknown number a may be found out either by an Arithmetical Operation, deducible out of the preceding Geometrical Construction of *Probl. 15.* or else after the said Equation is resolved into these three continual Proportionals,

$$aaaa + 6aa = 135;$$

$$aa + 6 : \sqrt{135} :: \sqrt{135} : aa$$

The lesser extreme aa may first be discovered after the manner of the Arithmetical Solution of the foregoing *Probl. 12.* of this *Chapt.* and then the square Root of that number found out for the value of aa shall be the number a sought. All which will be manifest by the following Operation and Diagram.

Suppos.

64. $AF, FG, FB, \div \div \div$ viz. $AF : FG :: FG : FB.$

65. $AF \sqsubset FB.$

66. $\square FG = 135.$

67. $AF - FB = 6 = EF.$

Req. to find out.

68. FB the lesser extreme, which answers to aa in the Analogy above express'd in 63°.

Operation Arithmetical.

69. By *Suppos.* in 60° : : $EF = 6.$
 70. Therefore : : $\frac{1}{2} EF = CF = 3 = CE.$
 71. And consequently, : : $\square CF = 9.$
 72. By *Suppos.* in 66° : : $\square FG = 135.$
 73. The sum of the two last preceding Equations gives (per *prop. 47. Elem. 1.*) : : $\square CG = 144.$
 74. The square Root of the Equation in 73° gives : : $CG = 12 = CB.$
 75. And by subtracting the Equation in 70°, from that in 74°, : : $FB = 9 = aa.$
 76. Wherefore : : $\sqrt{9} = 3 = a.$
 77. Moreover, out of the premises ariseth the following Canon, for the Arithmetical Resolution of all Biquadratic Equations falling under the same Form with that before express'd in 62°, and here-under repeated, where a represents a number sought, but b and m two given numbers, viz.

If : : $aaaa + bbba = mmmm,$

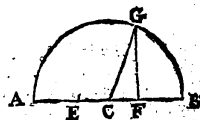
Then : : $a = \sqrt[4]{\frac{1}{2}bbbb + mmmm - \frac{1}{2}bb}$

Coroll. 3.

78. From the premises also it may be infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in continual proportion, such, that the mean is a Square known, and in each of the extremes one and the same unknown Square is found Affirmative, (that is, with the Sign $+$ prefix to it,) together with some known Square having either $+$ or $-$ prefix to it, or when in each of the extremes one and the same unknown Square is found negative, (that is, with the sign $-$ set before it,) together with some known Square having the sign $+$ prefix to it, then in either of those Cases, the side of that unknown Square may be found out Geometrically by the preceding Construction of *Probl. 15.* as will be manifest by the four following Examples.

Example 1.

79. : : $\left\{ \begin{array}{l} \text{Suppose} : : aa + bb : mm :: mm : aa + cc, \\ \text{And consequently, } \sqrt{aa + bb} : m :: m : \sqrt{aa + cc}; \end{array} \right.$



In which Analogies, let aa represent an unknown Square whose side is a , but bb , mm and cc three known Squares, whose sides are b , m and c , suppose also that b is greater than c , and consequently $aa + bb \sqsubset aa + cc$. I say then, that the unknown side a shall be given by the preceding Construction of *Probl. 15.* For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb - cc$, which by *Suppos.* is given, therefore the side of a Square equal to that difference, to wit, $\sqrt{bb - cc}$, may be esteem'd the Base of a right-angled Triangle, by the help of which given Base, and of the right-line m , which by *Supposition* is a given mean Proportional between the Hypotenusal $\sqrt{aa + bb}$ and the Perpendicular $\sqrt{aa + cc}$: the Hypotenusal and Perpendicular shall be given severally (by the said *Probl. 15.*) and may be represented by h and p , whose Squares are hh and pp , then by equating bb to the greater extreme $aa + bb$, or pp , to $aa + cc$, the side a will be found equal to $\sqrt{hh - bb}$ or $\sqrt{pp - cc}$: which Roots are given, and equal to one another.

Example 2.

80. : : $\left\{ \begin{array}{l} \text{Suppose} : : aa - bb : mm :: mm : aa - cc, \\ \text{And consequently, } \sqrt{aa - bb} : m :: m : \sqrt{aa - cc}; \end{array} \right.$

In which Analogies, if (as before in *Example 1.*) we suppose aa to represent an unknown Square whose side is a ; also bb , mm and cc three known Squares, whose sides are b , m and c , and b to be greater than c , whence consequently $aa - cc \sqsubset aa - bb$, then the unknown side a shall be given by the preceding Construction of *Probl. 15.* For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb - cc$, which by *Supposition* is given, then the side of a Square equal to that difference, to wit, $\sqrt{bb - cc}$, may be esteem'd the Base of a right-angled Triangle, by the help of which given Base and of the right-line m , which by *Supposition* is a given mean Proportional between the Hypotenusal $\sqrt{aa - cc}$ and the Perpendicular $\sqrt{aa - bb}$: the Hypotenusal and Perpendicular shall be given severally by the said *Probl. 15.* and may be represented by h and p , whose Squares are hh and pp , then by equating bb to $aa - cc$ (the greater extreme), or pp to $aa - bb$, the unknown side or right-line a will be found equal to $\sqrt{hh + cc}$ or $\sqrt{pp + bb}$: which Roots are given, and equal to one another.

Example 3.

81. : : $\left\{ \begin{array}{l} \text{Suppose} : : aa + bb : mm :: mm : aa + cc, \\ \text{And consequently, } \sqrt{aa + bb} : m :: m : \sqrt{aa + cc}; \end{array} \right.$

Example 4.

82. : : $\left\{ \begin{array}{l} \text{Suppose} : : bb - aa : mm :: mm : cc - aa, \\ \text{And consequently, } \sqrt{bb - aa} : m :: m : \sqrt{cc - aa}; \end{array} \right.$

By what hath been said in the Explication of *Examples 1.* and *2.* it will not be difficult to conceive how to find out in like manner the unknown Root of right-line represented by a in the third and fourth *Examples*, where b , m and c are supposed to be right-lines severally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in any of the three preceding *Corollaries* of *Probl. 15.* such known Squares or Rectangles must first of all be converted into a simple Square, per *Probl. 2. Chap. 4.*

Probl. XVI.

The mean of three Squares in Continual proportion being given, as also a Square equal to the sum of the extremes, to find out the extremes.

Or

Or thus,

The Hypotenusal of a right-angled Triangle being given, as also a mean Proportional between the Base and Perpendicular, to find out the Triangle. But a right-angled Triangle arising out of the Application of the Square of the given Mean to the given Hypotenusal must not exceed half the Hypotenusal.

Suppos.

1. AB = the Hypotenusal of a right-angled Triangle is given.
2. M = a mean proportional between the Base and Perpendicular is given.
3. $\frac{M}{AB}$ not $\leq \frac{1}{2}AB$.

Req. to find out the Triangle.

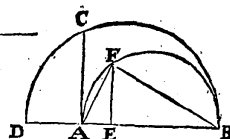
Construction.

4. Upon the point A , (one of the ends of the given Hypotenusal AB ,) erect $AC \perp AB$ and make $AC = M$ the given mean Proportional.
5. Let it be made (per *Probl. 7. of this Chapt.*) as AB to AC , so AC to a third Proportional, which suppose to be found AD , therefore, That is, (because $M = AC$,) $AB : AC :: AC : AD$.
6. Upon AB as a Diameter describe the Semicircle AFB .
7. Upon DB as a Diameter compos'd of AB and AD as one right-line, describe the Semicircle DCB .
8. By *Probl. 14. of this Chapt.* divide the given Hypotenusal AB into two such parts, that the line AD may be a mean proportional between the parts; which is possible to be done, for by *Construction in 5°*, $AD = \frac{M^2}{AB}$, and by *Supposition in 3°*, agreeable to the Determination annex'd to the *Problem*, $\frac{M}{AB}$ or AD not $\leq \frac{1}{2}AB$;
9. Suppose then the line AB to be cut in E , into two such parts AE and EB , that EF is a mean proportional between AE and EB , and that EF is equal to AD , therefore, $AE : EF$ (or AD), :: EF (or AD), : EB .
10. Draw the lines AF and BF , then shall AFB be the Triangle sought. Now we must shew that it will satisfy the *Problem*. First then, by supposition AB is equal to the given Hypotenusal; but that the angle AFB is a right-angle, and that the given right-line M is a mean proportional between AF and BF , (to wit, the Base and Perpendicular,) the following *Demonstration* will make manifest.

Demonstration.

11. Because by *Confr. in 6°*, $\angle AFB$ is a Semicircle.
12. Therefore, (per *prop. 31. Elem. 3.*) $\angle AFB$ is \perp . Which was to be Dem.
13. Again, because by *Confr. in 8°*, $EF = AD$.
14. Therefore, by taking in AB as a common altitude, it follows from *13°*, (per *prop. 1. Elem. 6.*) that $\square EF, AB = \square AD, AB$.
15. And because by *Confr. in 6°* and *8°*, $FE \perp AB$.
16. And it hath been proved in *12°*, $\angle AFB$ is \perp .
17. Therefore out of *15°* and *16°* (per *prop. 8. Elem. 6.*) $\triangle AEF$ and $\triangle AFB$ are like.
18. And consequently, (per *prop. 4. Elem. 6.*) $EF : AF :: FB : AB$.
19. And from *18°*, (per *prop. 17. Elem. 6.*) $\square EF, AB = \square AF, FB$.
20. Therefore from *14°* and *19°*, (per *Ax. 1.*) $\square AD, AB = \square AF, FB$.
21. And because by *Confr. in 5°*, $AB : M :: M : AD$.
22. And consequently (per *prop. 17. Elem. 6.*) $\square AD, AB = \square M$.
23. Therefore out of *20°* and *22°*, (per *Ax. 1.*) $\square AF, FB = \square M$.

24. There



24. Therefore from *23°*, (per *prop. 14. Elem. 6.*) $\square AF : M :: M : FB$. Which was to be Dem. And therefore the *Problem* is satisfied.

Coroll. 1.

25. From the premises it may be infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of Three Planes in Continual proportion, such, that one of the extremes is an unknown Square, the mean a known Square, and the other extreme the excess whereby some known Square exceeds the said unknown Square; then the Side of the said unknown Square shall be given by the preceding Geometrical Construction of *Probl. 16.*

As, for example, supposing aa to represent an unknown Square, whose side is a ; also mm and bb two known Squares, whose sides are m and b , let this Analogy be propos'd, viz.

$$bb - aa : mm :: mm : aa.$$

Then (per *prop. 22. Elem. 6.*) the Sides of proportional Squares are Proportionals also, therefore

$$\sqrt{bb - aa} : m :: m : a.$$

26. Which three continual Proportionals last express being well observed, it will be manifest that the extremes, to wit, $\sqrt{bb - aa}$ and a may be esteem'd the Base and Perpendicular of a right-angled Triangle, whose Hypotenusal is b . Now in that right-angled Triangle, the Hypotenusal b is given by *Supposition*, as also in a right-line, which is a mean Proportional between the Base and Perpendicular, and therefore the Base and Perpendicular shall be given severally by the preceding *Construction of Probl. 16.* either of which right lines, viz. either the Base or Perpendicular to found out may be taken for the line represented by a in the Analogy propos'd in *25°*. For, viewing the Diagram belonging to *Probl. 16.*

28. Suppose $\angle AFB = \perp$ in $\triangle AFB$.
29. Suppose also $FA : M :: M : FB$.
30. And consequently (per *prop. 22. Elem. 6.*) $\square FA : M :: M : \square FB$.
31. Then put $mm = \square M$.
32. Also, $bb = \square AB$.
33. And $aa = \square FB$.
34. Then from *28°*, *32°* and *33°*, it will be manifest (per *prop. 47. Elem. 1.*) that $bb - aa = \square FA (= \square AB - \square FB)$.
35. Also from *30°*, *31°*, *33°*, *34°*, $bb - aa : mm :: mm : aa$.
36. Which Analogy is the same with that propos'd in *25°*, and it will also be produced by putting $aa = \square FA$, (the Positions in *31°* and *32°* remaining unalter'd.)
37. For then it follows that $bb - aa = \square FB (= \square AB - \square FA)$.
38. And because by *Suppos. in 30°*, $\square FB : M :: M : \square FA$.
39. Therefore from the premises $bb - aa : mm :: mm : aa$.

This it appears, that from either of the two right-lines found out by *Probl. 16.* one and the same Analogy may be constituted, in which respect 'tis said to be Ambiguous.

39. It is also manifest, that if the Terms of the Analogy in *25°* be suppos'd to represent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this Biquadratic Equation will be produced, viz.

Where, if we suppose $bb = 5$, also $mm = 2$, and a to stand for some number unknown, that Biquadratic Equation may be express'd thus, viz.

$$5 - aa : 2 :: 2 : aa.$$

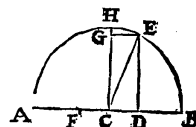
40. In which last Equation the unknown number a may be found out either by an Arithmetical Operation deducible from the preceding Geometrical Construction of *Probl. 16.* or else, after the said Equation is resolv'd into these three continual Proportionals,

The two values of aa may be found out after the manner of the Arithmetical Solution of the foregoing *Probl. 13.* of this *Chapt.* and consequently the Square Root of each number

number found out for the value of aa being extracted, there will arise two numbers, each of which may be taken for the number a in the Equation in 39° . All which will be evident by the following Operation and Diagram.

- Suppos.*
 41. $AD, DE, DB \div \div$, viz. $AD : DE :: DE : DB$.
 42. $DE = 2$, whence $\square DE = 4$.
 43. $AD + DB = 5 = AB$.

- Req. to find*
 44. AD and DB in numbers, (signified by aa in the Equation in 39° .)



Operation Arithmetical.

45. By *Suppos.* in 43° , $AB = 5$.
 46. Therefore, $\frac{1}{2}AB = CE = 2\frac{1}{2} = AC = CB$.
 47. And consequently, $\square CE = 6\frac{1}{4}$.
 48. By *Supposition* in 42° , $\square DE = 4$.
 49. And by subtracting the Equation in 48° from that in 47° , $\square CD = 2\frac{1}{4}$.
 50. The square Root of the last Equation gives $CD = 1\frac{1}{2}$.
 51. The sum of the Equations in 46° and 50° gives $AD = 4 = aa$.
 52. And consequently, $\sqrt{4} = 2 = a$.
 53. Again, by subtracting the Equation in 50° from that in 46° , $DB = 1 = aa$.
 54. And consequently, $\sqrt{1} = 1 = a$.
 Whence 'tis manifest, that the number signified by a in the Analogy in 40° (or in the Equation in 39°) may be either 2 or 1.
 55. Moreover, out of the premises ariseth the following Canon, for the Arithmetical Resolution of all Biquadratic Equations falling under the same Form with that before propos'd in 39° , and here-under repeated, where a represents a number sought, but b and m two numbers given, and subject to the Determination annex'd to *Probl. 16. viz. $\frac{mm}{b}$* must not be greater than $\frac{1}{2}b$, and consequently mm not greater than $\frac{1}{4}bb$.

If $bbaa - aaaa = mmmm$.

Then, $a = \sqrt{\frac{1}{2}bb + \sqrt{\frac{1}{4}bbbb - mmmm}}$.

Or, $a = \sqrt{\frac{1}{2}bb - \sqrt{\frac{1}{4}bbbb - mmmm}}$.

Coroll. 2.

56. From the premises it may be also infer'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in Continual proportion, such, that the mean is a known Square, and also one of the extremes is the excess of some unknown Square above a known Square, and the other extreme is the excess of some known Square above the said unknown Square; or, when one of the extremes is the excess of a known Square above an unknown Square, and the other is the sum of the same unknown Square and a known Square; in each of those Cases, the side of that unknown Square shall be given by the preceding *Construction* of *Probl. 16*.

Example 1.

57. *Suppose* $aa - dd : 4mm :: 4mm : cc - aa$,
 And consequently, $\sqrt{aa - dd} : 2m :: 2m : \sqrt{cc - aa}$.

In which Analogies, if we suppose aa to represent an unknown Square whose side is a ; but dd , $4mm$ and cc three known Squares, whose sides are d , $2m$ and c , the unknown side or right line a shall be given by the preceding *Construction* of *Probl. 16*. For the sum of the extremes in the first of those Analogies is $cc - dd$, which is given by *Supposition*; then the side of a Square equal to that sum, to wit, $\sqrt{cc - dd}$: may be esteem'd the Hypotenusal of a right-angled Triangle, by the help of which given Hypotenusal and the right-line $2m$, which by *Supposition* is a given mean Proportional between the Base and Perpendicular, (represented by $\sqrt{aa - dd}$; and $\sqrt{cc - aa}$;) the Base and Perpendicular shall be given severally by the said *Probl. 16*, and may be represented by f and g ,

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f and g , whose Squares are ff and gg ; then by equating ff to $aa - dd$, or gg to $cc - aa$, one value of the right-line a sought shall be given; again, by equating gg to $aa - dd$, or ff to $cc - aa$, another value of the right-line a shall be given: So two lines are found out, either of which may be taken for the line a sought, and therefore the Analogy above-propos'd in *Example 1.* and all others of the same kind are said to be Ambiguous.

Example 2.

58. *Suppose* $bb - aa : mm :: mm : aa + cc$,
 And consequently, $\sqrt{bb - aa} : m :: m : \sqrt{aa + cc}$.

By what hath been said in the Explication of *Example 1.* 'tis very easie to conceive how to find out in like manner the unknown Root or right-line represented by a in *Example 2.* where b , m and c are suppos'd to be right-lines severally given.

59. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in either of the three preceding *Corollaries* of *Probl. 16*. such known Squares or Rectangles must first of all be converted into a simple Square, per *Probl. 2. Chap. 4.*

CHAP. VI.

The manner of finding out such Right-lines and Squares as are represented by Algebraical Fractions, the Quantities constituting those Fractions being given severally.

Probl. I.

Suppos.

1. $a = \frac{bb}{c}$. An Equation propos'd.

2. b and c are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolv'd into these Proportionals, viz. $c : b :: b : a$.

In which Analogy, the two first Terms are right lines given by *Supposition*, therefore a third Proportional to them shall be given also, (per *Probl. 7. Chap. 5.*) which third Proportional is the line a sought.

Probl. II.

Suppos.

1. $a = \frac{bd}{c}$. An Equation propos'd.

2. b , c and d are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolv'd into this Analogy, viz. $c : b :: d : a$.

In which Analogy, the three first Terms are right lines given by *Supposition*, therefore (per *Probl. 8. Chap. 5.*) the fourth Proportional, to wit, the line a shall be given also.

Probl. III.

Suppos.

1. $a = \frac{bb + bd}{c - f}$. An Equation propos'd.

2. b , c , d and f are right lines given severally.

$E e$

Req.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolv'd into this } $c - f . b + d :: b : a$.
 Analogy,
 In which Analogy the three first Terms are given, for first, c and f being given by
Supposition, a right line equal to $c - f$ shall be given also, (per *Probl. 3. Chap. 5.*)
 secondly, b and d being given by *Supposition*, a right line equal to $b + d$ shall be given,
 (per *Probl. 1. Chap. 4.*) and lastly, the fourth Proportional line a sought shall be given,
 per *Probl. 8. Chap. 5.*

Probl. IV.

Suppos.

1. $a = \frac{bb - dd}{c + f}$. An Equation propos'd.

2. b, c, d and f are right lines given.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolv'd into this } $c + f . b + d :: b - d . a$.
 Analogy,
 4. In which Analogy the three first Terms are given, and therefore the fourth proportional
 line a sought shall be given also by *Probl. 8. Chap. 5.*
 5. In like manner, if } $a = \frac{bb + 2bc + cc}{d + f}$.
 6. Then this Analogy will discover the line a sought, } $d + f . b + c :: b + c . a$.
 viz. } $a = \frac{bb - 2bc + cc}{d + f}$.
 7. Again, if } $d + f . b - c :: b - c . a$.
 8. Then supposing $b < c$, the line a shall be given }
 by this Analogy, viz.

Probl. V.

Suppos.

1. $a = \frac{bc + df}{f}$. An Equation propos'd.

2. b, c, d, f, g are right lines given.

Req. to find the line a .

Construction.

3. First reduce df to a Rectangle, which shall have } $b . d :: f : b$.
 b for one of its sides, viz. let it be made (per }
Probl. 8. Chap. 5.) as b to d , so f to a fourth Pro- }
 portional, which may be called h ; therefore }
 4. Therefore by comparing the Rectangle of the } $bb = df$.
 extremes to the Rectangle of the means, }
 5. Then by setting bb in the place of df in the Equa- } $a = \frac{bc + bb}{g}$.
 tion propos'd, it will be converted into this, viz. }
 6. Which last Equation may be resolv'd into this } $g . b :: c + b . a$.
 Analogy,

But in the Analogy last express'd, the three first Terms are given by *Supposition* and *Construc-
 tion*, therefore (per *Probl. 8. Chap. 5.*) the fourth proportional line a sought shall be given
 also. The same line a may be found out divers other ways, as the industrious Learner will
 easily perceive.

Probl. VI.

Suppos.

1. $a = \frac{bcdf}{gcd - gbk}$. An Equation propos'd.

2. b, c, d, f, g, h and k are right lines given.

Req. to find the line a .

Constr.

Construction.

3. First, reduce bc to a Rectangle which shall have } $g . b :: c . l$.
 g for one of its sides, viz. let it be made (per }
Probl. 8. Chap. 5.) as g to b , so c to a fourth }
 Proportional, call it l , therefore, }
 4. Therefore, by comparing the Rectangle of the } $gl = bc$.
 extremes to the Rectangle of the means, }
 5. Therefore from 1° and 4°, by setting gl in the place } $a = \frac{glf}{gcd - gbk} = \frac{ldf}{cd - bk}$.
 of bc in the Equation propos'd, it gives }
 6. Again, reduce hk to a Rectangle that shall have } $dm = hk$.
 d for one of its sides, viz. let it be made as d to h , }
 so k to a fourth, which may be called m , therefore }
 7. Therefore from 5° and 6°, by setting dm in the } $a = \frac{ldf}{cd - dm} = \frac{lf}{c - m}$.
 place of hk in the Fraction in the latter part of }
 the 5th step, it gives }
 8. Therefore by resolving the latter Fraction in the } $c - m . l :: f . a$.
 7th step into Proportionals, this Analogy aritheth, }
 viz. }

In which Analogy the three first Terms are given by *Supposition* and *Construction*, there-
 fore the fourth Proportional, to wit, the line a sought shall be given also, by *Probl. 8.*
Chap. 5. of this Book.

Probl. VII.

Suppos.

1. $a = \frac{abb}{cc}$. An Equation propos'd.

2. b, c, d are right lines given.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolv'd into this } $cc . bb :: d . a$.
 Analogy, viz. }

In which Analogy, the three first Terms are given by *Supposition*, and qualified according
 to the tenour of *Probl. 10. Chap. 5.* therefore the Fourth Term, that is, the line a sought
 shall be given also by that Problem.

Probl. VIII.

Suppos.

1. $a = \frac{bcd}{ff}$. An Equation propos'd.

2. b, c, d, f , are right lines given.

Req. to find the line a .

Construction.

3. First, the Equation propos'd may be resolv'd into } $ff . bc :: d . a$.
 this Analogy, }
 4. Then let ff be reduced to a Rectangle that shall } $bg = ff$.
 have b for one of its sides, viz. let it be made }
 (per *probl. 7. Chap. 5.*) as b to f , so f to a third }
 Proportional, which may be called g , therefore }
 5. Then by taking bg instead of ff for the first Term } $bg . bc :: d . a$.
 of the Analogy in the third step, that Analogy will }
 be converted into this, }
 6. Whence, by casting away the common altitude b , } $g . c :: d . a$.
 this Analogy aritheth, }

In which last Analogy the three first Terms are right lines given by *Supposition* and *Con-
 struction*, therefore the fourth proportional line a shall be given also.

E c 2

Probl. IX.

Probl. IX.

Suppos.

$$1. a = \frac{2bce}{bb - cc}. \text{ An Equation propos'd.}$$

2. b and c are right lines given.Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this
 Analogy $bb - cc : bb :: 2c : a$,
 4. Then (by *Probl. 4. Chap. 5.*) find a Square equal to
 $bb - cc$, which Square may be called dd , this being
 set in the place of $bb - cc$, (the first Term of the pre-
 ceding Analogy,) will give these Proportionals, *viz.*
 $dd : bb :: 2c : a$.
 5. Reduce dd to a Rectangle that shall have b for one of
 its sides, *viz.* let it be made as b to d , so d to a third
 Proportional, which may be called f , therefore $bf = dd$.
 6. Then by taking bf instead of dd , the Analogy in the
 4th step will be converted into this, $bf : bb :: 2c : a$.
 7. Whence, by rejecting the common altitude b , this
 Analogy aritheth, $f : b :: 2c : a$.
 In which last Analogy the three first Terms are right lines given by *Supposition* and
Construction; therefore the fourth Proportional line a shall be given also, *per Probl. 8. Chs.*

Probl. X.

Suppos.

$$1. a = \frac{bbcc - dfff}{dff - bcc}. \text{ An Equation propos'd.}$$

2. b, c, d, f , are right lines given.Req. to find the line a .

Construction.

3. Let it be made (*per Probl. 8. Chap. 5.*) as c to f , so
 d to a fourth proportional line, call it g , therefore, $cg = df$.
 4. Then setting cg in the place of df in the Equation pro-
 pos'd, and expunging c out of the Numerator and
 Denominator, it gives $a = \frac{bbcc - gdf}{gf - bc}$.
 5. Again, let it be made as c to f , so g to a fourth pro-
 portional line, call it k , therefore, $ck = gf$.
 6. Then setting ck in the place of gf in the Fraction in
 the fourth step, and casting c out of the Numerator and
 Denominator, there aritheth $a = \frac{bb - kd}{k - b}$.
 7. Again, let it be made as k to b , so b to a third pro-
 portional line, call it m , therefore, $km = bb$.
 8. Then setting km in the place of bb in the Numerator of
 the Fraction in the 6th step, there will thence arise $a = \frac{km - kd}{k - b}$.
 9. Again, let it be made as $k - b$ to $m - d$, so k to a fourth
 Proportional, which shall be the line a sought, *viz.* $k - b : m - d :: k : a$.
 10. From the preceding *Construction* 'tis evident that the desired line a may be found out
 by these four following Rules of Three, *viz.*

$$\begin{array}{l} 1. \left\{ \begin{array}{l} c : f :: d : g. \\ c : f :: g : k. \\ c : k : b :: m : a. \end{array} \right. \\ 2. \left\{ \begin{array}{l} c : f :: d : g. \\ c : f :: g : k. \\ c : k : b :: m : a. \end{array} \right. \\ 3. \left\{ \begin{array}{l} c : f :: d : g. \\ c : f :: g : k. \\ c : k : b :: m : a. \end{array} \right. \\ 4. \left\{ \begin{array}{l} c : f :: d : g. \\ c : f :: g : k. \\ c : k : b :: m : a. \end{array} \right. \end{array}$$

11. But the Numerator and Denominator of the Algebraical Fraction in the Equation
 propos'd in this *Problem* do manifestly shew, that the lines b, c, d, f must be given
 with this Caution or Determination, *viz.*

$$f = \frac{bc}{d}; \text{ but } f \neq \sqrt{\frac{bcc}{d}}.$$

That

That is to say, if it be made as d to b , so c to a fourth Proportional, the line f must
 be greater than that fourth Proportional; but if (by *Probl. 11. Chap. 5.*) it be made
 as d to b , so cc to another Square, then the line f must be greater than the side of that
 latter Square.

Now if the line f be given within those limits, then k will be greater than b , and m
 greater than d , as the last Analogy in the 10th step requires. The same limits of f may
 be easily infer'd also from the four Analogies in the 10th step.

An Example in Numbers.

$$\text{If } \left\{ \begin{array}{l} b = 40, \\ c = 24, \\ d = 16, \\ f = 48; \end{array} \right\} \text{ then by the Analogies in the 10th step you will find } \left\{ \begin{array}{l} g = 32, \\ k = 64, \\ m = 25, \\ a = 24. \end{array} \right.$$

Thence it follows that $a = 24 = \frac{bbcc - dfff}{dff - bcc}$, the Equation propos'd.

Probl. XI.

Suppos.

$$1. aa = \frac{bbd}{c}. \text{ An Equation propos'd.}$$

2. b, c, d are right lines given.Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this
 Analogy, $aa : b :: d : c$.
 In which Analogy, the three first Terms are given, and qualified according to the tenour
 of *Probl. 11. Chap. 5.* therefore the line a shall be given also by that *Problem*.

Probl. XII.

Suppos.

$$1. aa = \frac{bdf}{c}. \text{ An Equation propos'd.}$$

2. b, c, d and f are right lines given.Req. to find the line a .

3. The Fraction $\frac{bdf}{c}$ signifies a Rectangle contain'd under a right line equal to $\frac{bd}{c}$ and
 the line f , therefore first, according to the *Construction* of *Probl. 2.* of this Chapter,
 find a right line equal to $\frac{bd}{c}$, which line may be called g , therefore,

$$aa = fg = \frac{bdf}{c}.$$

4. Then by *Probl. 9. Chap. 5.* reduce the Rectangle fg to a Square, which may be called
 mm , whose side is m , therefore,
 $aa = mm (= fg)$ And consequently, $m = a$ the line sought

Probl. XIII.

Suppos.

$$1. aa = \frac{bbcc}{dd}. \text{ An Equation propos'd.}$$

2. b, c and d are right lines given.Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this
 Analogy, $aa : bb :: cc : dd$.
 4. But the sides of proportional Squares are also Propor-
 tionals, therefore from 3^d, $a : b :: c : d$.

In which last Analogy the three first Terms are right lines given, therefore the fourth
 proportional line a sought shall be given also.

Probl. XIV.

Suppos.

Probl. XIV.

1. $aa = \frac{2bcd}{f}$. An Equation propos'd.2. b, c, d, f, g are right lines given.Req. to find the line a .

Construction.

3. The Equation propos'd may be resolv'd into this } $gg \cdot 2bc :: df \cdot aa$
 Analogy, }
 4. By *Probl. 9. Chap. 5.* reduce $2bc$ to a Square, which } $gg \cdot bb :: kk \cdot aa$
 may be called bb , reduce likewise df to a Square, which }
 you may call kk , then the preceding Analogy will }
 be converted into this, } $g \cdot b :: k \cdot a$
 5. But the sides of proportional Squares are also Pro-
 portionals, therefore, from 4^o ,

In which last Analogy the three first Terms are right lines given, and therefore the fourth proportional line a sought shall be given also.

Many other ways might be shewn to resolv'd (or effect) most of the preceding Problems of this Chapter; but for brevity sake, I leave them to be found out by the industrious Learner, who by the help of those before deliver'd will also easily perceive how to solve other Problems of like nature: And now having explain'd all such things as are materially necessary by way of Preparation to the Resolution and Composition of Plane Problems, I shall proceed to Examples, which I have divided into four Classes or Forms, contain'd in the four following Chapters.

C H A P. VII.

The first Classis of Examples of the Resolution and Composition of Plane Problems, to wit, such whose Construction may be perform'd by drawing only Right and Circular lines.

IN which Examples, the Resolution ends either in an Analogy whose three first Terms are right lines known, and the fourth gives the right line sought; or else it ends in a simple Equation between the right line sought, and one or more right lines known. What is meant by Mathematical Resolution and Composition, I have hinted by Definitions in the beginning of *Chap. 1. Book I.* of this Treatise, and now I come to expound and illustrate the same by Examples, after I have recommended a few things by way of Caution and Direction to Learners.

First, Let the Analyst take care to understand the import and meaning of a Problem propounded, lest by too much haste he lose his labour, or be too forward in censuring the Proposer, when the fault is in himself, for many undertake to be Correctors of others, when they themselves have indeed more need of correction.

Secondly, Forasmuch as the most part of Problems propos'd in Geometrical Figures have need of Preparation, let the Analyst endeavour, before he begins the Algebraical Resolution, to find out as much as he can by the Synthetical Method, which proceeds by a Series of Consequences deduced altogether from known Quantities; and sometimes it will be convenient to premise one or more preparatory Propositions to render the Resolution of a Problem propos'd, the more simple and intelligible.

Thirdly, When the Resolution of a Geometrical Problem is begun, the like care must be taken to keep every step thereof in the simplest Terms, for avoiding Equations of higher Powers than the nature of the Problem requires, especially such as exceed Geometrical Dimensions; for example, in the Resolution of a Plane Problem, no Term of any Analogy or Equation ought to exceed two Dimensions, viz. every Term must be either a right Line or a Plane, for 'tis improper to introduce Solids in the Resolution or Composition of a Plane Problem.

Fourthly,

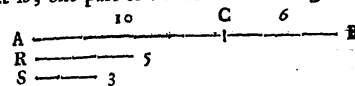
Fourthly, The Scope or aim of the Analyst in solving a Problem must be, first, to find out a Canon to direct how the Construction of the Problem may be effected by the Quantities given, and then the Construction being finish'd to form a Demonstration Synthetically, that may clearly prove the Problem to be fully satisfied. But although a Canon rightly found out by the Algebraick Art bids that only to be done which is possible, yet oftentimes in the Construction even of a Plane Problem, such objections will start up against the possibility of the Construction, as cannot be solved by any thing apparent either in the Canon, or in the proposition of the Problem: As, if a plain Triangle be to be made of three right lines, whereof one is rightly found out by Construction according to the direction of the Canon, and the other two are also discovered by the help of Quantities given and found out, yet before a Triangle can be made of those three right lines, it must be proved, that every two of them being joyn'd together as one right line, are longer than the third, which Proof may happen to be a more difficult work than the invention of the Canon: And therefore when any doubt ariseth concerning the possibility of any particular Construction, and it doth not clearly appear whether such Construction can be done or not, an ingenious Enquiry must be made to discover what is absolutely necessary to be given or granted to make that possible to be done, which the Problem requires, or the Canon bids to be done: Upon which Search, oftentimes one or more Determinations or Cautions will be found necessary to limit the Quantities given in the Problem, that its Construction may meet with no Impediment. Examples of Determinations will appear in divers Problems in this and the following Chapters.

Fifthly, After all necessary Determinations are premis'd, and the Construction of a Problem is finish'd, it remains to demonstrate that the Quantity or Quantities found out by the Construction will satisfy the Problem: But the Demonstration of the Solution of a Plane Problem, if its Construction be Algebraically found out in such manner that no Term of any Analogy or Equation in the Resolution exceeds Geometrical dimensions; may be formed by a repetition of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution; and the Demonstration of a Theorem may be formed by the steps of the Algebraick Resolution in a direct order, that is, by proceeding forward from the beginning to the end of the Resolution: All which will be copiously illustrated by the Resolutions and Compositions of Problems in this and the following Chapters.

Sixthly and lastly, I desire the Reader to take notice that in the Resolution of a Problem, I use the small Italian letters, a, b, c, d , &c. assuming always (some Vowels, a, e, o , &c. to represent a line sought, and Consonants, as b, c, d , &c. to signify lines given or known: But in the Composition of a Problem, that is, in its Construction and Demonstration, I use the Roman Capital letters, A, B, C, D , &c. to express things known, that the Resolution and Composition may be compared to one another without Confusion.

Problem I.

To divide a given right line into two parts which shall be in a given Reason, that is, one part to the other as two right lines given.



Suppos.

1. $b = AB$ a right line given to be cut into two parts.2. $r = R$ } the Terms of the given Reason of the parts sought.3. $s = S$ }

Req. to find

4. $AC \cdot CB$ such parts of AB , that $AC + CB = AB$, also $AC \cdot CB :: R \cdot S$.

Resolution.

5. For one of the parts sought put x .6. Therefore from 1^o and 5^o the other part is $b - x$.

7. And according to the tenour of the Problem propos'd,

8. There-

8. Therefore by *Composition of Reason converse*, (defin'd in *Self. 4. Chap. 3.*) $r + s : r :: b : a$.
Which last Analogy gives this

C A N O N.

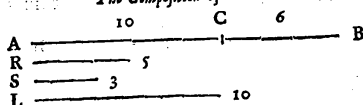
9. As the sum of the Terms of the given Reason is to the first Term; (that is, which of the two you please;) so is the line given to be divided, to one of the parts desired, which part subtracted from the line given to be divided leaves the other part.
10. *Note.* Although the Analogy in the seventh step may be converted into an Equation, (by comparing the Rectangle of the extremes to the Rectangle of the means,) from which, after due Reduction, the Analogy in the eighth step will arise, yet in Geometrical Demonstrations, which require a Contemplation upon Schemes or Figures, an Analogy in right lines is more simple, and easier to be understood than an Equation between Planes; or Solids; and therefore 'tis more usual with Geometricians in their Argumentations, to proceed as much as is possible from one Analogy to another, by Composition, Division, and other ways of arguing about Proportions, (defin'd in *Chap. 3. of this Book*;) that at length an Analogy may arise, when there is a possibility, wherein the three first Terms are given to find a fourth Proportional, which gives the Quantity sought; but there will be very often a necessity of converting an Analogy into an Equation, when known Quantities cannot be otherwise separated from unknown, as will hereafter appear by variety of Examples.

Concerning the Composition of a Geometrical Problem.

11. The *Composition* of a Problem consists of two parts, to wit, *Construction*, (or *Determination*;) and *Demonstration*; the former finds out that which is required to be done or found out, and the latter proves that that which is done or found out will satisfy the Problem propounded.

But before the *Construction* be begun, if the Problem be not universal, such Determinations (or Cautions) as are needful to limit the given Quantities, that the Problem may be possible must be annex to it, and the truth and reason of such Determinations made manifest; for 'tis the Office of him that undertakes to solve a Problem to determine what can, and what cannot be done; and if that which is required be possible, then to shew how, and how many ways it may be done: Now the Algebraical Art is an excellent Guide to shew the way leading to those ends; for first, the *Canon* resulting from the Resolution doth for the most part discover all such Determinations as are necessary to limit the given Quantities that the Problem may be possible, and directs also how its *Construction* may be made by working only with given Quantities. And lastly, if no Term of any Analogy or Equation in the Resolution exceeds Geometrical Dimensions, a Demonstration of the Solution of the Problem may be form'd out of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end of the Resolution to its beginning. But these things will best appear by Examples, and therefore I shall proceed to

The Composition of Probl. 1.



Suppos.

12. AB is a right line given to be cut into two parts.
13. R and S are the Terms of the given Reason of the parts sought.

Req. to find

14. AC and CB such parts of AB, that $AC + CB = AB$. Also,
15. $AC : CB :: R : S$.

Construction.

16. Let it be made (per *Probl. 8. Chap. 5.*) as $R + S$ to R, so AB to a fourth Proportional line, which may be called L, therefore

$$R + S : R :: AB : L.$$

17. From AB cut off $AC = L$, which is possible to be done if AB be greater than L, but

but $R + S$ the first Term of the last preceding Analogy is evidently greater than R the second Term, therefore (per *Schol. Prop. 14. Elem. 5.*) the third Term AB shall be greater than the fourth L; and consequently $AC = L$ may be cut off from AB. That done, the given line AB is divided in the point C into two parts AC, CB; which will satisfy the Problem. For first, $AC + CB = AB$, and that AC is to CB as R to S, I shall demonstrate by a retrograde repetition of the steps of the Resolution in manner following.

18. *Req. demonstr.* $R : S :: AC : CB$.

Demonstration.

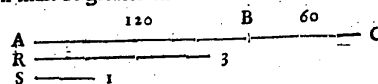
19. Because by *Constr.* in 16°, $R + S : R :: AB : L$.
20. And by *Constr.* in 17°, $AC = L$.
21. Therefore from 19°, by taking AC instead of L, $R + S : R :: AB : AC$.
That is, in 8°, (the last step of the Resolution,) $r + s : r :: b : a$.
22. Therefore from the Analogy in 21°, by *Division of Reason converse*, (defin'd in *Self. 8. Ch. 3.*) $R : S :: AC : CB$. *concluf.*
That is, in 7°, $r : s :: a : b - a$.

Which was to be done.

Note. In forming a *Demonstration* by a repetition of the steps of the Resolution in a backward order, it must be observed as a perpetual Rule, That when in the Resolution you pass forward from one step to another by *Composition of Reason*, in the *Demonstration* you are to return backward by *Division of Reason*; and when you pass by *Division of Reason* in the Resolution, you are to return by *Composition of Reason* in the *Demonstration*; also, *Addition* in the one, answers to *Subtraction* in the other. All which will be evident in the following Problems.

Probl. II.

To a given right line to add another right line, that the given with the added may have a given Reason to the line added. But the first Term of the Reason must be greater than the latter.



Suppos.

1. $b = AB$ a right line given to be increased.
2. $\{ r = R \}$ the Terms of the given Reason.
3. $\{ s = S \}$
4. BC a right line, such, that $AB + BC : BC :: R : S$.

Resolution.

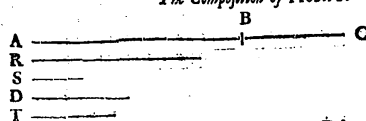
5. For the line sought put a .
6. Which added to the given line b makes $b + a$.
7. Then according to the import of the Problem, $r : s :: b + a : a$.
8. Therefore by *Division of Reason*, (defin'd in *Self. 7. Chap. 3.*) $r - s : s :: b : a$.

Hence this

Canon.

9. As the difference of the Terms of the given Reason is to the lesser Term, so is the line given to be increased, to the increase sought.

The Composition of Probl. 2.



$$\begin{aligned} AB &= 120 \\ R &= 3 \\ S &= 2 \\ D &= 1 \\ T &= 60 \\ BC &= 60 \end{aligned}$$

Fi

Suppos.

- Suppos.*
 10. AB is a right line given.
 11. R and S are the Terms of a given Reason.
 12. $R \sqsubset S$.
 13. $D = R - S$.

Req. to find
 14. BC a right line, such, that . . . $AB \vdash BC . BC :: R . S$.

Construction.
 15. Let it be made (per Probl. 8. Chap. 5.) as }
 D (or $R - S$) to S, so AB to a fourth pro- } $D . S :: AB . T$.
 portional line, call it T, therefore, . . . }

16. Let AB be continued to C, so, that . . . $BC = T$.
 Now the line BC (or T) being found out by the help of the given lines, AB, R and S, according to the direction of the Canon, we must shew that it will satisfy the Probl. therefore,
 17. . . . *Req. demonstr.* . . . $R . S :: AC . (AB \vdash BC) . BC$.

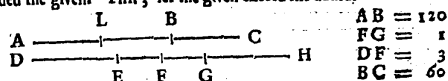
Demonstration.
 18. Because by Constr. in 15° and 16° , . . . $R - S . S :: AB . BC$.
 19. Therefore by Compos. of Reason, . . . $R . S :: AC . BC$.
 Which was to be done.

Note. In passing from the first step of this Demonstration, (which is the last step in the Resolution,) to the second, the Argumentation is made by Composition of Reason, because in passing to the last step of the Resolution from the last but one, it was argued by Division of Reason; agreeable to the Note at the end of the preceding Probl. 1.

Probl. III.

To a given right line to add another right line, that the Difference of the given and added may have a given Reason to their Summ. But the first Term of the Reason must be less than the latter Term.

This Problem hath two Cases; for either the given right line shall exceed the added, or the added the given. First, let the given exceed the added,



- Suppos.*
 1. $b = AB$ a right line given.
 2. $\begin{cases} r = FG \\ s = DF \end{cases}$ the Terms of the given Reason.
 3. $r \sqsupset s$.

Req. to find
 4. BC a right line, such, that $AB - BC . AB \vdash BC :: FG . DF$.

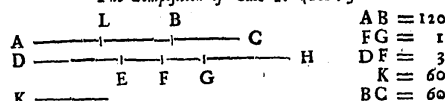
Resolution.
 5. For the line sought to be added put . . . a .
 6. Therefore the excess of the given line b above the line sought shall be . . . $b - a$.
 7. And the sum of the given line and the line sought shall be . . . $b + a$.
 8. Therefore, according to the tenour of the Problem propounded, this Analogy arithet, . . . $r . s :: b - a . b + a$.
 9. Therefore by Composition of Reason, . . . $s \vdash r . s :: 2b . b + a$.
 10. And by doubling the Consequents, . . . $s \vdash r . 2s :: 2b . 2b + 2a$.
 11. And inversly, . . . $2s . s \vdash r :: 2b \vdash 2a . 2b$.
 12. And by Division of Reason, . . . $s - r . s \vdash r :: 2a . 2b$.
 13. And inversly, . . . $s \vdash r . s - r :: 2b . 2a$.
 14. But a simple quantity is to a simple, as the double of the former to the double of the latter, therefore . . . $b . a :: 2b . 2a$.

15. And

15. And out of 13° and 14° , (per prop. 11. Elem. 5.) $s \vdash r . s - r :: b . a$.
 Hence this $CANON$.

16. As the summ of the Terms of the given Reason is to their difference, so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.
 17. *Note.* The line sought may easily be discovered by the Analogy in the ninth step; where the three first Terms being known, the fourth is known by Consequence; and since that fourth Term is evidently compos'd of the given line and the line sought, the given line subtracted from that known fourth Proportional shall necessarily give the line sought; whence 'tis manifest, that the Argumentation continued from the ninth step to the end of the Resolution is not of necessity, but only to shew how the line sought may be purely the fourth Proportional of an Analogy whose three first Terms are known, and consequently the line sought is known also: Which way of arguing by Analogies is more proper, (when it may be used,) than that by Equations, as hath before been hinted in Sect. 10. Probl. 1. of this Chapter.

The Composition of Case 1. Probl. 3.



- Suppos.*
 18. AB is a right line given.
 19. FG and DF are the Terms of a given Reason.
 20. $FG \sqsupset DF$.

Req. to find
 21. BC a right line, such, that . . . $AB - BC . AB \vdash BC :: FG . DF$.

Construction.
 22. Let it be made as $DF \vdash FG$ (that is, DG,) to $DF - FG$, so AB to a fourth proportional line, call it K, therefore, . . . $DF \vdash FG ; DF - FG :: AB : K$.
 23. Let AB be continued to C, so, that . . . $BC = K$.
 24. Now the line BC, or K being found out by the help of the given lines AB, DF, FG, (according to the direction of the Canon,) we must shew that it will satisfy the Problem, viz. that the difference of AB and BC is to their summ, as FG to DF; but this Analogy, (after I have premis'd a few things to contract the Demonstration,) I shall make manifest by a repetition of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution.

Prepar.
 25. From AB cut off $AL = BC = K$, which is possible to be done, for the first Term of the Analogy in 22° is evidently greater than the second, and therefore (per Schol. Prop. 14. Elem. 5.) the third Term AB shall be greater than the fourth K, or BC; suppose therefore
 26. Thence it follows that . . . $LB = AB - BC$.
 27. Let DF be continued to H, so, that . . . $DF = FH$.
 28. From DF cut off $FE = FG$, which is possible to be done, for by Supposition in 20° , $DF \sqsupset FG$; suppose therefore
 29. Therefore by subtracting the last Equation from that in 27° , . . . $DE = GH = DF - FG$.
 30. . . . *Req. demonstr.* . . . $FG . DF :: AB - BC . AB \vdash BC :: LB . AC$.

Demonstration.
 31. By Constr. in 22° and 23° , . . . $DF \vdash FG . DF - FG :: AB : BC$.
 That is in 15° , (the last step of the Resolution,) $s \vdash r . s - r :: b . a$.
 32. But there is the same Reason of the double to the double, as of the simple to the simple, therefore, . . . $2AB : 2BC :: AB : BC$.

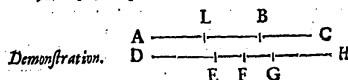
Ff 2

That

- That is, in the 14th step, . . . $2b \cdot 2a :: b \cdot a$.
33. Therefore out of 31° and 32° $DF+FG \cdot DE-FG :: 2AB \cdot 2BC$.
(*per Prop. 11. Elem. 5.*) $s+r \cdot s-r :: 2b \cdot 2a$.
34. Therefore inversely, . . . $DE-FG \cdot DF+FG :: 2BC \cdot 2AB$.
That is, in the 13th step, . . . $s-r \cdot s+r :: 2a \cdot 2b$.
35. Therefore by *Compos. of Reason*, . . . $2DF \cdot DE+FG :: 2AB+2BC \cdot 2AB$.
That is, in the 11th step, . . . $2s \cdot s+r :: 2b+2a \cdot 2b$.
36. Therefore inversely, . . . $DF+FG \cdot 2DF :: 2AB \cdot 2AB+2BC$.
That is, in the 10th step, . . . $s+r \cdot 2s :: 2b \cdot 2b+2a$.
27. Therefore by halving the Consequents, in the 9th step, . . . $DF+FG \cdot DF :: 2AB \cdot AB+BC$.
That is, in the 9th step, . . . $s+r \cdot s :: 2b \cdot b+a$.
38. Therefore by *Division of Reason*, . . . $FG \cdot DF :: AB-BC \cdot AB+BC$.
That is, in the 8th step, . . . $r \cdot s :: b-a \cdot b+a$.
39. Therefore from 38° and 26°, $FG \cdot DF :: LB \cdot AC$.
(*per Ax. 6. Chap. 2.*)

Which was to be Dem. Therefore the Problem is satisfied.

40. *Note.* Under every step of this Demonstration, I have set the correspondent step of the Resolution, that the Learner having respect to the *Note* at the end of *Probl. 1.* of this Chapter, may clearly perceive how the Demonstration is form'd out of the steps of the Resolution in a Retrograde order, that is, by returning backwards from the end to the beginning of the Resolution; for the first step in the Demonstration answers to the last in the Resolution, the second in the Demonstration, to the last but one in the Resolution, and so backwards in the Resolution, until the Analogy that was first assumed in the Resolution be positively and infallibly proved to be true. But after the Demonstration is in that manner discovered, the Algebraical steps must be omitted: So when the foregoing Demonstration beginning at the 31st step, is freed from the Analogies express'd by the small *Italian* letters belonging to the Resolution, and contracted by the help of the preparatory Equations in the 26th and 29th steps, respect also being had to the Diagram, there will arise this following



41. Because by *Constr.* in 22° and 23°, $DG \cdot DE :: AB \cdot BC$.
42. And by *prop. 15. Elem. 1.* $2AB \cdot 2BC :: AB \cdot BC$.
43. Therefore, *per prop. 11. Elem. 5.* $DG \cdot DE :: 2AB \cdot 2BC$.
44. And inversely, $DE \cdot DG :: 2BC \cdot 2AB$.
45. And by *Composition*, $2DF \cdot DG :: 2AC \cdot 2AB$.
46. And inversely, $DG \cdot 2DF :: 2AB \cdot 2AC$.
47. And by halving the Consequents, $DG \cdot DF :: 2AB \cdot AC$.
48. Wherefore by *Division of Reason*, $FG \cdot DF :: LB \cdot AC$.

Which in 30° was *Req. dem.*

49. Thus you have seen the first Case of *Probl. 3.* effected and demonstrated Synthetically, or by way of *Composition*, which argues altogether with known quantities; but the substance of the *Composition*, to wit, the *Construction and Demonstration*, was found out Analytically, or by way of *Resolution*, which from an Assumption of the quantity sought as if it were known or granted, together with the help of one or more known quantities, proceeds by Consequences, until in Conclusion the quantity so assumed or feigned to be known, is found equal to some quantity certainly known, and is therefore known also.

But it may be objected, that *Demonstrations* formed by the steps of Algebraical Resolution are for the most part rude and prolix; this I grant, but experience shews, that a *Demonstration* so found out may oftentimes be easily contracted, or, at least, give light to find out others more succinct and elegant. And since my purpose is, to shew the Learner a general and ready way of forming the *Demonstrations* of such *Theorems*, and *Solutions* of *Problems* as he finds out by ALGEBRA; when no Term of any Analogy or Equation

Equation in any step of the Resolution exceeds Geometrical Dimensions, I shall very seldom digress from the steps of the Resolution.

The Resolution of Case 2a. Probl. 3.

In this Case the line sought is suppos'd to exceed the given line AB.

50. For the given line AB put (as before) b .
51. And for the line sought, a .
52. Therefore the excess of the line sought above the given is $a-b$.
53. And the sum of both lines is $a+b$.
54. Therefore, according to the tenour of *Probl. 3.* this Analogy ariseth, $s \cdot s :: a-b \cdot a+b$.
55. Therefore by *Composition of Reason*, $s+r \cdot s :: 2a \cdot a+b$.
56. And by doubling the Consequents, $s+r \cdot 2s :: 2a \cdot 2a+2b$.
57. And inversely, $2s \cdot s+r :: 2a+2b \cdot 2a$.
58. And by *Division of Reason*, $s-r \cdot s+r :: 2b \cdot 2a$.
59. And because there is the same Reason of the simple to the simple, as of the double to the double, therefore $b \cdot a :: 2b \cdot 2a$.
60. And out of 58° and 59°, (*per prop. 11. Elem. 5.*) $s-r \cdot s+r :: b \cdot a$.

Hence this

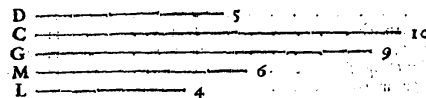
CANON.

61. As the Difference of the Terms of the given Reason is to their Summ, so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.

The *Composition* of this latter Case differing but little from the former, I shall leave it as an exercise to the Learner.

Probl. IV.

The difference of the extremes of three proportional right lines being given, as also the summ of the mean and lesser extreme, to find the Proportionals.



Suppos:

1. G, M, L $\frac{M}{L}$, viz. $G \cdot M :: M \cdot L$.

2. $G \subset L$.

3. $d = G-L$ is given.

4. $c = M+L$ is given.

Req. to find G, M, L:

Resolution.

5. For the lesser extreme Proportional put a .
6. Therefore out of 3° and 5°, the greater extreme is $d+a$.
7. And from 4° and 5°, the mean is $c-a$.
8. Therefore according to the tenour of the Problem, $d+a \cdot c-a :: b-a \cdot a$.
9. Therefore by *Composition of Reason*, $d+c \cdot c-a :: c \cdot a$.
10. And alternately, $d+c \cdot c :: c-a \cdot a$.
11. Wherefore by *Composition*, $d+2c \cdot c :: c+a \cdot a$.

Hence this

CANON.

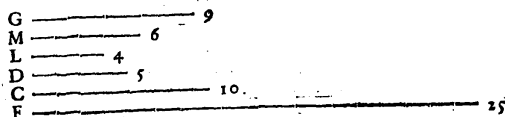
12. As the sum of the difference and the extremes of the double sum of the mean and lesser extreme, is to the sum of the mean and lesser extreme, so is the last mentioned sum to the lesser extreme. Therefore the lesser extreme sought is given.

13. After

13. After a Canon is found out by the Algebraical Art, it may be propounded in the form of a Theorem, whose Demonstration may be made by a repetition of the steps of the Resolution in a direct order, (not in a retrograde,) viz. by proceeding from the beginning to the end of the Resolution; as for example, the last preceding Canon may be proposed in the form of a Theorem, and demonstrated thus.

THEOREM 1.

14. If three right lines be Proportionals, the sum of the mean and lesser extreme, shall be a mean proportional between the lesser extreme, and the sum of the difference of the extremes and the double sum of the mean and lesser extreme.



Suppos.

15. $G, M, L \div \div$; viz. $G : M :: M : L$.
16. $G \sqsubset L$.

Prepar.

17. Make $D = G - L$, therefore $D + L = G$.
18. Make $C = M + L$, therefore $C - L = M$.
19. Make $F = D + 2C = G + 2M + L$.
20. . . . Req. demonstr. $F : C :: C : L$.

Demonstration.

21. By Suppos. in 15° , $G : M :: M : L$
22. Therefore out of $17^\circ, 18^\circ$ and 21° , by } $D + L : C - L :: C - L : L$
exchanging equal right lines, }
That is, in 8° , (the first Analogy in the } $d + a : c - a :: c - a : l$
Resolution,) }
23. Therefore from 22° , by Composition of } $D + C : C - L :: C : L$
Reason, }
That is, in 9° , $d + c : c - a :: c - a : l$
24. Therefore alternately, $D + C : C :: C - L : L$
That is, in 10° , $d + c : c :: c - a : a$
25. Therefore by Composition, $D + 2C : C :: C - L : L$
26. But by Constr. in 19° , $F = D + 2C$.
27. Therefore from 25° and 26° , $F : C :: C : L$

Which was to be Dem.

But that Demonstration, after the letters of the Resolution are cast away, may be compendiously reduced unto this that follows, respect being had to the Suppositions; and Preparation in $15^\circ, 16^\circ, 17^\circ, 18^\circ, 19^\circ$.

28. Req. demonstr. $G + 2M + L : M + L :: M + L : L$

Demonstration.

29. By Suppos. in 15° , $G : M :: M : L$
30. Therefore by Composition, $G + M : M :: M + L : L$
31. And alternately, $G + M : M + L :: M : L$
32. Wherefore by Constr. $G + 2M + L : M + L :: M + L : L$
Which was to be Dem.

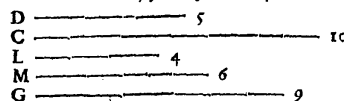
Hence arith.

THEOREM 2.

33. If three right lines be Proportionals, the sum of the mean and lesser extreme shall be a mean Proportional between the lesser extreme, and the aggregate of the sum of the extremes and double sum of the mean.

The

The Composition of Probl. 4.



Suppos.

34. $D =$ the difference of the extremes of three Proportionals is given.
35. $C =$ the sum of the mean and lesser extreme is given.

Req. to find the Proportionals.

Construction.

36. Let it be made (by Probl. 7. Chap. 5.) as } $D + 2C : C :: C : L$,
 $D + 2C$ to C , so C to a third proportional }
line, which suppose to be found L , therefore, }
37. By which Analogy, (and per Schol. Prop. 14. } $M + L = C$.
Elem. 5.) the mean C is greater than L , therefore }
find a right line M equal to $C - L$, thence }
it follows that $G - L = D$.
38. Find a right line G equal to $D + L$, therefore, } $G - L = D$.
39. So by the help of the given lines D and C , according to the direction of the Canon }
in the preceding twelfth step, three right lines are found out, to wit, L , M and G , }
which shall be the three Proportionals required. Now we must shew that they will }
satisfy the Problem. First then, 'tis manifest by Construction in 38° , that the difference }
of the extremes G and L is equal to the given difference D . Secondly, by Construction }
in 37° , the sum of the mean M and the lesser extreme L is equal to the given sum C . }
It remains only to prove that the said G, M and L are Proportionals, viz. that as G is }
to M , so M to L ; but this Analogy may be made manifest by a Repetition of the steps }
of the preceding Resolution in a retrograde order, that is, by returning backwards from }
the end to the beginning of the Resolution, in manner following.

40. Req. demonstr. $G : M :: M : L$

Demonstration.

41. Because by Constr. in 36° , $D + 2C : C :: C : L$.
That is, in 11° , (the last step of the Resolution,) } $d + 2c : c :: c : l$.
42. Therefore by Division of Reason, $D + C : C :: C - L : L$.
That is, in the tenth step, $d + c : c :: c - a : a$.
43. Therefore alternately, $D + C : C - L :: C : L$.
That is, in the ninth step, $d + c : c - a :: c : a$.
44. Therefore, by Division of Reason, $D + L : C - L :: C - L : L$.
That is, in the eighth step, $d + a : c - a :: c - a : a$.
45. And because by Constr. in 38° and 37° , $G = D + L$ Also, $M = C - L$.
46. Therefore out of 44° and 45° , by exchanging } $G : M :: M : L$
equal right lines,
Which was to be Dem. And therefore the Problem is satisfied.

Another way of resolving the foregoing Probl. 4.

47. The same things being supposed and given as before } $a = M$.
in $1^\circ, 2^\circ, 3^\circ$ and 4° of this Probl. put a for the mean }
proportional sought; viz. suppose } $c - a (= L)$
48. Therefore out of 4° and 47° the lesser extreme shall be } $c - a (= L)$
49. And by adding a the given difference of the extremes }
to the said lesser extreme $c - a$, the greater extreme } $d + c - a (= G)$
shall be }
50. Therefore according to the tenor of the Probl. this } $d + c - a : a :: a : c - a$.
Analogy aritheth out of $47^\circ, 48^\circ, 49^\circ$, $d + c : a :: c : c - a$.
51. Therefore by Composition of Reason, $d + c : c :: c : c - a$.
52. And alternately, $d + c : c :: c - a : a$.
53. And inversely, $d + c : c - a :: c : a$.
54. Therefore by Composition of Reason, $d + 2c : d + c :: c + a : a$.
Which

Which last Analogy gives this *CANON*.

55. As the aggregate of the difference of the extremes and the double sum of the mean and lesser extreme, is to the aggregate of the difference of the extremes and the sum of the mean and lesser extreme; so is the sum of the mean and lesser extreme to the mean. Therefore the difference of the extremes of three Proportionals being given, as also the sum of the mean and lesser extreme, the mean shall be also given by the Canon last exprest. The Demonstration whereof, and the Composition of the Problem according to this latter way of Resolution being very easie, I shall leave the same to the Learners exercise.

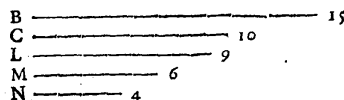
Probl. V.

The sum of the first and second of three Proportionals being given, as also the sum of the second and third, to find the Proportionals.

Suppos.

1. L, M, N are \div ; viz. $L : M :: M : N$.
2. $b = L - M$ is given.
3. $c = M - N$ is given.

Req. to find L, M, N.



Resolution.

4. Put a for the first Proportional sought, viz. $a = L$.
5. Therefore out of 2^d and 4th, the mean is $b - a (= M)$.
6. And by subtracting the said mean $b - a$ from the given sum c , the remainder gives the third Proportional, to wit, $c + a - b (= N)$.
7. Therefore (according to the *Probl.*) these must be Proportionals, viz. $a : b - a :: b - a : c + a - b$.
8. Therefore inversely, $b - a : a :: c + a - b : b - a$.
9. And by Composition of Reason, $b : a :: c : b - a$.
10. And alternately, $b : c :: a : b - a$.
11. And inversely, $c : b :: b - a : a$.
12. Therefore by Composition of Reason, $b + c : b :: b : a$.

Which last Analogy gives this

CANON.

13. As the aggregate of the sum of the first and second Proportionals and sum of the second and third, is to the sum of the first and second; so is the last mentioned sum to the first Proportional.

Therefore if the sum of the first and second of three Proportionals be given, as also the sum of the second and third, the mean shall be also given by the said Canon; whence also this

THEOREM.

14. If three right lines be Proportionals, the sum of the first and second is a mean Proportional, between the first, and the aggregate of the sum of the first and second, and sum of the second and third.

Which Theorem may easily be demonstrated by a repetition of the steps of the Resolution in a direct order, after the manner of demonstrating the Theorem in 14th of the foregoing *Probl.* 4. but for brevity sake I shall leave the Demonstration to the Learners practice, and proceed to the Composition of *Probl.* 5.

The Composition of Probl. 5.

Suppos.

15. $B =$ the sum of the first and second of three Proportionals is given.
16. $C =$ the sum of the second and third Proportionals is given.

Req.

Req. to find the Proportionals. [In the preceding Diagram.]

17. By *Probl. 7. Chap. 5.* let it be made as $B + C$ to B , so B to a third Proportional, suppose it be L , therefore

$$B + C : B :: B : L.$$

18. Make $M = B - L$, whence $L + M = B$, but that $B < L$; as that effecton requires, is manifest by *Construction* in 17th; for $B + C$ the first Term of the Analogy in 17th is greater than B in the second, and therefore (per *Schol. Prop. 14. Elem. 5.*) the third Term, which is also B , shall be greater than L the fourth, therefore 'tis possible to cut off from B the right line L , and a right line will remain, which may be called M .

19. Make $N = C + L - B$, which is possible to be done if $C + L < B$; but that $C + L < B$, I prove thus,

By *Constr.* in 17th, $B + C : B :: B : L$.

And therefore (per *Schol. Prop. 25. Elem. 5.*) $B + C + L < 2B$.

Consequently, by equal subtraction of B , $C + L < B$.

Which was to be proved. Therefore 'tis possible from the sum of the right lines C and L to cut off the right line B , and a right line will remain, which may be called N .

20. I say L , M and N are the three Proportionals required. Now we must shew that they will satisfy the Problem.

21. First then, the sum of the right lines L and M is (by *Construction* in 18th) equal to the given sum B .

22. Secondly, that the sum of the right lines M and N is equal to the given sum C , I prove thus,

By *Constr.* in 18th, $M = B - L$.

And by *Constr.* in 19th, $N = C + L - B$.

Therefore by adding the two last Equations together, $M + N = C$.

Which was to be proved. It remains to shew that the said three right lines L , M and N are Proportionals, but that will be made manifest by the following Demonstration, which is formed out of the preceding Resolution by a repetition of the steps thereof in a retrograde order, viz. by returning backwards from the end to the beginning of the Resolution.

23. $b : c :: L : M :: M : N$.

Demonstration.

24. Forasmuch as by *Construction* in 17th, $B + C : B :: B : L$.

That is, in 12th, (the last step of the *Resolut.*) $b + c : b :: b : a$.

25. Therefore by Division of Reason, $c : b :: b - a : a$.

That is, in 11th, $c : b :: b - a : a$.

26. Therefore inversely, $B : C :: L : B - L$.

That is, in 10th, $b : c :: a : b - a$.

27. Therefore alternately, $B : L :: c : B - L$.

That is, in 9th, $b : b - a :: c : b - a$.

28. And because by *Construction* in 18th, $M = B - L$.

29. Therefore out of 27th and 28th, $B : L :: C : M$.

30. And because it hath been proved in 18th that $B < L$.

31. Therefore out of 29th by Division of Reason, $B - L : L :: C - M : M$.

That is, in 8th, $b - a : a :: c + a - b : b - a$.

32. Therefore inversely, $L : B - L :: M : C - M$.

That is, in 7th, $a : b - a :: b - a : c + a - b$.

33. But by *Constr.* in 18th, $M = B - L$.

34. And by what hath been proved in 22th, $N = C - M$.

35. Therefore out of 32th, 33th and 34th, $L : M :: M : N$.

Which was to be Dem. and therefore the Problem is satisfied.

Another way of resolving the foregoing Probl. 5.

36. The same things being given and supposed, as before in 15th, $a = M$.

37. Therefore out of 2nd and 36th, the first Proportional shall be $b - a (= L)$.

38. And out of 3rd and 36th, the third Proportional shall be $c - a (= N)$.

G g

39. There-

39. Therefore out of 36° , 37° and 38° , according to the tenour of *Probl. 5*.
 40. Therefore inversely,
 41. And by *Composition of Reason*,
 42. And alternately,
 43. Wherefore by *Composition of Reason*,
 Which last Analogy gives this

C A N O N.

44. As the aggregate of the given sum of the first and second Proportionals, and the given sum of the second and third, is to the sum of the second and third, so is the sum of the first and second, to the mean Proportional sought.

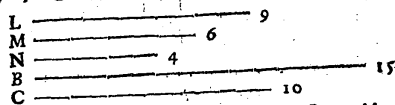
Which Canon, if it be propounded in the form of a Theorem, may be demonstrated by a repetition of the steps of the Resolution in a direct order. But leaving that and the Composition of *Probl. 5*, according to the latter Resolution, to the Learners exercise, I shall demonstrate the following Theorem by a repetition of the steps of the latter Resolution in a retrograde order.

T H E O R E M.

45. If three right lines be such, that the aggregate of the sum of the first and second and sum of the second and third, is to the sum of the second and third as the sum of the first and second, to the second: those three lines shall be Proportionals, viz. As the first is to the second, so is the second to the third.

Suppos.

46. L, M, N, are three right lines.
 47. $B = L + M$, whence $B - M = L$.
 48. $C = M + N$, whence $C - M = N$.
 49. $B + C = L + C :: B + M$.



50. . . . *Req. demonstr.* L, M, N are $++$, viz. $L : M :: M : N$.

Demonstration.

51. Because by *Suppos.* in 49°,
 52. Therefore by *Division of Reason*,
 53. And alternately,
 54. Therefore by *Division of Reason*,
 55. And inversely,
 56. But by *Suppos.* in 47°,
 57. And by *Suppos.* in 48°,
 58. Therefore out of 55° , 56° and 57° ,
 Which was to be dem.

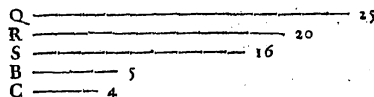
Probl. VI.

The difference of the greater extreme and mean of three Proportionals being given, as also the difference of the mean and lesser extreme, to find the Proportionals. But the first difference must be greater than the latter.

Suppos.

1. Q, R, S are $++$, viz. $Q : R :: R : S$.
 2. $Q - R$.
 3. $b = Q - R$ is given.
 4. $c = R - S$ is given.
 Reg. to find Q, R, S;

Q —



Resolution.

5. Put a for the mean Proportional sought, viz. $a = R$.
 6. Therefore out of 3° and 5° , the greater extreme shall be $a + b (= Q)$.
 7. And out of 4° and 5° , the lesser extreme shall be $a - c (= S)$.
 8. Therefore out of 6° , 5° and 7° , this Analogy will arise, (according to the import of the Problem,) viz. $a + b : a :: a - c$.
 9. Therefore by *Division of Reason*,
 10. And alternately,
 11. And by *Conversion of Reason*,
 12. And inversely,
 Which last Analogy gives this

C A N O N.

13. As the excess by which the given difference of the greater extreme and mean exceeds the given difference of the mean and lesser extreme, is to the difference of the greater extreme and mean; so is the difference of the mean and lesser extreme; to the mean Proportional sought, whence the extremes will be easily discovered.

Which Canon, if it be propounded in the form of a Theorem, may be easily demonstrated by a repetition of the steps of the Resolution in a direct order, but leaving that to the Learner's practice, I shall demonstrate the following Theorem by a retrograde repetition of the steps of the Resolution.

T H E O R E M 1.

14. If three right lines be such, that the excess by which the excess of the first above the second exceeds the excess of the second above the third, be to the excess of the second above the third, as the excess of the first above the second is to the second, then those three right lines shall be Proportionals, viz. As the first is to the second, so the second to the third.

Suppos.

15. Q, R, S are three right lines.
 16. $Q - R$.
 17. $R - S$.
 18. $B = Q - R$, whence $Q = B + R$.
 19. $C = R - S$, whence $S = R - C$.
 20. $B - C = C :: B : R$.
 21. . . . *Req. demonstr.* Q, R, S are $++$, viz. $Q : R :: R : S$.

Demonstration.

22. Because by *Suppos.* in 20°,
 23. Therefore alternately,
 24. And inversely,
 25. And by *Conversion of Reason*,
 26. And alternately,
 27. And by *Compof. of Reason*,
 28. But by *Supposition* in 18°,
 29. Also by *Suppos.* in 19°,
 30. Therefore out of 27° , 28° and 29° , by exchange of equal right lines,
 Which was to be Dem.

31. The Determination annex'd to *Probl. 6*. to wit, That the given difference of the greater extreme and mean must be greater than the given difference of the mean and lesser extreme, is discovered by the preceding Canon in 13°, and is necessarily to be prescribed for limiting the said differences, that they may be capable of constructing the Problem, as will be manifest by the subsequent

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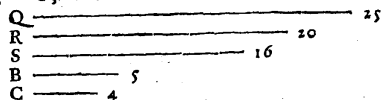
L E M M A.

L E M M A.

32. If three right lines be Proportionals, the difference of the greater extreme and mean, is greater than the difference of the mean and lesser extreme.

Suppos.

33. Q, R, S are \div ; viz. $Q : R :: R : S$.
 34. $Q \sqsubset R$, and consequently, $R \sqsubset S$.
 35. $B = Q - R$, whence $B + R = Q$.
 36. $C = R - S$, whence $R - C = S$.



37. . . . *Req. demonstr.* . . . $B \sqsubset C$.

Demonstration.

38. Because by *Suppos.* in 33° , $Q : R :: R : S$.
 39. And by *Suppos.* in 35° and 36° , $B + R = Q$ And $R - C = S$.
 40. Therefore out of 38° and 39° , $B + R : R :: R : R - C$.
 41. And by *Division of Reason*, $B : R :: C : R - C$.
 42. But $R \sqsubset R - C$, therefore from 41° , (per } . . . $B \sqsubset C$.
Schol. prop. 14. Elem. 5.)

Which was to be Demonstr.

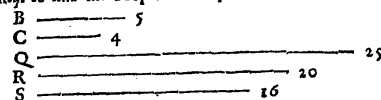
The Determination being demonstrated, I shall proceed to

The Composition of Probl. 6.

Suppos.

43. $B =$ the difference of the greater extreme and mean of three Proportionals is given.
 44. $C =$ the difference of the mean and lesser extreme is given.
 45. $B \sqsubset C$. (Determination.)

Req. to find the Proportionals.



Construction.

46. By *Probl. 8. Chap. 5.* let it be made as $B - C$ to B , so C to a fourth Proportional which may be called R , therefore,

$$B - C : B :: C : R.$$

which fourth Proportional R shall be greater than the third C , because the second B is greater than the first $B - C$.

47. Make $Q = R + B$, whence, $Q - R = B$.
 48. Make $S = R - C$, whence, $R - S = C$, which effect is possible, for by the Analogy in 46° , it is manifest that $R \sqsubset C$.
 49. I say Q, R and S are the three Proportionals required: Now we must shew that they will satisfy the Problem; First then, by *Construction* in 47° , the excess of Q above R is equal to the given difference B ; secondly, by *Constr.* in 48° , the excess of R above S is equal to the given difference C . So it remains only to prove that Q, R and S are Proportionals, in this order, viz. $Q : R :: R : S$, but that is made manifest by the subsequent Demonstration, which is form'd out of the preceding Resolution, by a repetition of the steps thereof in a retrograde (not in a direct) order.

50. . . . *Req. demonstr.* . . . $Q : R :: R : S$.

Demonstration.

51. Because by *Constr.* in 46° , which answers to the } $B - C : B :: C : R$
 last step of the Resolution, to wit, 12° ,
 52. Therefore inversely, $B : B - C :: R : C$.
 53. And by *Conversion of Reason*, $B : C :: R : R - C$.
 54. And alternately, $B : R :: C : R - C$.
 55. There-

55. Therefore by *Composition of Reason*, $B + R : R :: R : R - C$.
 56. But by *Constr.* in 47° , $Q = B + R$.
 57. And by *Constr.* in 48° , $S = R - C$.
 58. Therefore out of $55^\circ, 56^\circ, 57^\circ$, $Q : R :: R : S$.
 Which was to be dem. therefore that is done which was required by *Probl. 6*.

Another way of resolving the foregoing *Probl. 6*.

59. The same things being given and supposed as before }
 in $1^\circ, 2^\circ, 3^\circ, 4^\circ$ of *Probl. 6*. for the lesser extreme }
 of the three Proportionals sought, put a .
 60. To which lesser extreme if you add the given difference }
 c , it makes the mean, to wit, $a + c$.
 61. And by adding the given difference b to the mean }
 Proportional, it gives the greater extreme, to wit, }
 62. Therefore according to the import of the Problem, }
 these must be Proportionals, viz. $a : a + c :: a + c : a + c + b$.
 63. Therefore inversely, $a + c : a :: a + c + b : a + c$.
 64. And by *Division of Reason*, $c : a :: b : a + c$.
 65. And by *altern and inverse Reason*, $b : c :: a + c : a$.
 66. Wherefore by *Division of Reason*, $b - c : c :: c : a$.

Which last Analogy gives this

C A N O N.

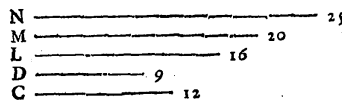
67. As the excess whereby the given difference of the greater extreme and mean exceeds the given difference of the mean and lesser extreme, is to the difference of the mean and lesser extreme; so is the difference last mentioned, to the lesser extreme sought: whence the mean and greater extreme are easily discovered.

Probl. VII.

The difference of the extremes of three Proportionals being given, as also a right line whose Square is equal to the difference between the Square of the mean and the Square of one of the extremes, to find the Proportionals.

Suppos.

1. N, M, L are \div ; viz. $N : M :: M : L$.
 2. $N \sqsubset L$.
 3. $d = N - L$ is given.
 4. $e = \sqrt{M^2 - L^2}$ is given; and consequently,
 5. $\alpha = M - L$ is given.

Req. to find N, M, L .

Resolution.

6. Put a for the lesser extreme sought, viz. $a = L$.
 7. Therefore out of 3° and 6° , the greater extreme shall be $a + d (= N)$.
 8. And out of 6° and 7° , the Rectangle contained under the extremes, or the Square of the mean, is equal to $aa + da$.
 9. And out of 6° , the Square of the lesser extreme is aa .
 10. Which Square aa being subtracted from $aa + da$, (to wit, }
 from the Square of the mean,) the remainder shall be the difference }
 of the said Squares, viz. da .
 11. But the difference last mentioned must be equal to the given }
 difference cc , therefore $da = cc$.
 12. Which Equation may be resolved into this Analogy, $d : c :: c : a$.

Hence

Hence this

CANON.

13. As the given difference of the extremes, is to the given right line whose Square is equal to the difference of the Squares of the mean and lesser extreme; so is the same right line to the lesser extreme fought.

Or thus,

As the given difference of the extremes, is to the given right line whose Square is equal to the difference of the Squares of the mean and greater extreme; so is the same right line to the greater extreme fought.

Hence this

THEOREM.

14. If three right lines be Proportionals, the difference of the Squares of the mean and lesser extreme is equal to the Rectangle contained under the difference of the extremes and the lesser extreme.

Or thus,

The difference of the Squares of the mean and greater extreme, is equal to the Rectangle contained under the difference of the extremes and greater extreme.

The Composition of Probl. 7.

Suppos.

15. D = the difference of the extremes of three Proportionals is given.
16. C = a right line, whose Square is equal to the difference of the Squares of the mean and lesser extreme is given.

Req. to find the Proportionals.

$$\begin{array}{rcl} N & \text{---} & 25 \\ M & \text{---} & 20 \\ L & \text{---} & 16 \\ D & \text{---} & 9 \\ C & \text{---} & 12 \end{array}$$

Construſion.

17. By Probl. 7. Chap. 5. let it be made as D to C, so C to a third Proportional, which suppose to be the right line L, therefore,
 $D : C :: C : L$

18. By Probl. 2. Chap. 5. find a right line, as M, such, that its Square may be equal to the Square of L together with the Square of C, therefore,

$$\square M = \square L + \square C;$$

And consequently, $\square M - \square L = \square C$.

19. Make $N = L + D$; whence, $N - L = D$.

20. I say N, M and L are the three Proportionals required. Now we must shew that they will satisfy the Problem.

21. First then, by Construſion in 19°, the excess of N above L is equal to D the given difference of the extremes.

22. Secondly, the excess by which the Square of the mean M exceeds the Square of the lesser extreme L, is (by Constr. in 18°) equal to the Square of C, to wit, the given difference of the Squares of the mean and lesser extreme.

23. It remains only, to prove that the said three right lines N, M and L are Proportionals, in this order, viz. As N is to M, so M to L; But that is made manifest by the subsequent Demonstration, which is formed by a retrograde repetition of the steps of the preceding Resolution.

18. . . . Req. demonstr. . . . N . M :: M . L.

Demonstration.

25. Because by Construſion in 17°, $D : C :: C : L$.
That is, in 12°, (the last step of the Resolut.) . . . $d : c :: c : a$.
26. Therefore, per 17. prop. 6. Elem. . . . $\square D, L = \square C$.
That is, in 11° . . . $da = cc$.
27. Again, because by Constr. in 19°, $N = L + D$.
28. Therefore (per prop. 1. Elem. 6.) by drawing L as a common altitude into each part of the Equation in 27°.
29. And consequently out of 28° and 26°, by exchange of equal Rectangles, . . . $\square NL = \square L + \square C$.

30. But

30. But by Constr. in 18°, . . . $\square M = \square L + \square C$.
Therefore out of 29° and 30°, per 1. Ax. Chap. 2. . . . $\square NL = \square M$.
31. Therefore out of 31°, per 14. prop. 6. Elem. . . . $N . M :: M . L$.

Which was to be Demonstr. therefore the Problem propounded is satisfied.

Having by the preceding Examples of Resolution and Composition given the Learner a taste of the manner of arguing by Analogies, which is the best way when the nature of a Problem will admit the same, I shall now proceed to Examples of arguing partly by Equations, and partly by Analogies. But it must be remembered, that when in the Resolution you pass from one step to another by Addition, in the Demonstration of the Problem you must return by Subtraction: For Addition in the Resolution requires Subtraction in the Composition, and Subtraction in the one, Addition in the other; also Composition of Reason in the one requires Division of Reason in the other, as before hath been said in the Note at the end of Probl. 1. of this Chapter.

Probl. VIII.

A right line equal to the sum of three proportional right lines being given, as also a right line whose Square is equal to the sum of the Squares of all the said Proportionals, to find out the Proportionals severally. But the first of those lines given must be greater than the latter, yet not greater than the right line arising out of the Application of the triple Square of the said latter line to the first.

$$\begin{array}{rcl} L & \text{---} & 8 \\ M & \text{---} & 4 \\ N & \text{---} & 2 \\ B & \text{---} & 14 \\ C & \text{---} & \sqrt{84} \end{array}$$

Suppos.

1. L, M, N $\frac{8}{2}$, viz. L . M :: M . N.
2. $b = L + M + N$ is given.
3. $c = \sqrt{\square L + \square M + \square N}$ is given; therefore,
4. $cc = \square L + \square M + \square N$ is given also.

Req. to find L, M, N.

Resolution.

5. For the mean Proportional fought put a, viz. $a = M$.
suppose
6. Therefore from 2° and 5°, the sum of the extremes shall be $b - a = L + N$.
7. Therefore the Square of the sum of the extremes is $bb + aa - 2ba = \square L + \square N + 2 \square LN$.
8. And the Square of the mean Proportional, or the Rectangle of the extremes is $aa = \square M = \square LN$.
9. Which Square or Rectangle in 8°, being subtracted from the Square in 7°, leaves the sum of the Squares of all the three Proportionals, viz. $bb - 2ba = \square L + \square M + \square N$.
10. But by Supposition in 4°, $cc = \square L + \square M + \square N$.
11. Therefore from 9° and 10°, (per Ax. 1. Chap. 2.) $bb - 2ba = cc$.
12. And by adding 2ba to each part of the Equation in 11°, this ariseth $bb = cc + 2ba$.
13. And by subtracting cc from each part of the Equation in 12°, $bb - cc = 2ba$.
14. And because (per Theor. 8. Chap. 4.) $bb - cc = b + c \times b - c$.
15. Therefore from 13° and 14°, (per Ax. 1. Chap. 2.) $2ba = b + c \times b - c$.
16. Therefore by resolving the last preceding Equation into Proportionals, it shall be $2b : b + c :: b - c : a$.

From

From the ninth step arith.

THEOREM 1.

17. If three right lines be Proportionals, the excess whereby the Square of their sum exceeds twice the Rectangle made of that sum and the mean proportional, shall be equal to the sum of the Squares of all the three proportionals.

From the last step of the Resolution arith.

THEOREM 2.

18. If three right lines be Proportionals, then this Analogy will attend them, viz. As the double sum of all the three Proportionals is to the simple sum increased with the side of a Square equal to the sum of the Squares of the three Proportionals; so is the excess whereby the sum of the three Proportionals exceeds the said side, to the mean Proportional.

Therefore the sum of three Proportionals being given, as also the sum of their Squares, the mean Proportional shall be given also by the preceding Theor. 2. whence the sum of the extremes is consequently given. And lastly, the sum of the extremes being given, as also the mean, the extremes shall be given severally, by Probl. 13. Chap. 5.

But to solve this Probl. 8. Arithmetically, the following Canon, (deducible from the 13th step,) will be more ready than Theor. 2.

CANON.

19. From the Square of the given sum of three Proportionals, subtract the given sum of their Squares, and divide the remainder by the double of the first given sum; so shall the Quotient be the mean Proportional; which subtracted from the sum of all three, leaves the sum of the extremes. And lastly, the sum of the extremes being given, as also the mean, the extremes shall be given severally, by Theor. in 21^o of Probl. 13. Chap. 5. But for the greater evidence, I shall demonstrate the truth of the preceding Theor. 1. and 2. and consequently the Canon, by the steps of the foregoing Resolution in a direct order, viz. by proceeding from the beginning to the end of the Resolution.

Suppos.

$$20. L, M, N \div \div, \text{ viz. } L : M :: M : N.$$

$$21. B = L + M + N.$$

$$22. C = \sqrt{L^2 + M^2 + N^2}.$$

$$23. \square C = \square L + \square M + \square N.$$

$$\begin{array}{r} L \quad \text{—————} 8 \\ M \quad \text{—————} 4 \\ N \quad \text{—————} 2 \\ B \quad \text{—————} 14 \\ C \quad \text{—————} \sqrt{84} \end{array}$$

$$24. \dots \text{Reg. demonstr.} \dots \left\{ \begin{array}{l} \text{Theor. 1. } \square B - 2 \square BM = \square C. \\ \text{Theor. 2. } 2B \cdot B + C :: B - C, M \end{array} \right.$$

Demonstration.

$$25. \text{By Suppos. in } 21^o, \dots B = L + M + N.$$

$$26. \text{Therefore by subtracting } M \text{ from each part, } B - M = L + N.$$

$$27. \text{And by squaring each part in } 26^o, \text{ this Equation will arise, (per Theor. 5, and 2. of Chap. 4.) } \square B + \square M - 2 \square BM = \square L + \square N + 2 \square LN.$$

$$28. \text{And from } 20^o, \text{ (per prop. 17. Elem. 6.) } \square M = \square LN.$$

$$29. \text{Therefore by subtracting the Equation in } 28^o \text{ from that in } 27^o, \text{ this will manifest, (per Ax. 9, & 6. Chap. 2.) } \square B - 2 \square BM = \square L + \square M + \square N.$$

$$30. \text{But by Suppos. in } 23^o, \dots \square C = \square L + \square M + \square N.$$

$$31. \text{Therefore from } 29^o \text{ and } 30^o, \text{ (per Ax. 1. Chap. 2.) } \square B - 2 \square BM = \square C.$$

Which was Theor. 1. to be demonstr.

$$32. \text{Again, by adding } 2 \square BM \text{ to each part of the Equation in } 31^o, \text{ this arith. } \square B = \square C + 2 \square BM.$$

33. And

33. And by subtracting $\square C$ from each part of the Equation in 32^o, $\square B - \square C = 2 \square BM.$
 34. Bar by Theor. 8. Chap. 4. $\square B - \square C = \square B - \square C \times \frac{B - C}{B}.$
 35. Therefore from 33^o and 34^o (per Ax. 1.) $2 \square BM = \square B - \square C \times \frac{B - C}{B}.$
 36. Therefore, (per prop. 14. Elem. 6.) $2B \cdot B + C :: B - C \cdot M.$

Which was Theor. 2. to be dem. Therefore the truth of both the preceding Theorems is made manifest; also the Canon in 19^o is evident from the Equation in 33^o, by Application of each part thereof unto 2 B.

37. I shall in the next place, in order to the Composition of Probl. 8. before propounded, demonstrate the Determination annex'd to it for limiting the lines given, that they may be capable of effecting the Problem.

$$\left\{ \begin{array}{l} B < C, \\ \text{Determination. } B \text{ not } < \frac{3 \square C}{B}. \end{array} \right.$$

That is, the line given for the sum of three Proportionals must be greater than that given right line whose Square is equal to the sum of the Squares of all the three Proportionals, yet not greater than the right line arising out of the Application of the triple of the said Square, to the right line given for the sum of the three Proportionals.

38. The Scope of the Determination is, to remove two Objections that may be brought against the Construction of the Problem (in the following 76th and 87th steps,) unless the given lines be limited as the Determination prescribes, whose first part, to wit, that $B < C$ is discovered by the last step of the Resolution, and already demonstrated. The latter part of the Determination, to wit, that the given line B ought not to be greater than $\frac{3 \square C}{B}$, is neither apparent in the proposition of the Problem, nor

in either of the Theorems resulting from the Resolution; but that it is a property adherent to three Proportionals, I shall demonstrate by the following Lemma, and afterwards shew that it is necessary to make the Problem possible.

LEMMA.

39. If three right lines be Proportionals, their sum shall sometimes be equal to the right line arising out of the Application of the triple sum of all their Squares to their said sum; sometimes less, but never greater than the right line arising by the said Application.
 40. Two Cases are to be demonstrated to prove this Lemma, for three Proportionals are either equal between themselves, or unequal. In the first Case, 'tis easy to perceive that the Square of the sum of three Proportionals is equal to the triple sum of their Squares; for supposing N, N, N to represent three Proportionals, their sum is 3N, the Square whereof is 9N², which is manifestly equal to 3N² + 3N² + 3N², to wit, the triple sum of the Squares of the said three equal Proportionals N, N, N, be applied to (or divided by) 3 N the sum of the same Proportionals, the line (or Quotient) arising by that Application shall necessarily be 3 N, (the sum of the said Proportionals.) Therefore the first Case of the Lemma is manifest.

Suppos. in Case 2.

$$41. L, M, N \div \div, \text{ viz. } L : M :: M : N.$$

$$42. L < M, \text{ and consequently } M < N.$$

Prepar.

$$43. \text{By Probl. 1. Chap. 5. make } B = L + M + N.$$

$$44. \text{By Probl. 2. Chap. 5. make } C = \sqrt{L^2 + M^2 + N^2}.$$

$$45. \text{Thence it follows, that } \square C = \square L + \square M + \square N.$$

$$46. \text{By Probl. 8. Chap. 5. let it be made, } B : C :: 3C \text{ (to a fourth) } T.$$

$$47. \text{Thence it follows, that } T = \frac{3 \square C}{B}.$$

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number; then half the sum of those two numbers and half their difference shall be the Sides or Roots of the two Squares sought.

As, if 5 be given for the difference of two Squares sought, I take 5 and 1; for the Product of their multiplication is 5; then the half of their sum is 3, and the half of their difference is 2; lastly, the Squares of the said 3 and 2 are 9 and 4, the Squares sought; for their difference is 5, as was prescribed.

Again, the same number 5 being given for the difference of two Squares, take a number at pleasure, as 2, by this divide the given number 5, the Quotient is $\frac{5}{2}$; therefore the Product of the multiplication of the Divisor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the sum and half the difference of the said 2 and $\frac{5}{2}$, to wit, $\frac{9}{2}$ and $\frac{1}{2}$ shall be the sides of the Squares sought; and consequently the Squares themselves are $\frac{81}{4}$ and $\frac{1}{4}$, whose difference is 5, as was desired.

After the same manner innumerable pairs of Squares may be found out in Rational numbers, and the difference of each pair shall be equal to one and the same given number.

The reason of the Canon may be made manifest by this

Theorem.

The Product made by the multiplication of any two unequal numbers is equal to the difference of two Squares, to wit, of the Square of half the sum, and the Square of half the difference of the same two unequal numbers.

As, if a be the greater, and b the lesser of two numbers, then

The Square of $\frac{1}{2}a + \frac{1}{2}b$ is $\frac{1}{4}aa + \frac{1}{4}ab + \frac{1}{4}bb$,

The Square of $\frac{1}{2}a - \frac{1}{2}b$ is $\frac{1}{4}aa - \frac{1}{4}ab + \frac{1}{4}bb$,

The difference of those two Squares is ab .

Which difference is manifestly the Product of the multiplication of the two proposed numbers a and b . Wherefore the Theorem, and consequently the Canon first given is manifest.

The Definition of Binomial I.

When the greater Name (or part) of a Binomial is a Rational number, and the lesser part is a Surd square Root of some Rational number, and the square Root of the difference of the Squares of the parts is a Rational number, the sum of the two parts is called a First Binomial.

Explication.

Let this Binomial be proposed, $3 + \sqrt{5}$

The Squares of the Names, or parts, are $\begin{matrix} 9 \\ 5 \end{matrix}$

The difference of those Squares is 4

The square Root of that difference is 2

Because the greater part 3 is a Rational number, and the lesser part $\sqrt{5}$ is a Surd square Root of a Rational number 5, and the difference of the Squares of the parts, viz. 4, is a Square whose Root 2 is a Rational number; the Binomial proposed, to wit, $3 + \sqrt{5}$ is called a First Binomial.

How to find out two such numbers as may constitute a First Binomial.

1. By the Canon of the preceding Question at the beginning of this 15. *Self.* find out two Square numbers whose difference may be some Rational number not a Square, such are these Squares, 9 and 4 .
2. Their difference is 5 .
3. Take some Rational number at pleasure for the greater part of the Binomial sought, as 6 .
4. Then say, by the Rule of Three, If 9 the greater of the two Squares found out in the first step, give 5 the difference in the second, what shall 36 the Square of the number taken in the third give? whence the fourth Proportional will be found 20, the square Root whereof is the lesser part, to wit, $\sqrt{20}$.
5. I say, the sum of the two numbers found out in the third and fourth steps, is a first Binomial, to wit, $6 + \sqrt{20}$.

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The Definition of Binomial II.

When the lesser part of a Binomial is a Rational number, and the greater part is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the parts is Communurable to the greater part; the sum of the two Parts is called a Second Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{18} + 4$

The Squares of the Parts are $\begin{matrix} 18 \\ 16 \end{matrix}$

The Difference of those Squares is 2

The square Root of that Difference is $\sqrt{2}$

Because the lesser Part 4 is a Rational number, and the greater Part $\sqrt{18}$ is the Surd square Root of a Rational number 18, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{2}$, is Communurable to the greater Part $\sqrt{18}$; (for according to the Definition in *Self.* 7. of this *Chapt.* $\sqrt{2} \cdot \sqrt{18} :: 1 : 3$, that is, as a Rational number to a Rational number;) the proposed number $\sqrt{18} + 4$ is a Second Binomial.

How to find out two such numbers as may constitute a Second Binomial.

1. By the foregoing Canon find out two square numbers whose Difference may be some Rational number not a Square; such are these Squares, 9 and 4 .
2. Their Difference is 5 .
3. Take some Rational number at pleasure for the lesser Part of the Binomial sought, as 10 .
4. Then say, If 5 the Difference in the third step, gives 9 the greater of the two Squares in the first, what shall 100 the Square of the number taken in the third give? whence you will find 180, whose square Root shall be the greater Part, viz. $\sqrt{180}$.
5. I say the sum of the two numbers found out in the third and fourth steps is a Second Binomial, viz. $\sqrt{180} + 10$.

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the Parts is Communurable to the greater Part; the sum of the two Parts is called a Third Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{50} + \sqrt{32}$

The Squares of the Parts are $\begin{matrix} 50 \\ 32 \end{matrix}$

The Difference of those Squares is 18

The square Root of that Difference is $\sqrt{18}$

Because the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd square Roots of two Rational numbers 50 and 32, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{18}$, is Communurable to the greater Part $\sqrt{50}$; (for $\sqrt{18} \cdot \sqrt{50} :: 3 : 5$, that is, as a Rational number to a Rational number;) the proposed number $\sqrt{50} + \sqrt{32}$ is a Third Binomial.

How to find out two such numbers as may constitute a Third Binomial.

1. Find out two Square numbers whose Difference may be some Rational number not a Square, such are these Squares, 9 and 4 .
2. Their Difference is 5 .
3. Take some Rational number not a Square, which may exceed the said Difference 5 by an Unit or two, viz. by 1, when the said Difference increased with 1 makes not a Square; but by 2, when the Difference increased with 1 makes a Square: so in this Example, I take 6, because $5 + 1$ makes not a Square, $5 + 2$ makes a Square.
4. Again, take some Rational number at pleasure, as 12 .

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5. The Square thereof is 144
 6. Then say, If 6 the number taken in the third step, gives 9 the greater of the two Squares in the first; what shall 144 the square number in the fifth give? whence the fourth Proportional is 216, whose square Root, to wit $\sqrt{216}$ shall be the greater Part;
 7. Say again, If the said Square 9 gives 5 the Difference in the second step; what shall 216 the fourth Proportional found out in the sixth give? whence you will find 120, whose square Root, to wit $\sqrt{120}$ shall be the lesser Part;
 8. I say, the sum of the two numbers found out in the sixth and seventh steps is a third Binomial, to wit, $\sqrt{216} + \sqrt{120}$

The Definition of Binomial IV.

When the greater Part of a Binomial is a Rational number, and the lesser Part is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part; the sum of the two Parts is called a Fourth Binomial.

Explication.

Let this Binomial be proposed, $5 + \sqrt{12}$
 The Squares of the Parts are 25 and 12
 The Difference of those Squares is 13
 The square Root of that Difference is $\sqrt{13}$

Because the greater Part 5 is a Rational number, and the lesser Part $\sqrt{12}$ is a Surd square Root of a Rational number 12; and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{13}$, is Incommensurable to the greater Part 5; (for $\sqrt{13}$ hath not such proportion to 5 as a Rational number to a Rational number;) the number $5 + \sqrt{12}$ above proposed is a Fourth Binomial.

How to find out two such numbers as may constitute a Fourth Binomial.

1. Take any square number, as 9
 2. Divide that square number 9 into two numbers not Squares, as into 6 and 3
 3. Take a Rational number at pleasure for the greater Part of the Binomial sought, as 5
 4. Then say, If 9 the square number in the first step, give 6 the greater of the two numbers in the second; what shall 36 the Square of the number taken in the third give? so the fourth Proportional will be found 24, whose square Root, to wit $\sqrt{24}$, shall be the lesser Part;
 5. I say, the sum of the two numbers found out in the third and fourth steps, is a Fourth Binomial, viz. $5 + \sqrt{24}$

The Definition of Binomial V.

When the lesser Part of a Binomial is a Rational number, and the greater Part is a Surd square Root of some Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part; the sum of the two Parts is called a Fifth Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{6} + 2$
 The Squares of the Parts are 6 and 4
 The Difference of those Squares is 2
 The square Root of that Difference is $\sqrt{2}$

Because the lesser Part 2 is a Rational number, and the greater Part $\sqrt{6}$ is a Surd square Root of a Rational number 6, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{2}$, is Incommensurable to the greater Part $\sqrt{6}$; (for $\sqrt{2} \cdot \sqrt{6} :: 1 \cdot \sqrt{3}$, not as a Rational number to a Rational number,) the proposed number $\sqrt{6} + 2$ is a Fifth Binomial.

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How to find out two such numbers as may constitute a Fifth Binomial.

1. Take any square number, as 9
 2. Divide that square number 9 into two numbers not Squares, as into 6 and 3
 3. Take a Rational number at pleasure for the lesser Part of the Binomial sought, as 2
 4. Then say, If 6 the greater of the two numbers in the second step, gives 9 the square number in the first; what shall 4 the Square of the Rational number taken in the third give? whence you will find the fourth Proportional 6, whose square Root, to wit, $\sqrt{6}$, shall be the greater Part sought;
 5. I say, the sum of the two numbers found out in the third and fourth steps is a Fifth Binomial, viz. $2 + \sqrt{6}$

The Definition of Binomial VI.

When each of the two Parts of a Binomial is a Surd square Root of some Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part; the sum of the two Parts is called a Sixth Binomial.

Explication.

Let this Binomial be proposed, $\sqrt{5} + \sqrt{3}$
 The Squares of the Parts are 5 and 3
 The Difference of the Squares of the Parts is 2
 The square Root of that Difference is $\sqrt{2}$

Because the two Parts $\sqrt{5}$ and $\sqrt{3}$ are Surd square Roots of two Rational numbers 5 and 3, and the square Root of the Difference of the Squares of the Parts, viz. $\sqrt{2}$, is Incommensurable to the greater Part $\sqrt{5}$; (for $\sqrt{2}$ hath not such proportion to $\sqrt{5}$ as a Rational number to a Rational number;) the number $\sqrt{5} + \sqrt{3}$ above proposed is a Sixth Binomial.

How to find out two such numbers as may constitute a Sixth Binomial.

1. Take two such Prime numbers that their sum may not be a Square, as 7 and 3
 2. Their sum is 12
 3. Take also any square number, as 9
 4. Take again some Rational number at pleasure, as 6
 5. The Square thereof is 36
 6. Then say, If 9 the square number taken in the third step, gives 12 the sum of the two Prime numbers in the first; what shall 36 the Square in the fifth step give? whence you will find 48, whose square Root, to wit, $\sqrt{48}$, shall be the greater Part;
 7. Say again, If 12 the sum of the two Prime numbers in the first step, gives 7 the greater of those Prime numbers; what shall 48 the fourth Proportional found out in the sixth step give? whence you will find 28, whose square Root, viz. $\sqrt{28}$ shall be the lesser Part;
 8. I say, the sum of the two numbers found out in the sixth and seventh steps is a Sixth Binomial, viz. $\sqrt{48} + \sqrt{28}$

If of every one of those six Binomials the lesser Part be subtracted from the greater, by interposing the sign $-$, the six Remainders answer to the six Lines which Euclid in Prop. 86, 87, 88, 89, 90, 91. of his Tenth Elem. calls Apotomes or Residual lines; as,

I.	$3 - \sqrt{5}$	By changing $+$ into $-$, is made Residual.	I.	$3 - \sqrt{5}$
II.	$\sqrt{18} - 4$		II.	$\sqrt{18} - 4$
III.	$\sqrt{50} - \sqrt{32}$		III.	$\sqrt{50} - \sqrt{32}$
IV.	$5 - \sqrt{12}$		IV.	$5 - \sqrt{12}$
V.	$\sqrt{6} - 2$		V.	$\sqrt{6} - 2$
VI.	$\sqrt{5} - \sqrt{3}$		VI.	$\sqrt{5} - \sqrt{3}$

The precedent Constructions of the said six Binomials are demonstrated in Prop. 49, 50, 51, 52, 53, 54. of 10. Elem. Euclid.

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Now

Now if any Binomial or Residual be given, we may easily find out another of the same kind in this manner, *viz.* For the first and fourth Binomials, if it be made as the greater Name or Part to the lesser, so any Rational number assumed for the greater Part of a new first or fourth Binomial, to a fourth Proportional number, this number shall be the lesser Part of the new first or fourth Binomial. But for the second and fifth, if it be made as the lesser Part to the greater, so any Rational number taken for the lesser Part of a new second or fifth Binomial to a fourth Proportional, the number so produced shall be the greater Part of the new second or fifth Binomial. And lastly, for the third and sixth Binomials, if it be made as the greater Part to the lesser, (each of which is a Surd square Root,) so any Surd square Root assumed for the greater Part of a new third or sixth Binomial, to a fourth Proportional, there will come forth the lesser Part of a new third or sixth Binomial. (The reason of this Operation is manifest, *per Prop. 15. Elem. 10. Euclid.*) And, after a new Binomial is found out, its correspondent Residual is also made, by changing the sign $+$ into $-$, as before hath been said.

As, for example, if a first Binomial $3 + \sqrt{5}$ be proposed, to find another like to it; I take a Rational number at pleasure, as 8, for the greater Part of the Binomial sought; then by the Rule of Three, as 3 is to $\sqrt{5}$, so 8 to a fourth Proportional, to wit, $\sqrt{\frac{128}{3}}$, for the lesser Part sought, therefore $8 - \sqrt{\frac{128}{3}}$ shall be a new first Binomial, and $8 - \sqrt{\frac{128}{3}}$ a new first Residual; and so of the rest.

SECT. XVI. Concerning the extraction of the Square Root out of Binomials and Residuals constituted in such manner as hath been shewn in the preceding Sect. 15.

Every one of the Binomials and Residuals whose Construction hath been shewn in the preceding Sect. 15. hath a square Root, that is, such a Binomial or Residual that if it be multiplied into it self will produce the given Binomial or Residual, as may be evidently collected out of Prop. 55, 56, 57, 58, 59, and 60. Also out of Prop. 92, 93, 94, 95, 96, and 97. of the Tenth Book of Euclid's Elements.

As, for example, a Binomial of the first kind, suppose $7 + \sqrt{48}$ hath a square Root, to wit, $2 + \sqrt{3}$; for this being squared (or multiplied into it self) produceth that Binomial $7 + \sqrt{48}$; whose greater Part 7 is composed of 4 and 3 the Squares of the Parts of the Root $2 + \sqrt{3}$; and the lesser Part $\sqrt{48}$ is the double of the Product made by the multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2 + \sqrt{3}$: all which is evident by the multiplication of $2 + \sqrt{3}$ into it self. The like effect will be found in every one of the rest of the Binomials constituted in the preceding Sect. 15. Therefore if a Binomial be proposed, and its square Root desired, there is given the sum of the Squares of the Parts of the Root, (which sum is the greater Part of the Binomial proposed;) and the double of the Product of the Parts of the Root, (which double Product is the lesser Part of the Binomial proposed,) to find out the two Parts of the Root severally. And therefore in order to the Extraction of the square Root of a Binomial, it will be requisite to search out a Canon for the solving of this following

QUEST.

The sum of the Squares of two numbers being given; as also (c) the double Product of the multiplication of the same two numbers; to find the numbers severally.

RESOLUTION.

1. For one of the two numbers sought put a
 2. Then for as much as the double of the Product of their multiplication is given c, therefore the Product it self is $\frac{c}{2}$
 3. Which Product divided by the first number a gives the other number $\frac{c}{2a}$
 4. Therefore the Square of the first number is a^2
 5. And the Square of the other number is $\frac{cc}{4aa}$
 6. Therefore the sum of the Squares of the two numbers is $aa + \frac{cc}{4aa}$
7. Which

7. Which sum must be equal to $\frac{1}{2}$, the given sum of the Squares; hence this Equation; $aa + \frac{cc}{4aa} = b$
8. From which Equation, after due Reduction, there will arise $baa - aaaa = \frac{1}{2}cc$
9. And from the last Equation (*per Canon in Sect. 10. Chap. 15. Book 1.*) there will arise this following Canon, to find out the two numbers sought, *viz.*

CANON 1.

$$\sqrt{\frac{b}{2} + \sqrt{\frac{1}{4}bb - \frac{1}{4}cc}} = \text{the greater number,}$$

$$\sqrt{\frac{b}{2} - \sqrt{\frac{1}{4}bb - \frac{1}{4}cc}} = \text{the lesser number.}$$

That is, in words,

From a quarter of the Square of the given sum of the Squares; subtract a quarter of the Square of the double Product given; then add and subtract the square Root of that Remainder to and from half the given sum of the Squares; so shall the square Roots of the Summ and Remainder of that Addition and Subtraction be the two numbers sought:

$$10. \text{ Moreover, because } \frac{b + \sqrt{bb - cc}}{2} = \frac{1}{2}b + \sqrt{\frac{1}{4}bb - \frac{1}{4}cc}$$

$$11. \text{ Therefore, } \sqrt{\frac{b + \sqrt{bb - cc}}{2}} = \sqrt{\frac{1}{2}b + \sqrt{\frac{1}{4}bb - \frac{1}{4}cc}}$$

$$12. \text{ Likewise because } \frac{b - \sqrt{bb - cc}}{2} = \frac{1}{2}b - \sqrt{\frac{1}{4}bb - \frac{1}{4}cc}$$

$$13. \text{ Therefore, } \sqrt{\frac{b - \sqrt{bb - cc}}{2}} = \sqrt{\frac{1}{2}b - \sqrt{\frac{1}{4}bb - \frac{1}{4}cc}}$$

14. Therefore from the eleventh and thirteenth steps another Canon ariseth to solve the Question; *viz.*

CANON 2.

$$\sqrt{\frac{b + \sqrt{bb - cc}}{2}} = \text{the greater number,}$$

$$\sqrt{\frac{b - \sqrt{bb - cc}}{2}} = \text{the lesser number.}$$

That is, in words,

From the Square of the given sum of the Squares subtract the Square of the double Product given; then add and subtract the square Root of the Remainder to and from the given sum of the Squares: so shall the square Root of half the Summ and Remainder of that Addition and Subtraction be the two numbers sought.

By the help of either of those Canons we may extract the square Root of a Binomial or Residual; but I shall use the latter only, whence ariseth

A General Rule for the Extraction of the Square Root out of Binomials and Residuals:

From the Square of the greater part of a given Binomial or Residual, subtract the Square of the lesser; then add the square Root of the Remainder to the greater part; and subtract it also from the same; lastly, connect the square Roots of the half of that Summ and Remainder by the sign $+$ if a Binomial be proposed, but by $-$ if a Residual; so you have the desired square Root of the given Binomial or Residual.

The practice of this Rule will be shewn in the following Examples.

Example 1.

Let it be required to extract the square Root of this first Binomial, $27 + \sqrt{704}$

The Operation.

1. From the Square of the greater part 27, *viz.* from 729
2. Subtract the Square of the lesser part $\sqrt{704}$, to wit, 704
3. The Remainder is 25
4. The square Root of that Remainder is 5

5. To which square Root add the greater part 27
6. The sum is 32
7. The half of that sum is 16
8. The square Root of the said half sum is the greater part of the Root fought, to wit, 4
9. Then from the greater part of the given Binomial, viz. from 27
10. Subtract the square Root before found in the fourth step, to wit, 5
11. The Remainder is 22
12. The half of which Remainder is 11
13. The square Root of the said half Remainder is the lesser part of the Root fought, to wit, $\sqrt{11}$
14. I say, the two Names or parts in the eighth and thirteenth steps being connected by $+$ shall be the square Root fought, to wit, $4 + \sqrt{11}$

But if $-$ instead of $+$ be prefix to the lesser part of the said Root, it will give $4 - \sqrt{11}$, which is the square Root of the first Residual or Apotome $27 - \sqrt{704}$.

The former of those two Roots answers to the Irrational line called (in Prop. 37, and 55. lib. 10. Elem. Euclid.) a Binomial line, and the latter answers to the Irrational line called (in Prop. 74, and 92.) an Apotome or Residual line.

The Proof of the Root above extracted out of the first Binomial, is made by multiplying the Root into it self; thus,

- The sum of the Squares of the parts of $4 + \sqrt{11}$, 16 + 11, that is, 27
 the Root found out is
 The Product of the same parts multiplied one into the other is $4\sqrt{11}$, that is, $\sqrt{176}$
 The double of the said Product is $8\sqrt{11}$, that is, $\sqrt{704}$
 The sum of the said sum of the Squares of the parts and the double Product is $27 + \sqrt{704}$

Whence it is manifest that $27 + \sqrt{704}$ is the Square of $4 + \sqrt{11}$, therefore this is the true square Root of that first Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the Parts, the Remainder $27 - \sqrt{704}$ is the Square of $4 - \sqrt{11}$; therefore this is the square Root of that first Residual.

Example 2.

Let it be required to extract the square Root out of this second Binomial $\sqrt{12} + 6$

The Operation.

1. From the Square of the greater part $\sqrt{12}$, viz. from 12
2. Subtract the Square of the lesser part 6, to wit, 36
3. The Remainder is 24
4. The square Root of that Remainder is $\sqrt{24}$
5. To which square Root add the greater part, (by the Rule in Sect. 8. of this Chap.) $\sqrt{12}$
6. The Sum is $\sqrt{48}$
7. The half of which Summ is $\sqrt{12}$
8. The square Root of that half Summ is the greater part of the Root fought, to wit, $\sqrt{(4)12}$
9. Again, from the greater part of the given Binomial, viz. from $\sqrt{12}$
10. Subtract the square Root before found in the fourth step, (by the said Rule in Sect. 8.) viz. $\sqrt{12}$
11. The Remainder is $\sqrt{27}$
12. The half of which Remainder is, $\sqrt{27}$
13. The square Root of the said half Remainder is the lesser part of the Root fought, to wit, $\sqrt{(4)2}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by the sign $+$ shall be the Root fought, to wit, $\sqrt{(4)12} + \sqrt{(4)2}$

And if $-$ instead of $+$ be prefix to the lesser part of the said Root, it will give $\sqrt{(4)12} - \sqrt{(4)2}$, which is the square Root of the second Residual $\sqrt{12} - 6$.

The

The former of those two Roots answers to the irrational line called (in Prop. 38, & 56. lib. 10. Elem. Euclid.) a first Binomial; and the latter answers to the irrational line called (in Prop. 75, & 93.) a first Medial Residual.

The Proof of the Root above extracted out of the second Binomial.

- The Squares of the Parts of $\sqrt{(4)12} + \sqrt{(4)2}$ the Root found out, are $\sqrt{12}$ and $\sqrt{2}$
 Which Squares added together, (as in Example 6. Sect. 8. of this Chap. is manifest,) makes the sum $7\sqrt{2}$, that is, $\sqrt{12}$
 The Product of the Parts, viz. $\sqrt{(4)12}$ into $\sqrt{(4)2}$ is $\sqrt{(4)8}$; that is, 3
 The double of the said Product is 6
 Therefore the sum of the sum of the Squares of the Parts and the said double Product is $\sqrt{12} + 6$

Whence it is manifest that $\sqrt{12} + 6$ is the Square of $\sqrt{(4)12} + \sqrt{(4)2}$; therefore this is the true square Root of that second Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the Parts, the Remainder $\sqrt{12} - 6$ is the Square of $\sqrt{(4)12} - \sqrt{(4)2}$; therefore this is the square Root of that second Residual.

Example 3.

Let it be required to extract the square Root of this third Binomial $\sqrt{12} + \sqrt{80}$

The Operation.

1. From the Square of the greater part $\sqrt{12}$, viz. from 12
2. Subtract the Square of the lesser part, to wit, 80
3. The Remainder is 68
4. The square Root of that Remainder is $\sqrt{68}$
5. To which square Root add the greater part $\sqrt{12}$
6. The sum is $\sqrt{12} + \sqrt{68}$
7. The half of which sum is $\sqrt{12}$
8. The square Root of that half sum is the greater part of the Root fought, to wit, $\sqrt{(4)12}$
9. Again, from the greater part of the given Binomial, viz. from $\sqrt{12}$
10. Subtract the square Root before found in the fourth step, to wit, $\sqrt{12}$
11. The Remainder is $\sqrt{60}$
12. The half of which Remainder is $\sqrt{15}$
13. The square Root of the said half Remainder is the lesser part of the Root fought, to wit, $\sqrt{(4)15}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by $+$, shall be the square Root fought; to wit, $\sqrt{(4)12} + \sqrt{(4)15}$

And if $-$ instead of $+$ be prefix to the lesser part of the said Root, it gives $\sqrt{(4)12} - \sqrt{(4)15}$, which is the square Root of the third Residual $\sqrt{12} - \sqrt{80}$.

The former of those two Roots answers to the irrational line called (in Prop. 39, & 57. lib. 10. Elem. Euclid.) a second Binomial; and the latter answers to the irrational line called (in Prop. 76, & 94.) a second Medial Residual.

The Proof of the Root above extracted out of the third Binomial.

- The Squares of the parts of $\sqrt{(4)12} + \sqrt{(4)15}$, the Root found out, are $\sqrt{12}$ and $\sqrt{15}$
 Which Squares added together, make $7\sqrt{2}$, that is, $\sqrt{12}$
 The Product of the parts, viz. $\sqrt{(4)12}$ into $\sqrt{(4)15}$, is $\sqrt{(4)180}$; that is, $\sqrt{180}$
 The double of the said Product is $\sqrt{180}$
 Therefore the sum of the sum of the Squares of the parts and the said double Product is $\sqrt{12} + \sqrt{80}$

Whence it is manifest, that $\sqrt{12} + \sqrt{80}$ is the Square of $\sqrt{(4)12} + \sqrt{(4)15}$; therefore this is the square Root of that third Binomial: which was to be proved. Moreover,

Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $\sqrt{2\frac{1}{2}} - \sqrt{80}$ is the Square of $\sqrt{(4)}\frac{1}{2} - \sqrt{(4)}\frac{1}{5}$; therefore this is the Square Root of that third Residual.

Example 4.

Let it be required to extract the square Root of this fourth Binomial: $7 + \sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, viz. from . . . 49
2. Subtract the Square of the lesser part $\sqrt{20}$, to wit, . . . 20
3. The Remainder is . . . 29
4. The Square Root of that Remainder is . . . $\sqrt{29}$
5. To which Square Root add the greater part . . . 7
6. The Summ is . . . $7 + \sqrt{29}$
7. The half of which Summ is . . . $\frac{7}{2} + \sqrt{\frac{29}{4}}$
8. The square Root of that half Summ is the greater part of the Root sought, to wit, . . . $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}}$
9. Again, from the greater part of the given Binomial, viz. from . . . 7
10. Subtract the square Root before found in the fourth step, to wit, . . . $\sqrt{29}$
11. The Remainder is . . . $7 - \sqrt{29}$
12. The half of which Remainder is . . . $\frac{7}{2} - \sqrt{\frac{29}{4}}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, . . . $\sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$
14. I say, the two parts in the eighth and thirteenth steps, (the former of which is a Binomial, and the latter a Residual,) being connected by +, shall be the Square Root sought, to wit, . . . $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} + \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$

Which Root answers to the irrational line called (in Prop. 40, & 58. lib. 10. Elem. Euclid.) a Major line.

And if the lesser Name of the said Root be subtracted from the greater, by interpoling the sign —, it gives $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} - \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$: which is the Root of the fourth Residual $7 - \sqrt{20}$, and answers to the irrational line called (in Prop. 77, & 95. lib. 10. Elem. Euclid.) a Minor line.

The Proof of the Root above extracted out of the fourth Binomial.

- The Squares of the parts of the Root found out are . . . $\frac{7}{2} + \sqrt{\frac{29}{4}}$ and $\frac{7}{2} - \sqrt{\frac{29}{4}}$
 Therefore the sum of the Squares of the parts is . . . $\frac{7}{2} + \frac{7}{2}$, that is, 7
 The Product of the parts will be found (by Rule 2. Self. 12. of this Chapt.) . . . $\sqrt{\frac{29}{4}} - \frac{29}{4}$: that is, $\sqrt{29}$
 The double of the said Product is . . . $\sqrt{20}$
 Therefore the sum of the said sum of the Squares of the parts and the double Product is . . . $7 + \sqrt{20}$.

Whence it is manifest that $7 + \sqrt{20}$ is the Square of $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} + \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$: therefore this is the square Root of that fourth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $7 - \sqrt{20}$ is the Square of $\sqrt{\frac{7}{2} + \sqrt{\frac{29}{4}}} - \sqrt{\frac{7}{2} - \sqrt{\frac{29}{4}}}$: therefore this is the square Root of that fourth Residual $7 - \sqrt{20}$.

Example 5.

Let it be required to extract the square Root out of this fifth Binomial, $\sqrt{20} + 4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, viz. from . . . 20
2. Subtract the Square of the lesser part 4, to wit, . . . 16
3. The Remainder is . . . 4
4. The square Root of that Remainder is . . . 2
5. To which square Root add the greater part . . . $\sqrt{20}$

6. The sum is . . . $\sqrt{20} + 2$
7. The half of that sum is . . . $\sqrt{5} + 1$
8. The square Root of the said half sum is the greater part of the Root sought, to wit, . . . $\sqrt{\sqrt{5} + 1}$
9. Again, from the greater part of the given Binomial, viz. from . . . $\sqrt{20}$
10. Subtract the square Root before found in the fourth step, to wit, . . . 2
11. The Remainder is . . . $\sqrt{20} - 2$
12. The half of which Remainder is . . . $\sqrt{5} - 1$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, . . . $\sqrt{\sqrt{5} - 1}$
14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual,) being connected by +, shall be the Square Root sought, to wit, . . . $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$

Which Root answers to the irrational line called (in Prop. 41, & 59. lib. 10. Elem. Euclid.) a line containing in Power a Rational and a Medial Rectangle: And if the lesser Name of the said Root be subtracted from the greater, by the interpolation of the sign —, it gives $\sqrt{\sqrt{5} + 1} - \sqrt{\sqrt{5} - 1}$: which is the square Root of the fifth Residual $\sqrt{20} - 4$, and answers to the irrational line which (in Prop. 78, & 96. lib. 10.) is called a line making with a Rational Space the whole Space Medial.

The Proof of the Root above extracted out of the fifth Binomial.

- The Squares of the parts of $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$. . . $\sqrt{5} + 1$ and $\sqrt{5} - 1$
 the Root found out, are . . . $\sqrt{5} + 1$ and $\sqrt{5} - 1$
 Therefore the sum of the said Squares of the parts is . . . $\sqrt{5} + \sqrt{5}$; that is, $\sqrt{20}$
 The Product of the parts multiplied one into the other (according to Rule 2. Self. 12. of this Chapt.) is . . . $\sqrt{5} - 1$: that is, 2
 The double of the said Product is . . . 4
 Therefore the sum of the said sum of the Squares of the parts and double Product is . . . $\sqrt{20} + 4$.

Whence it is manifest that $\sqrt{20} + 4$ is the Square of $\sqrt{\sqrt{5} + 1} + \sqrt{\sqrt{5} - 1}$: therefore this is the square Root of that fifth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $\sqrt{20} - 4$ is the Square of $\sqrt{\sqrt{5} + 1} - \sqrt{\sqrt{5} - 1}$: Therefore this is the square Root of the said fifth Residual $\sqrt{20} - 4$.

Example 6.

Let it be required to extract the square Root of this sixth Binomial, $\sqrt{20} + \sqrt{8}$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, viz. from . . . 20
2. Subtract the Square of the lesser part $\sqrt{8}$, to wit, . . . 8
3. The Remainder is . . . 12
4. The square Root of that Remainder is . . . $\sqrt{12}$
5. To which square Root add the greater part . . . $\sqrt{20}$
6. The sum is . . . $\sqrt{20} + \sqrt{12}$
7. The half of which sum is . . . $\sqrt{5} + \sqrt{3}$
8. The square Root of the said half sum is the greater part of the Root sought, to wit, . . . $\sqrt{\sqrt{5} + \sqrt{3}}$
9. Again, from the greater part of the given Binomial, viz. from . . . $\sqrt{20}$
10. Subtract the square Root before found in the fourth step, to wit, . . . $\sqrt{12}$
11. The Remainder is . . . $\sqrt{20} - \sqrt{12}$
12. The half of that Remainder is . . . $\sqrt{5} - \sqrt{3}$
13. The

13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, $\sqrt{5} - \sqrt{3}$;
 14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual) being connected by $+$, shall be the square Root sought, to wit, $\sqrt{5} + \sqrt{3}$;
 15. $\sqrt{5} + \sqrt{3}$;
 16. $\sqrt{5} + \sqrt{3}$;
 17. $\sqrt{5} + \sqrt{3}$;
 18. $\sqrt{5} + \sqrt{3}$;
 19. $\sqrt{5} + \sqrt{3}$;
 20. $\sqrt{5} + \sqrt{3}$;
 21. $\sqrt{5} + \sqrt{3}$;
 22. $\sqrt{5} + \sqrt{3}$;
 23. $\sqrt{5} + \sqrt{3}$;
 24. $\sqrt{5} + \sqrt{3}$;
 25. $\sqrt{5} + \sqrt{3}$;
 26. $\sqrt{5} + \sqrt{3}$;
 27. $\sqrt{5} + \sqrt{3}$;
 28. $\sqrt{5} + \sqrt{3}$;
 29. $\sqrt{5} + \sqrt{3}$;
 30. $\sqrt{5} + \sqrt{3}$;
 31. $\sqrt{5} + \sqrt{3}$;
 32. $\sqrt{5} + \sqrt{3}$;
 33. $\sqrt{5} + \sqrt{3}$;
 34. $\sqrt{5} + \sqrt{3}$;
 35. $\sqrt{5} + \sqrt{3}$;
 36. $\sqrt{5} + \sqrt{3}$;
 37. $\sqrt{5} + \sqrt{3}$;
 38. $\sqrt{5} + \sqrt{3}$;
 39. $\sqrt{5} + \sqrt{3}$;
 40. $\sqrt{5} + \sqrt{3}$;
 41. $\sqrt{5} + \sqrt{3}$;
 42. $\sqrt{5} + \sqrt{3}$;
 43. $\sqrt{5} + \sqrt{3}$;
 44. $\sqrt{5} + \sqrt{3}$;
 45. $\sqrt{5} + \sqrt{3}$;
 46. $\sqrt{5} + \sqrt{3}$;
 47. $\sqrt{5} + \sqrt{3}$;
 48. $\sqrt{5} + \sqrt{3}$;
 49. $\sqrt{5} + \sqrt{3}$;
 50. $\sqrt{5} + \sqrt{3}$;
 51. $\sqrt{5} + \sqrt{3}$;
 52. $\sqrt{5} + \sqrt{3}$;
 53. $\sqrt{5} + \sqrt{3}$;
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 55. $\sqrt{5} + \sqrt{3}$;
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 65. $\sqrt{5} + \sqrt{3}$;
 66. $\sqrt{5} + \sqrt{3}$;
 67. $\sqrt{5} + \sqrt{3}$;
 68. $\sqrt{5} + \sqrt{3}$;
 69. $\sqrt{5} + \sqrt{3}$;
 70. $\sqrt{5} + \sqrt{3}$;
 71. $\sqrt{5} + \sqrt{3}$;
 72. $\sqrt{5} + \sqrt{3}$;
 73. $\sqrt{5} + \sqrt{3}$;
 74. $\sqrt{5} + \sqrt{3}$;
 75. $\sqrt{5} + \sqrt{3}$;
 76. $\sqrt{5} + \sqrt{3}$;
 77. $\sqrt{5} + \sqrt{3}$;
 78. $\sqrt{5} + \sqrt{3}$;
 79. $\sqrt{5} + \sqrt{3}$;
 80. $\sqrt{5} + \sqrt{3}$;
 81. $\sqrt{5} + \sqrt{3}$;
 82. $\sqrt{5} + \sqrt{3}$;
 83. $\sqrt{5} + \sqrt{3}$;
 84. $\sqrt{5} + \sqrt{3}$;
 85. $\sqrt{5} + \sqrt{3}$;
 86. $\sqrt{5} + \sqrt{3}$;
 87. $\sqrt{5} + \sqrt{3}$;
 88. $\sqrt{5} + \sqrt{3}$;
 89. $\sqrt{5} + \sqrt{3}$;
 90. $\sqrt{5} + \sqrt{3}$;
 91. $\sqrt{5} + \sqrt{3}$;
 92. $\sqrt{5} + \sqrt{3}$;
 93. $\sqrt{5} + \sqrt{3}$;
 94. $\sqrt{5} + \sqrt{3}$;
 95. $\sqrt{5} + \sqrt{3}$;
 96. $\sqrt{5} + \sqrt{3}$;
 97. $\sqrt{5} + \sqrt{3}$;
 98. $\sqrt{5} + \sqrt{3}$;
 99. $\sqrt{5} + \sqrt{3}$;
 100. $\sqrt{5} + \sqrt{3}$;

Which Root answers to the irrational line which (in Prop. 42, & 60. lib. 10. Elem. Eucl.) is called, a line containing in Power two *Medial Rectangles*: And, if the lesser part of the said Root be subtracted from the greater by the interpolation of the sign $-$, it gives $\sqrt{5} + \sqrt{3}$; $-\sqrt{5} - \sqrt{3}$: which is the Root of the sixth Residual $\sqrt{20} - \sqrt{8}$, and answers to the irrational line which (in Prop. 79, & 97. lib. 10. Euclid.) is called a line making with a *Medial Rectangle* a whole *Space Medial*.

The Proof of the Root above extracted out of the sixth Binomial.

The Squares of the parts of $\sqrt{5} + \sqrt{3}$;
 the Root sought, are $\sqrt{5} + \sqrt{3}$ and $\sqrt{5} - \sqrt{3}$;

Therefore the sum of the said Squares of the parts is $\sqrt{5} + \sqrt{3}$; that is, $\sqrt{20}$;

The Product of the parts multiplied one into the other is $\sqrt{5} - \sqrt{3}$; that is, $\sqrt{8}$;

The double of the said Product is $\sqrt{20} - \sqrt{8}$;

Therefore the sum of the said sum of the Squares of the parts and double Product is $\sqrt{20} - \sqrt{8}$;

Whence it is manifest that $\sqrt{20} - \sqrt{8}$ is the Square of $\sqrt{5} + \sqrt{3}$; $-\sqrt{5} - \sqrt{3}$; therefore this is that square Root of the sixth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the parts, the Remainder $\sqrt{20} - \sqrt{8}$ is the Square of $\sqrt{5} + \sqrt{3}$; $-\sqrt{5} - \sqrt{3}$; therefore this is the square Root of that sixth Residual.

Note. In every Binomial and Residual constituted according to the preceding Sect. 11, the square Root of the difference of the Squares of the Names or parts is equal to the difference of the Squares of the parts of the Root of the Binomial or Residual.

As in the first Binomial $27 + \sqrt{704}$, whose square Root hath before been found $4 + \sqrt{11}$; the Square of 27 , to wit, 729 , exceeds 704 , the Square of $\sqrt{704}$, by 25 , whose square Root 5 is equal to the difference of the Squares of the parts of the Root of the Binomial proposed, to wit, the difference between 16 and 11 .

This property may be demonstrated thus, let $b + \sqrt{d}$ represent a Binomial Root whose greater part is b ; then the Square of that Root is $bb + 2b\sqrt{d} + d$, this divided into its Names or parts makes the Binomial $bb + d$ more $2b\sqrt{d}$; then the Squares of the parts of this Binomial are $bbb + 2bbd + dd$ and $2b\sqrt{d}$, and the difference of these Squares is $bbb - bbd + dd$, whose square Root $b - \sqrt{d}$ is manifestly the difference of the Squares of the parts of the Root $b + \sqrt{d}$ first proposed: which was to be shewn. The like property may be demonstrated in a Residual.

How to extract the Square Root out of a Binomial design'd by Letters, if it hath a Binomial Root.

By the same general Rule which hath before been exercis'd in extracting the square Root out of Binomials express'd by Numbers, we may extract the square Root out of a Binomial design'd by Letters, when it hath a binomial Root, as will be evident by the following Examples; where for the more apparent distinction of the parts of the given Binomial, instead of $+$ I set the word [more] between the parts, and instead of $-$ I set the word [less] between the parts of a given Residual.

Example 1.

Let it be required to extract the square Root out of $bb + d$ more $2b\sqrt{d}$;

The Operation.

1. From the Square of the greater part, (which suppose to be $bb + d$), viz. from $bbb + 2bbd + dd$;
 2. Subtract the Square of the lesser part $2b\sqrt{d}$, to wit, $2b\sqrt{d}$;
 3. The

3. The Remainder is $bbb - 2bbd - dd$;
 4. The square Root of that Remainder is $bb - d$;
 5. To which square Root add the greater part, to wit, $bb + d$;
 6. The Summ is $2bb$;
 7. The half of that Summ is bb ;
 8. The square Root of that half Summ is the greater part of the Root sought, to wit, b ;
 9. Then from the greater part of the given Binomial, viz. from $bb + d$;
 10. Subtract the square Root before found in the fourth step, to wit, $bb - d$;
 11. The Remainder is $2d$;
 12. The half of which Remainder is d ;
 13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, \sqrt{d} ;
 14. I say, the two parts in the eighth and thirteenth steps being connected by the sign $+$ shall be the square Root sought, to wit, $b + \sqrt{d}$;

Which Root being squared, or multiplied into it self, will evidently produce the given Binomial $bb + d$ more $2b\sqrt{d}$.

Example 2.

Let it be required to extract the square Root out of $mm + \frac{pxx}{m}$ more $x\sqrt{4pm}$;

The Operation.

1. From the Square of the greater part $mm + \frac{pxx}{m}$;
 viz. from $mmmm + 2mpxx + \frac{ppxx}{m}$;
 2. Subtract the Square of the lesser part $x\sqrt{4pm}$, to wit, $4mpxx$;
 3. The Remainder is $mmmm - 2mpxx + \frac{ppxx}{m}$;
 4. The square Root of that Remainder is $mm - \frac{pxx}{m}$;
 5. To which square Root add the greater part, to wit, $mm + \frac{pxx}{m}$;
 6. The Summ is $2mm$;
 7. The half of which Summ is mm ;
 8. The square Root of the said half Summ is the greater part of the Root sought, to wit, m ;
 9. Again, from the greater part of the given Binomial, viz. from $mm + \frac{pxx}{m}$;
 10. Subtract the square Root before found in the fourth step, to wit, $mm - \frac{pxx}{m}$;
 11. The Remainder is $\frac{2pxx}{m}$;
 12. The half of which Remainder is $\frac{pxx}{m}$;
 13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, $\sqrt{\frac{pxx}{m}}$ or $x\sqrt{\frac{p}{m}}$;
 14. I say, the two parts in the eighth and thirteenth steps, being connected by $+$, shall be the square Root sought, to wit, $m + x\sqrt{\frac{p}{m}}$;

Which Binomial Root being squared or multiplied into it self, will produce the given Binomial.

Example 3.

Let it be required to extract the square Root out of $a + b\sqrt{ab}$ more $2ab$;

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The Operation.

1. From the Square of the greater part, viz. from a^2 . . . a^2
 2. Subtract the Square of the lesser part, to wit, b^2 . . . $a^2 - b^2$
 3. The Remainder is $a^2 - b^2$. . . $a^2 - b^2$
 4. The Square Root of that Remainder is $\sqrt{a^2 - b^2}$. . . $\sqrt{a^2 - b^2}$
 5. To which Square Root add the greater part, to wit, a . . . $a + \sqrt{a^2 - b^2}$
 6. The Summ is $a + \sqrt{a^2 - b^2}$. . . $a + \sqrt{a^2 - b^2}$
 7. The half of that Summ is $\frac{a + \sqrt{a^2 - b^2}}{2}$. . . $\frac{a + \sqrt{a^2 - b^2}}{2}$
 8. The Square Root of the said half Summ is the greater part of the Root sought, to wit, $\sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}}$. . . $\sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}}$
 9. Again, from the greater part of the given Binomial, viz. from a . . . $a - \sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}}$
 10. Subtract the square Root before found in the fourth step, to wit, $\sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}}$. . . $a - \sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}}$
 11. The Remainder is $\frac{a - \sqrt{a^2 - b^2}}{2}$. . . $\frac{a - \sqrt{a^2 - b^2}}{2}$
 12. The half of which Remainder is $\frac{a - \sqrt{a^2 - b^2}}{4}$. . . $\frac{a - \sqrt{a^2 - b^2}}{4}$
 13. The Square Root of the said half Remainder is the lesser part of the Root sought, to wit, $\sqrt{\frac{a - \sqrt{a^2 - b^2}}{4}}$. . . $\sqrt{\frac{a - \sqrt{a^2 - b^2}}{4}}$
 14. I say, the two parts in the eighth and thirteenth steps, being connected by $+$, shall be the Square Root sought, to wit, $\sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b^2}}{4}}$. . . $\sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b^2}}{4}}$
 15. Which Binomial Root may be also exprest thus, $\sqrt{(4)aa + (4)abb}$. . . $\sqrt{(4)aa + (4)abb}$
- The Proof may be made by multiplying the Root found out into it self.

Example 4.

Again, if the Square Root of this Residual be desired, $\sqrt{a^2 - b^2}$ left $2\sqrt{abcd}$.
 The Root being extracted by the precedent method, will $\sqrt{a^2 - b^2} = \sqrt{d^2 - bc^2}$.
 be found
 Which Root may be also exprest thus, $\sqrt{(4)aa - (4)abb}$

But if it happen that when the Square of the lesser part of the given Binomial or Residual is subtracted from the Square of the greater part, the square Root of the Remainder and the greater part are not commensurable, (according to the Definition before given in *Self. 7.* of this *Chap.*) there is no more to be done in such case, but to prefix before the given Binomial or Residual the sign $\sqrt{}$, with a line drawn over both its parts, to denote the universal square Root of the given Binomial or Residual. As: to extract the square Root out of this Residual $\sqrt{\frac{1}{4}aa + \frac{1}{4}bb}$, I write $\sqrt{\frac{1}{4}aa + \frac{1}{4}bb} = \frac{1}{2}a$: which kind of Roots are commonly called Universal.

SECT. 17. Questions to exercise the foregoing Rules of this Chapter.

QUEST. 1.

To divide 100 into two such parts, that if each part be divided by the other part, the sum of the Quotients may make 3.

RESOLUTION.

1. For one of the parts sought put x . . . x
2. Then consequently the other part is $100 - x$. . . $100 - x$
3. Therefore, according to the import of the Question, this Equation ariseth, viz. $\frac{x}{100 - x} + \frac{100 - x}{x} = 3$. . . $\frac{x}{100 - x} + \frac{100 - x}{x} = 3$
4. Which Equation duly reduced gives $100x - x^2 = 20000$. . . $100x - x^2 = 20000$
5. Wherefore by resolving the said Equation by the Canon in *Self. 10. Chap. 15. Book 1.* the two values of x , which are the desired parts of 100, will be found these, to wit, $x = \frac{50 \pm \sqrt{50^2 - 10000}}{1}$. . . $x = \frac{50 \pm \sqrt{50^2 - 10000}}{1}$

6. The

6. The sum of the said parts or numbers found out is manifestly 100, so it remains only to prove that

$$\frac{50 + 10\sqrt{5}}{50 - 10\sqrt{5}} + \frac{50 - 10\sqrt{5}}{50 + 10\sqrt{5}} = 3$$

The Proof.

7. To add those two surd Fractions in the sixth step into one sum, reduce them to a common Denominator, viz. multiply $50 + 10\sqrt{5}$ by $50 - 10\sqrt{5}$, and the Product (by the first of the three compendious Rules in *Self. 10.* of this *Chap.*) will be found $3000 + 1000\sqrt{5}$
8. Likewise, multiply $50 - 10\sqrt{5}$ by $50 + 10\sqrt{5}$, and the Product (by the second of the said three Rules) will be $3000 - 1000\sqrt{5}$
9. Then take the sum of those two Products for the Numerator of a Fraction, or a Dividend, to wit, 6000
10. Also multiply the two Denominators of the surd Fractions in the sixth step one by the other, (according to the last of the three Rules above cited,) and take the Product for a Denominator, or Divisor, viz. 2000
11. Lastly, the Numerator in the ninth step being set over the Denominator in the tenth gives the sum of the two surd Fractions or Quotients in the sixth step, viz. $\frac{6000}{2000} = 3$

Which sum is manifestly 3, as was to be proved.

Another Proof.

The Quotient that ariseth by dividing $50 + 10\sqrt{5}$ by $50 - 10\sqrt{5}$, (according to the Rule of Division in the sixth branch of *Self. 11.* of this *Chap.*) is $\frac{3}{2} + \sqrt{\frac{5}{4}}$
 Likewise, the Quotient that ariseth by dividing $50 - 10\sqrt{5}$ by $50 + 10\sqrt{5}$ is $\frac{3}{2} - \sqrt{\frac{5}{4}}$
 The sum of those two Quotients is manifestly 3, (as before.)

QUEST. 2.

To divide a given number, suppose 6, into three such unequal numbers in continual proportion, that the sum of the Squares of the extremes may be to the Square of the mean in a given proportion, but the first term of this proportion must exceed the double of the latter term. Let it therefore be desired that the sum of the Squares of the extremes may be to the Square of the mean as 3 to 1.

RESOLUTION.

1. For the mean Proportional put a . . . a
2. Then because the sum of all the three Proportionals must make 6, and the mean is a , the sum of the extremes shall be $6 - a$. . . $6 - a$
3. Therefore the Square of the sum of the extremes is $36 - 12a + a^2$. . . $36 - 12a + a^2$
4. But (by *Theor. 3. Chap. 6.* of this Book) the Square of the sum of the extremes of three numbers continually proportional is equal to the Squares of the extremes, together with the double Square of the mean; therefore from the Square in the third step I subtract $2a^2$ (the double Square of the mean,) and there remains the sum of the Squares of the extremes, to wit, $36 - 12a - a^2$. . . $36 - 12a - a^2$
5. But (according to the Question) the sum of the Squares of the extremes must be equal to the triple Square of the mean; therefore from the fourth and first step this Equation ariseth, viz. $36 - 12a - a^2 = 3a^2$. . . $36 - 12a - a^2 = 3a^2$
6. From which Equation after due Reduction this ariseth, viz. $4a^2 - 12a + 9 = 0$. . . $4a^2 - 12a + 9 = 0$
7. Therefore by resolving the last Equation, (according to the Canon in *Self. 6. Chap. 15.*) the value of a , that is, the mean Proportional sought will be discovered, viz. $\frac{3}{2} - \frac{1}{2} = \frac{1}{2}$. . . $\frac{3}{2} - \frac{1}{2} = \frac{1}{2}$

8. And

8. And from the seventh and second steps the sum $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{4}} =$ sum of the extremes, of the extremes will be also made known, viz.
9. Then, (as is manifest by *Quest. 4. Chap. 16. Book 1.*) the sum of the extremes of three numbers continually proportional being given, as also the mean, the extremes shall be given severally by this following

CANON.

From the Square of half the sum of the extremes subtract the Square of the mean, and extract the Square Root of the Remainder; then this Square Root being added to, and subtracted from the said half sum, will give the extremes severally. Therefore,

10. From the Square of the half of $\frac{1}{2} - \sqrt{\frac{1}{4}}$, that is, from $\frac{1}{4} - \frac{1}{4}\sqrt{\frac{1}{4}}$
11. Subtract the Square of $\sqrt{\frac{1}{4}} - \frac{1}{2}$, viz. $\frac{1}{4} - \frac{1}{4}\sqrt{\frac{1}{4}}$
12. The Remainder is $\frac{1}{4} - \frac{1}{4}\sqrt{\frac{1}{4}}$
13. Then the Square Root of that Remainder being extracted, (by the General Rule before delivered in *Self. 16. of this Chap.* for extracting the Square Root out of Binomials,) will be found $\sqrt{\frac{1}{4}} - \frac{1}{2}$
14. Which Square Root added to the half of $\frac{1}{2} - \sqrt{\frac{1}{4}}$, gives the greater extreme sought, to wit, $\frac{1}{2}$
15. But the said Square Root subtracted from the half of $\frac{1}{2} - \sqrt{\frac{1}{4}}$, leaves the lesser extreme, to wit, $\frac{1}{2} - \sqrt{\frac{1}{4}}$
16. Wherefore, (in the seventh, fourteenth and fifteenth steps,) three numbers continually proportional are found out, viz. $\frac{1}{2}$, $\sqrt{\frac{1}{4}} - \frac{1}{2}$, and $\frac{1}{2} - \sqrt{\frac{1}{4}}$, whose sum is 6; and the sum of the Squares of the extremes is equal to the triple of the Square of the mean, as will appear by

The Proof.

First, the Product made by the multiplication of the first and third numbers one into the other, that is, of $\frac{1}{2}$ into $\frac{1}{2} - \sqrt{\frac{1}{4}}$, is $\frac{1}{4} - \frac{1}{4}\sqrt{\frac{1}{4}}$, which is also the Square of the second number $\sqrt{\frac{1}{4}} - \frac{1}{2}$, (as will easily appear by Multiplication;) therefore the said three numbers are Proportionals.

Secondly, the sum of the said three proportional numbers is 6; for the mean $\sqrt{\frac{1}{4}} - \frac{1}{2}$ added to $\frac{1}{2} - \sqrt{\frac{1}{4}}$ the lesser extreme, makes 3, to which adding the greater extreme $\frac{1}{2}$, the sum is 6.

Thirdly, the sum of the Squares of the extremes $\frac{1}{2}$ and $\frac{1}{2} - \sqrt{\frac{1}{4}}$, is equal to the triple of the Square of the mean $\sqrt{\frac{1}{4}} - \frac{1}{2}$; for the said sum, as also the said triple Square will by Multiplication be found $\frac{1}{4} - 9\sqrt{\frac{1}{4}}$. Therefore all the conditions in the Question are satisfied.

But that the necessity of the Determination annexed to the Question may be made manifest, it remains to prove, That if three unequal numbers be in continual proportion, the sum of the Squares of the extremes is greater than the double of the Square of the mean: Therefore,

Let three unequal numbers in continual proportion be exposed, a, \sqrt{ac}, e ::

Then their Squares shall be also Proportionals, (*per 22. Prop. 6. Elem. Euclid.*) viz. $aa : ae :: ae : ee$

Therefore (by 25. *Prop. 5. Elem. Euclid.*) $aa + ee < 2ae$.

But $aa + ee$ is the sum of the Squares of the extremes of the three Proportionals exposed, and $2ae$ is equal to the double Square of the mean Proportional, wherefore the Theorem is proved; and consequently the Determination is manifestly necessary to be annexed to the Question proposed, that there may be a possibility of finding out what is thereby desired. The Determination may also be easily infer'd from the Canon in the foregoing ninth step.

QUEST. 3.

What is the Product made by the continual multiplication of these four numbers one into another, which differ by an equal excess, to wit, Unity?

$$\left\{ \begin{array}{l} \sqrt{\frac{1}{2}} + \sqrt{101} - \frac{1}{2} \\ \sqrt{\frac{1}{2}} + \sqrt{101} - \frac{1}{2} \\ \sqrt{\frac{1}{2}} + \sqrt{101} - \frac{1}{2} \\ \sqrt{\frac{1}{2}} + \sqrt{101} - \frac{1}{2} \end{array} \right.$$

Answ.

Answ. The desired Product is exactly 100
For, (by the last of the three compendious Rules before delivered in *Self. 16. of this Chap.* for the multiplication of Binomials and Residuals,) the Product of the first and fourth numbers is $\sqrt{101} - \frac{1}{2}$
Likewise, the Product of the second and third number is $\sqrt{101} + \frac{1}{2}$
Lastly, the two last preceding Products being multiplied one into another (by the same Rule) make 100

QUEST. 4.

1. If a, b, c , be such Quantities that $aa + ca = b$
What is the value of a ?
2. Answ. By the Canon in *Self. 6. Chap. 15. Book 1.* $a = \sqrt{b + \frac{1}{4}cc} - \frac{1}{2}c$
By which value of a , the Equation propos'd may be expounded, as is manifest by the following

Demonstration.

3. If $a = \sqrt{b + \frac{1}{4}cc} - \frac{1}{2}c$
4. Then consequently by adding $\frac{1}{2}c$ to each part, $a + \frac{1}{2}c = \sqrt{b + \frac{1}{4}cc}$
5. And by multiplying each part of the last Equation into it self, $aa + ca + \frac{1}{4}cc = b + \frac{1}{4}cc$
6. Wherefore, by subtracting $\frac{1}{4}cc$ from each part, there remains $aa + ca = b$

Which was to be proved.

Note. This Demonstration is formed in the way of Composition by the steps of the Resolution of the same Question in *Self. 5. Chap. 15. Book 1.* but in a retrograde or backward order; for the first step in the Composition, (or Demonstration) is the last in the Resolution; the second step in the Composition is the last but one in the Resolution; and so by returning backwards by the steps of the Resolution, the Demonstration ends in the Equation propos'd to be resolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUEST. 5.

1. If a, b, k , be such Quantities that $aa - ba = k$
What is the value of a ?
2. Answ. By the Canon in *Self. 8. Chap. 15. Book 1.* $a = \frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + kb}$
By which value of a , the Equation propos'd may be expounded; as appears by the following

Demonstration.

3. If $a = \frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + kb}$
4. Then by subtracting $\frac{1}{2}b$ from each part, $a - \frac{1}{2}b = \sqrt{\frac{1}{4}b^2 + kb}$
5. And by multiplying each part of the last Equation into it self, $aa - ba + \frac{1}{4}bb = k + \frac{1}{4}bb$
6. Wherefore by subtracting $\frac{1}{4}bb$ from each part, $aa - ba = k$

Which was to be proved.

QUEST. 6.

1. If c and n be put for such known Quantities, that n not $\leq \frac{1}{2}cc$,
2. And if a be put for a Quantity unknown, and $ca - aa = n$;
What is the value of a ?
3. Answ. By the Canon in *Self. 10. Chap. 15. Book 1.* these two values of a will be found $a = \frac{1}{2}c \pm \sqrt{\frac{1}{4}cc - n}$

By each of which values of a , the Equation propos'd in the second step may be expounded, viz. if either $\frac{1}{2}c + \sqrt{\frac{1}{4}cc - n}$ or, $\frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$ be put equal to a , then $ca - aa = n$.

Definit.

Demonstration.

4. First, if $a = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$
 5. Then by subtracting $\frac{1}{2}c$ from each part, $a - \frac{1}{2}c = -\sqrt{\frac{1}{4}cc - n}$
 6. And by multiplying each part of the last Equation into it self, $aa - ca + \frac{1}{4}cc = \frac{1}{4}cc - n$
 7. And by adding ca to each part, $aa + \frac{1}{4}cc = \frac{1}{4}cc - ca - n$
 8. And by subtracting $\frac{1}{4}cc$ from each part, $aa = ca - n$
 9. And by adding n to each part, $aa + n = ca$
 10. Wherefore by subtracting aa from each part, $n = ca - aa$
 11. That is, $ca - aa = n$

Which was to be proved.

- Again, if $a = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - n}$
 12. Then by adding $\sqrt{\frac{1}{4}cc - n}$ to each part, $a + \sqrt{\frac{1}{4}cc - n} = \frac{1}{2}c$
 13. And by subtracting a from each part, $\sqrt{\frac{1}{4}cc - n} = \frac{1}{2}c - a$
 14. And by multiplying each part of the last Equation into it self, $\frac{1}{4}cc - n = \frac{1}{4}cc - ca + aa$
 15. And by adding ca to each part, $ca + \frac{1}{4}cc - n = \frac{1}{4}cc + aa$
 16. And subtracting $\frac{1}{4}cc$ from each part, $ca - n = aa$
 17. And by adding n to each part, $ca = aa + n$
 18. Wherefore by subtracting aa from each part, $ca - aa = n$

Which was to be proved.

QUEST. 7.

1. If b and c be put for such known Quantities, that c is greater than b , but less than $2b$, and if a be put for a Quantity unknown;
 2. And if $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$;
 What is the value of a ?

RESOLUTION.

3. Because the Squares of equal Quantities are also equal, by multiplying each part of the Equation in the second step into it self, this is produced, *viz.*

$$\frac{aa}{2} + \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}$$

4. Then to the end the Surd Quantity in the Equation in the third step may solely make one part of an Equation, let $\frac{aa}{2}$ be subtracted from each part of that Equation, and this will remain, *viz.*

$$\sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c} - \frac{aa}{2} = \frac{2baa - caa}{2c}$$

5. And to the end the Radical sign in the first part of the last Equation may vanish, let each part be multiplied by it self, so an Equation in Rational Quantities will be produced, *viz.*

$$\frac{a^4 - 9b^4}{4} = \frac{4bb^4 - 4bca^4 + cca^4}{4cc}$$

6. And by reducing the last Equation to a common Denominator $4cc$, and then by multiplying each part by the same $4cc$, this Equation in Integers will be produced, *viz.*

$$cca^4 - 9b^4cc = 4bb^4c - 4bca^4 + cca^4$$

7. And from the Equation in the last preceding step, after due reduction is made to make those Quantities wherein a^4 is found to possess one part, this following Equation ariseth, *viz.*

$$4bca^4 - 4bb^4c = 9b^4cc$$

8. Then by dividing each part of the last Equation by $4bc - 4bb$, to the end that a^4 may stand alone, this Equation ariseth, *viz.*

$$a^4 = \frac{9b^4cc}{4bc - 4bb} = \frac{9b^4cc}{4c - 4b}$$

9. But $\frac{9b^4cc}{4} \text{ into } \frac{b}{c-b} = \frac{9b^4cc}{4c-4b}$

10. There-

10. Therefore from the two last preceding Equations, by exchanging equal Quantities, this Equation ariseth, *viz.*

$$a^4 = \frac{9b^4cc}{4} \text{ into } \frac{b}{c-b}$$

11. And by extracting the square Root out of each part of the Equation in the tenth step, this ariseth;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}$$

12. Wherefore by extracting the square Root out of each part of the Equation in the eleventh step, the desired value of a is discovered, *viz.*

$$a = \sqrt{\frac{3bc}{2}} \text{ into } \sqrt{\frac{b}{c-b}}$$

An Example of Quest. 7. in Numbers.

13. If $b = 16$;
 14. And $c = 25$;
 15. And $a = \text{a number unknown}$;
 16. And if $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$;

What is the number a ?

17. Answer. From the thirteenth, fourteenth, and twelfth steps, $a = \sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; Then it will follow, that

$$\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (= 8\sqrt{2}, \text{ or, } \sqrt{512})$$

Note. The numbers to express the values of b and c must not be taken at pleasure but such, that the number c may exceed the number b , and be less than $2b$, as is prescribed in the Question, the former part of which Determination is discovered by the Denominator $c - b$ of the surd Fraction in the twelfth step, and the latter part of the Determination is manifest by the latter part of the Equation in the fourth step; where caa is to be subtracted from $2baa$, which cannot be done so as to leave a Remainder greater than nothing, unless c be less than $2b$.

SECT. XVIII. An Explication of Fran. van Schooten's General Rule, to extract what Root you please out of any Binomial in numbers; having such a Binomial Root as is desired.

Preparation.

First, if the given Binomial hath Fractions in it, it must be freed from them, by multiplying the Binomial by their Denominator. As, for example, to extract $\sqrt[4]{(3)}$, that is, the cubic Root, out of $\sqrt[4]{\frac{242}{125}} + 12\sqrt[4]{\frac{242}{125}}$, I multiply the Binomial by 5; and it makes $\sqrt[4]{968} + 25$; for $\sqrt[4]{242}$ multiplied by $\sqrt[4]{5}$, (that is, by 25) produceth $\sqrt[4]{968}$, and $12\sqrt[4]{25}$ into 25 makes 25. Likewise, if there be propos'd $\sqrt[4]{\frac{242}{125}} + \sqrt[4]{\frac{242}{125}}$, I first multiply it by $\sqrt[4]{5}$, and it makes $\sqrt[4]{242} + \sqrt[4]{25}$; then this Binomial multiplied by 2 produceth (as before) $\sqrt[4]{968} + 25$; and so of others.

Secondly, if neither of the two parts of the given Binomial be Rational, it must be reduced by Multiplication or Division to another Binomial that shall have one of its parts Rational; which Reduction may always be done by the multiplication of either part; but often times more briefly by the multiplication or division of the lesser number. As, for example, $\sqrt[4]{242} + \sqrt[4]{125}$ may be multiplied by $\sqrt[4]{242}$, and it makes $242 + \sqrt[4]{58808}$; but more compendiously by $\sqrt[4]{2}$, and there comes forth $22 + \sqrt[4]{486}$. After the same manner, $\sqrt[4]{33993} + \sqrt[4]{617578125}$ may be first multiplied by $\sqrt[4]{33993}$, and the Product again by $\sqrt[4]{33993}$, so there will be produced another Binomial whose Rational part is the absolute number 3993; but more briefly by $\sqrt[4]{(1)9}$, and there will

R R

B B

be produced another Binomial whose Rational part is 33; and yet more compendiously, if the Binomial proposed be divided by $\sqrt{(3)}$, there will arise $11 \div \sqrt{125}$.

But here is to be noted, that when one part of a Binomial is Rational, whether it be of a Binomial first given, or of another deduced (as above) from that given, then also the Square of the other part ought to be Rational, otherwise no Root can be extracted out of the Binomial or the other deduced from it.

Thirdly, to extract $\sqrt{(6)}$ out of a given Binomial qualified as above is supposed, we must first extract the Square Root, and then out of this the cubick Root, and to extract $\sqrt{(9)}$, we must first extract $\sqrt{(3)}$, and then out of the cubick Root found out we must again extract $\sqrt{(3)}$; and so of any other Root whose Index is a Composited number. But as to the extraction of the square Root out of a Binomial, a Rule hath been already given and exemplified in the preceding *Self*. 16. so that here there is need only that I shew how to extract $\sqrt{(3)}$, $\sqrt{(5)}$, $\sqrt{(7)}$, $\sqrt{(11)}$, and such like, whose Indices are Prime numbers.

Fourthly, to extract $\sqrt{(3)}$, $\sqrt{(5)}$, $\sqrt{(7)}$, or the like Root whose Index is a Prime number, we must first of all try whether out of the given Binomial there can be extracted a binomial Root which hath one part Rational, but that may be discovered by subtracting the Square of the lesser part of the given Binomial from the Square of the greater, and extracting the Root out of the Remainder; to wit, the cubick Root, if $\sqrt{(3)}$ be to be extracted out of the given Binomial; or the Root of the fifth Power, if $\sqrt{(5)}$ be to be extracted, and so of others: For if the Root of the said Remainder be not a Rational number, then the Binomial Root sought will certainly want a Rational part, *viz.* each of its parts will be surd; in which case, in order to extract that Root, the given Binomial must be multiplied by the difference of the Squares of the parts, if the Question be concerning the extraction of the cubick Root; or by the Square of the said difference, if $\sqrt{(5)}$ be sought, or by the Cube of the same difference, if $\sqrt{(7)}$ be required; or by the fifth Power of the said difference, if $\sqrt{(11)}$ be sought, and so of the rest. By which multiplication another Binomial will always be produced, wherein the Root of the difference of the Squares of the parts will be the same with the difference of the Squares of the parts of the former Binomial.

As, to extract the cubick Root out of $25 \div \sqrt{968}$; I first subtract 625, the Square of 25, from 968, the Square of $\sqrt{968}$, and there remains 343, whose cubick Root 7 is a Rational number: which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which hath one of its parts Rational.

Likewise, to extract the cubick Root out of $22 \div \sqrt{486}$, we must subtract 484, the Square of 22, from 486, and extract the cubick Root out of the Remainder 2; but because that cannot be done exactly, it shews that the cubick Root of $22 \div \sqrt{486}$ wants a Rational part; and therefore $22 \div \sqrt{486}$ must be multiplied by the said Remainder 2, that there may be a Binomial $44 \div \sqrt{1944}$, wherein the cubick Root of the difference of the Squares of the parts is 2.

So to extract $\sqrt{(5)}$ out of $11 \div \sqrt{125}$; because 121 the Square of 11 subtracted from 125 leaves 4, which considered as a fifth Power hath not an exact Rational Root, we must multiply $11 \div \sqrt{125}$ by 16 the Square of 4, that there may come forth $176 \div \sqrt{32000}$, where $\sqrt{(5)}$ of the difference of the Squares of the parts is 4.

Again, to extract $\sqrt{(7)}$ out of $33 \div \sqrt{114242}$, wherein the difference of the Squares of the parts is 2; because this 2 is not the seventh Power of any Rational number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it makes $2704 \div \sqrt{7311488}$, wherein the $\sqrt{(7)}$ of the difference of the Squares of the parts is 2.

THE RULE.

When a Binomial given, or another deduced from it (if need be) by the precedent Preparation, is such, that one of its parts, and the Square of the other part, as also the Root of the difference of the Squares of the parts, (to wit, the cubick Root when $\sqrt{(3)}$, or $\sqrt{(5)}$ when $\sqrt{(5)}$ is sought) are Rational whole numbers; then out of a Binomial so qualified, $\sqrt{(3)}$, or $\sqrt{(5)}$, or $\sqrt{(7)}$, &c. may be extracted, if it hath such a Root, in manner following, *viz.*

First, extract the Root of the difference of the Squares of the parts of the Binomial qualified as aforesaid, *viz.* the cubick Root, when $\sqrt{(3)}$ is sought; but $\sqrt{(5)}$ when $\sqrt{(5)}$, or $\sqrt{(7)}$, &c. which Root so extracted is to be reserved for a Dividend.

Secondly,

Secondly, find out a Rational number a little greater than the Root sought, with this caution, that the Rational number found out may not exceed the said Root above $\frac{1}{2}$, which may easily be done by Vulgar Arithmetick, and take the said Rational number for a Divisor.

Thirdly, divide the said Dividend by the said Divisor, and if the Rational part of the given Binomial be greater than the other part, add the Quotient to the said Rational Divisor, and the half of the greatest whole number contained in the sum shall be the Rational part of the Root sought; then from the Square of that Rational part subtract the Root of the difference of the Squares of the parts, (to wit, the Dividend first found out as above,) so the Remainder shall be the Square of the other part, when such a Root as was required can be extracted out of the given Binomial, which you may easily try, by multiplying this Root found out into it self, according to the degree of the Power represented by the given Binomial: for the Root found out being multiplied into it self cubically, if $\sqrt{(3)}$ was sought, or, five times into it self, if $\sqrt{(5)}$ was sought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be less than the other part, then after you have found out the Quotient as above, subtract it from the Rational Divisor, and the half of the greatest whole number contained in the Remainder shall be the Rational part of the Root sought; to the Square of which part if there be added the Dividend first found out as above, the sum will be the Square of the other part, when the Binomial proposed hath a Root; but by multiplying the Root found out into it self (as before) you may easily try whether it be a true Root or not.

Example 1. To extract the Cubick Root out of $20 \div \sqrt{392}$.

First, the difference of the Squares of the parts of the given Binomial, *viz.* the excess of 400, the Square of 20; above 392, the Square of $\sqrt{392}$ is 8, whose cubick Root 2 I reserve for a Dividend.

Secondly, I seek a Rational number that may be greater than the cubick Root of $20 \div \sqrt{392}$, (the given Binomial,) yet so that the excess may not be greater than $\frac{1}{2}$, to which end I extract the square Root of 392, and find it to be greater than 19, but less than 20; then to 20 the Rational part of the given Binomial I add 19 and 20 severally, and it makes 39 and 40; which are the nearest Rational whole numbers that can express the true value of the given Binomial; whence the cubick Root thereof will be found greater than 3, but less than 3 $\frac{1}{2}$; this 3 $\frac{1}{2}$, which, according to the Caution before given, exceeds the true cubick Root of the given Binomial by an excess not greater than $\frac{1}{4}$; I reserve for a Divisor.

Thirdly, I divide 2, the Dividend before reserved, by the said Divisor 3 $\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now because 20 the Rational part of the given Binomial is greater than the other part $\sqrt{392}$, I add the said Quotient $\frac{4}{7}$ to the said Divisor 3 $\frac{1}{2}$, and it makes the sum 4 $\frac{4}{7}$, wherein the greatest whole number is 4, whose half is 2 the Rational part of the Root sought; by the help of which Rational part, the other part is easily discovered; for if from 4 the Square of the said 2, you subtract 2, the cubick Root of the difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2 \div \sqrt{2}$ is the cubick Root of $20 \div \sqrt{392}$ the Binomial proposed, as will appear by the Proof: For $2 \div \sqrt{2}$ being multiplied into it self cubically produceth $20 \div \sqrt{392}$; and for the same reason, $2 \div \sqrt{2}$ is the cubick Root of $20 \div \sqrt{392}$.

Example 2. To extract the Cubick Root out of $44 \div \sqrt{1944}$.

First, the cubick Root of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the square Root of 1944 is greater than 44, but less than 45; these added severally to 44 the Rational part of the given Binomial, make 88 and 89, whose cubick Roots being extracted, do shew that the cubick Root of the given Binomial is greater than 4, but less than 4 $\frac{1}{2}$; this Rational number 4 $\frac{1}{2}$, which according to the Caution before given exceeds the true Root sought by an excess not greater than $\frac{1}{4}$, I take for a Divisor: Thirdly, I divide the said Dividend 2 by the said Divisor 4 $\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I subtract from the said 4 $\frac{1}{2}$; (I subtract, because 44 the Rational part of the given Binomial is less than the other part $\sqrt{1944}$;) and there remains 4 $\frac{1}{9}$; then the half of 4, the greatest whole number contained in 4 $\frac{1}{9}$, is 2, which is the Rational part of the Root sought: Lastly, to 4 the Square of the said 2, I add 2 the cubick Root of the difference of the Squares of the parts, and it makes 6 the Square of the other part. So that $2 \div \sqrt{6}$ is the cubick Root sought, as will appear by the Proof: For if it be multiplied into it self cubically, it

K k 2

produceth

produceth $44 \div \sqrt{1944}$ the Binomial proposed; and for the same reason, $\sqrt{6} - 2$ is the cubick Root of $\sqrt{1944} - 44$.

Example 3. To extract $\sqrt{(5)}$ out of $176 \div \sqrt{32000}$.

First, the difference of the Squares of the parts will be found 1924 , whose $\sqrt{(1)}$ is 4 for a Dividend: Secondly, the sum of the parts will be found greater than 354 , but less than 355 ; and consequently $\sqrt{(5)}$ of the sum of the parts is greater than 3 , but less than $3\frac{1}{2}$: Thirdly, by the said $3\frac{1}{2}$ I divide the said 4 , and the Quotient is $1\frac{1}{2}$, which I subtract from the said Divisor $3\frac{1}{2}$ (because the Rational part of the given Binomial is less than the other part) and there remains $2\frac{1}{2}$; then the half of 2 (the greatest whole number contained in $2\frac{1}{2}$) is 1 , the Rational part of the Root sought: Lastly, the Square of the said 1 , to wit, 1 , added to 4 (the $\sqrt{(5)}$ of the difference of the Squares of the parts of the given Binomial) makes 5 the Square of the other part. So that $1 \div \sqrt{5}$ is the $\sqrt{(5)}$ of the given Binomial $176 \div \sqrt{32000}$, at least if any $\sqrt{(5)}$ can be extracted out of the same; but $1 \div \sqrt{5}$ multiplied into it self five times makes $176 \div \sqrt{32000}$: therefore $1 \div \sqrt{5}$ is manifestly the desired $\sqrt{(5)}$ of $176 \div \sqrt{32000}$.

Example 4. To extract $\sqrt{(7)}$ out of $2704 \div \sqrt{731488}$.

First, the $\sqrt{(7)}$ of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the value of the given Binomial will be found greater than 5407 , but less than 5408 ; whence the $\sqrt{(7)}$ thereof will be discovered to be greater than 3 , but less than $3\frac{1}{2}$: Thirdly, by the said $3\frac{1}{2}$ I divide the Dividend before found 2 , and the Quotient is $\frac{2}{3}$, which I add to the Divisor $3\frac{1}{2}$ (because the Rational part 2704 is greater than the other part) and it makes the sum $4\frac{1}{3}$; and therefore $\frac{2}{3}$, the half of the greatest whole number contained in $4\frac{1}{3}$, is the Rational part of the Root sought: Lastly, from 4 , the Square of the said $\frac{2}{3}$, I subtract 2 , to wit, $\sqrt{(7)}$ of the difference of the Squares of the parts of the given Binomial, and there remains 2 the Square of the other part. So that $2 \div \sqrt{2}$ is the desired $\sqrt{(7)}$ of the given Binomial $2704 \div \sqrt{731488}$; for this is the seventh Power of $2 \div \sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial hath been multiplied or divided by some number, and thereby reduced to another Binomial, and the Root of this latter is found out, we must divide or multiply the Root found out by the Root of the number by which the Binomial was multiplied or divided; so there will come forth the Root of the given Binomial.

As, for example, because to extract the cubick Root out of $\sqrt{242} \div 12\frac{1}{2}$, we first multiplied this Binomial by 2 and found $25 \div \sqrt{968}$, whose cubick Root by the Rule before given will be found $1 \div \sqrt{8}$; this must be divided by $\sqrt{(3)}$, and the Quotient $\sqrt{(3)} \div \sqrt{(6)}$ shall be the cubick Root of $\sqrt{242} \div 12\frac{1}{2}$ the Binomial proposed.

But that the reason of the said Division by $\sqrt{(3)}$ may be more clearly appear, let there be put $d = 1 \div \sqrt{8}$, then it follows that $ddd = 25 \div \sqrt{968}$, and $\frac{ddd}{2} = \sqrt{242} \div 12\frac{1}{2}$ (the Binomial proposed). Therefore by extracting the cubick Root out of each part of the last Equation, there ariseth $\sqrt{(3)} \frac{ddd}{2}$, that is, $\frac{d}{\sqrt{(3)}^2} = \sqrt{(3)} \div \sqrt{242} \div 12\frac{1}{2}$.

But by supposition $d = 1 \div \sqrt{8}$; therefore $1 \div \sqrt{8}$ divided by $\sqrt{(3)}$, that is to say, $\sqrt{(3)} \div \sqrt{(6)}$ shall be the cubick Root of $\sqrt{242} \div 12\frac{1}{2}$: which was to be shewn.

Example 2. To extract $\sqrt{(3)}$ out of $\sqrt{242} \div \sqrt{242}$.

First, to prepare it for extraction, we multiplied by $\sqrt{5}$, and found $\sqrt{242} \div 12\frac{1}{2}$, whole $\sqrt{(3)}$ (as appears in the last preceding Example) is $\sqrt{(3)} \div \sqrt{(6)}$ 128 , which divided by $\sqrt{(6)}$ gives the Quotient $\sqrt{(6)} \div \sqrt{(6)} \div \sqrt{(6)} \div \sqrt{(6)}$ for the desired cubick Root of $\sqrt{242} \div \sqrt{242}$. The reason of which division by $\sqrt{(6)}$ may be thus manifested, let there be put $d = \sqrt{(3)} \div \sqrt{(6)}$ 128 ; then it follows that $ddd = \sqrt{242} \div 12\frac{1}{2} = \sqrt{242} \div \sqrt{242}$ into $\sqrt{5}$, whence $\frac{ddd}{\sqrt{5}} = \sqrt{242} \div \sqrt{242}$; therefore the cubick Root

of each part of the last Equation being extracted there ariseth $\sqrt{(3)} \frac{ddd}{\sqrt{5}}$, that is, $\frac{d}{\sqrt{(6)}^5}$ (for $\sqrt{(3)}$ of $\sqrt{5}$ is $\sqrt{(6)}$) $\Rightarrow \sqrt{(3)} \div \sqrt{242} \div \sqrt{242}$. But by supposition, $d = \sqrt{(3)} \div \sqrt{(6)}$ 128 ; therefore $\sqrt{(3)} \div \sqrt{242} \div \sqrt{242}$ is the cubick Root of $\sqrt{242} \div \sqrt{242}$: which was to be shewn.

$d = \sqrt{(3)} \div \sqrt{(6)}$ 128 ; therefore $\sqrt{(3)} \div \sqrt{(6)}$ 128 divided by $\sqrt{(6)}$ gives the true cubick Root of $\sqrt{242} \div \sqrt{242}$: which was to be shewn.

Example 3. To extract $\sqrt{(3)}$ out of $\sqrt{242} \div \sqrt{242}$.

First, (according to the second Rule of the precedent Preparation) I multiply it by $\sqrt{2}$, and there comes forth $22 \div \sqrt{486}$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44 \div \sqrt{1944}$, whose cubick Root (as before hath been shewn) is $2 \div \sqrt{6}$, which must be divided by $\sqrt{2}$ and there will come forth $\sqrt{2} \div \sqrt{3}$ for the cubick Root sought of $\sqrt{242} \div \sqrt{242}$. But to manifest the reason of dividing $\sqrt{2} \div \sqrt{3}$ by $\sqrt{2}$; let there be put $d = 2 \div \sqrt{6}$, then it follows that $ddd = 44 \div \sqrt{1944} = 22 \div \sqrt{486}$ into 2 , whence $\frac{ddd}{2} = 22 \div \sqrt{486}$; and this Equation divided by $\sqrt{2}$

(because in the Preparation we multiplied by $\sqrt{2}$) gives $\frac{ddd}{\sqrt{2}} = \sqrt{242} \div \sqrt{242}$; therefore $\sqrt{(3)}$ being extracted out of each part of the last Equation there ariseth $\sqrt{(3)} \frac{ddd}{\sqrt{2}}$, that is, $\frac{d}{\sqrt{(6)}^3}$ or $\frac{d}{\sqrt{2}}$ $= \sqrt{(3)} \div \sqrt{242} \div \sqrt{242}$. But by supposition, $d = 2 \div \sqrt{6}$; therefore $2 \div \sqrt{6}$ divided by $\sqrt{2}$, viz. the Quotient $\sqrt{2} \div \sqrt{3}$ shall be the cubick Root of $\sqrt{242} \div \sqrt{242}$: which was to be shewn.

Example 4. To extract $\sqrt{(5)}$ out of $\sqrt{(3)} 3993 \div \sqrt{(6)} 17578125$.

First, (according to the second preparatory Rule) I divide the given Binomial by $\sqrt{(3)}$, and then (according to the fourth preparatory Rule) I multiply the Quotient $\sqrt{(3)} 1331 \div \sqrt{(6)} 1953125$ by 16 , and there comes forth $176 \div \sqrt{32000}$, whose $\sqrt{(5)}$ (as hath been shewn) is $1 \div \sqrt{5}$. Now this Root $1 \div \sqrt{5}$ divided by $\sqrt{(15)}$ 16 ; and the Quotient multiplied by $\sqrt{(15)}$ will discover the true $\sqrt{(5)}$ of $\sqrt{(3)} 3993 \div \sqrt{(6)} 17578125$, the reason of which Division and Multiplication may be made manifest thus, let there be put $d = 1 \div \sqrt{5}$, then it follows that $ddd = 176 \div \sqrt{32000}$; and by dividing each part of the last Equation by 16 (because in the preparatory work we multiplied by 16) there ariseth $\frac{ddd}{16} = \sqrt{(3)} 1331 \div \sqrt{(6)} 1953125$; and by

multiplying each part of this Equation by $\sqrt{(3)}$, there will be produced $\frac{ddd \times \sqrt{(3)}}{16}$ $= \sqrt{(3)} 3993 \div \sqrt{(6)} 17578125$: Therefore $\sqrt{(5)}$ being extracted out of each part of the last Equation, there will arise $\sqrt{(5)} \frac{ddd \times \sqrt{(3)}}{16}$, that is, $\frac{d \sqrt{(3)}}{\sqrt{(15)}^3}$ equal to $\sqrt{(5)}$ of $\sqrt{(3)} 3993 \div \sqrt{(6)} 17578125$. But by supposition, $d = 1 \div \sqrt{5}$; therefore $1 \div \sqrt{5}$ multiplied into $\sqrt{(3)}$, and the Product divided by $\sqrt{(15)}$ 16 , or $\frac{1}{16} \div \sqrt{5}$ divided by $\sqrt{(15)}$ 16 , and the Quotient multiplied by $\sqrt{(15)}$ 16 produceth the true $\sqrt{(5)}$ of $\sqrt{(3)} 3993 \div \sqrt{(6)} 17578125$: which was to be shewn.

The Demonstration follows:

The certainty of the preceding Rule will be made manifest by the three following Propositions.

PROPOSITION I.

If a Binomial whereof one part and the Square of the other are Rational numbers be multiplied into it self cubically, there will be produced another Binomial, the Square of whose lesser part being subtracted from the Square of the greater part, leaves a cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root or first Binomial.

To make this manifest, let there be proposed the Binomial $b \div \sqrt{d}$, this multiplied into it self cubically produceth $bbb \div 3bb\sqrt{d} \div 3bd \div d\sqrt{d}$, to wit, the Cube of $b \div \sqrt{d}$. Here you are to note well, that although in that Cube there be four parts or members, yet they are to be esteemed but as two, one of which, to wit, $bbb \div 3bd$ may design a Rational number, and the other, $3bb\sqrt{d} \div d\sqrt{d}$ (or $bb \div d \times \sqrt{d}$) an irrational or surd number whose Square is Rational; whence it is manifest, first, that the Cube of a Binomial is also a Binomial, viz. $b \div \sqrt{d}$ multiplied into it self cubically, produceth this Binomial

Binomial $bbb + 3bd$ more $3bb/d + d/d$ (or $3bb + d \times d$); secondly, the Rational part $bbb + 3bd$ is manifestly composed of the Cube of the Rational part of the Root and of the triple Product made by the multiplication of the same Root into the Square of its other part; and lastly, the difference of the Squares of the said parts $bbb + 3bd$ and $3bb/d + d/d$ is equal to the Cube of $bb - d$, or of $d - bb$, viz. to the Cube of the difference of the Squares of the parts of the Root $b + d$: For the Squares of $bbb + 3bd$ and $3bb/d + d/d$ are $bbbbb + 6bbbd + 3bbd + 9bbbd + 3bbd + ddd$, and if these Squares be subtracted one from the other, the Remainder is either $bbbbb - 3bbbd + 3bbd - ddd$, which is the Cube of $bb - d$; or else the Remainder is $ddd - 3bbd + 3bbbd - bbb$, which is the Cube of $d - bb$.

To illustrate this Proposition by Numbers, let there be put $b = 2$, and $d = 6$; hence the Binomial $2 + \sqrt{6}$ multiplied into it self cubically produceth the Binomial $44 + \sqrt{1944}$, wherein the difference of the Squares of the parts (viz. the Remainder when 1936 the Square of 44 is subtracted from 1944 the Square of $\sqrt{1944}$) is 8, to wit, the Cube of the difference of the Squares of the parts of the binomial Root $2 + \sqrt{6}$.

Likewise this Binomial $2 + \sqrt{2}$ multiplied into it self cubically produceth the Binomial $20 + \sqrt{292}$, wherein the difference of the Squares of the parts, to wit, 8, is the Cube of the difference of the Squares of the parts of the Root $2 + \sqrt{2}$.

The same properties adhere also to a Residual Root, viz. the Cube of the Residual Root $b + d$ is also a Residual, to wit, $bbb + 3bd + 3bb/d + d/d$, (or $3bb + d \times d$), and the difference of the Squares of the parts of the latter Residual is equal to the Cube of the difference of the Squares of the parts of the Root or first Residual.

PROP. 2.

If a Binomial whereof one part and the Square of the other are Rational numbers, be multiplied by the difference of the Squares of the parts, the Product will be another Binomial, wherein the difference of the Squares of the parts is a Cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root multiplied.

To make this manifest, let there be proposed the Binomial $b + \sqrt{d}$, and suppose b greater than \sqrt{d} ; then $b + \sqrt{d}$ multiplied by $bb - d$, the difference of the Squares of the parts, will produce this Binomial, to wit, $bbb - bd$ more $bb\sqrt{d} - d\sqrt{d}$, the Squares of whose parts are $bbbbb - 2bbbd + bbdd$ and $bbbd - 2bbd + ddd$, then this latter Square subtracted from the former leaves $bbbbb - 3bbbd + 3bbd - ddd$, which is the Cube of $bb - d$ the difference of the Squares of the parts of the first Binomial $b + \sqrt{d}$. The same property would appear if we supposed b less than \sqrt{d} .

To illustrate this Proposition by Numbers, suppose $b = 2$, and $d = 486$; whence the Binomial $2 + \sqrt{486}$ multiplied by 2, the difference of the Squares of the parts, produceth the Binomial $44 + \sqrt{1944}$, wherein the difference of the Squares of the parts is 8, which is the Cube of 2, the difference of the Squares of the parts of the former Binomial $2 + \sqrt{486}$.

PROP. 3.

If the difference of the Squares of any two numbers be divided by a number which doth not exceed the sum of those two numbers above $\frac{1}{2}$; then the Quotient added to the said Divisor will give a number greater than the double of the greater of the said two numbers, but the excess will be less than unity: and if the said Quotient be subtracted from the said Divisor, the Remainder shall be greater than the double of the lesser of the two numbers, but this excess also shall be less than unity.

To manifest this, let a represent the greater of two numbers, and e the lesser; also, let b represent some Fraction not greater than $\frac{1}{2}$: then I say, first, $a + e + b + \frac{ae - ee}{a + e + b}$ is greater than $2a$; but the excess is less than 1, which I prove thus:

It is evident that $aa + ee + bb + 2ae + 2be + 2ba + aa - ee$ is greater than $2aa + 2ae + 2ba$; therefore by dividing each of those two Compound quantities by $a + e + b$, it follows that the first Quotient $a + e + b + \frac{ae - ee}{a + e + b}$ shall be greater than the latter

Quotient $2a$; and if this quantity be subtracted from that, the Remainder $\frac{2be + bb}{a + e + b}$ will be less than 1. For by supposition b is not greater than $\frac{1}{2}$; therefore $2be$ is less than $a + e$,

$a + e$, and bb less than b ; and consequently the Numerator $2be + bb$ is less than the Denominator $a + e + b$: wherefore $\frac{2be + bb}{a + e + b}$ is less than 1.

After the same manner it may be proved that $a + e + b - \frac{ae - ee}{a + e + b}$ is greater than $2e$; but this excess also shall be less than 1: which was to be shewn.

Now to apply the preceding three Propositions to the Demonstration of the Rule before given, let it be required to extract the Cubick Root out of the Binomial $100 + \sqrt{7803}$, whose Rational part 100 is greater than the other part $\sqrt{7803}$. Here we may suppose $bbb + 3bd$ to be 100, and $3bb/d + d/d$ (or $3bb + d \times d$) to be $\sqrt{7803}$; so that $bb + 3bd$ more $3bb + d \times d$ may design the given Binomial $100 + \sqrt{7803}$; and its Cubick root $b + \sqrt{d}$ the Root sought, whose greater part may be b , and the lesser \sqrt{d} : Then, according to the Rule

To extract $\sqrt{(3)}$ out of . . . $100 + \sqrt{7803}$:

First, from the Square of 100, that is, from . . .	>	10000
Subtract the Square of $\sqrt{7803}$, that is, . . .	>	7803
The Remainder is . . .	>	2197
The Cubick root of that Remainder is . . .	>	13 (= $bb - d$)

Which Root 13 is (by Prop. 1.) equal to the difference of the Squares of the parts of the Binomial Root sought.

Secondly, find out a Rational number greater than the sum of the parts of the Cubick root sought, with this Caution, that the excess may not be above $\frac{1}{2}$, viz.

To the greater part of the given Binomial, that is, to . . .	>	100
Add the nearest value in whole numbers of the other part . . .	>	88 or 89
$\sqrt{7803}$, that is, . . .	>	88 and 189

So the sum shews, that the value in whole numbers of the given Binomial falls between 88 and 189.

Whence the Cubick root of the given Binomial is greater than $5\frac{1}{2}$, but less than 6; so that the excess of 6 above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out (as above) 13 the true difference of the Squares of the parts of the Cubick root sought, and 6 a Rational number which exceeds not the true sum of the same parts above $\frac{1}{2}$; We may by the help of Prop. 3, and 1. find out the parts severally in this manner, viz.

Divide the said . . .	>	13
By the said . . .	>	6
And the Quotient is . . .	>	$2\frac{1}{2}$
Which added to the said Divisor 6, makes the sum . . .	>	$8\frac{1}{2}$

Which sum $8\frac{1}{2}$ doth (by Prop. 3.) exceed the double of the greater (to wit, the Rational) part of the Cubick Root sought, but the excess is less than 1; therefore $8\frac{1}{2}$ is less than the said double, but $8\frac{1}{2}$ is greater than the same: and consequently, because the said greater part is supposed to be a Rational whole number, the double thereof must necessarily be 3, (to wit,) the greatest whole number between $7\frac{1}{2}$ and $8\frac{1}{2}$; and therefore the said part it self is 4: which being found out, it is easy to find the other part. For, (by Prop. 1.) if from 16 the Square of the said greater part 4, there be subtracted 13, the Cubick root of the difference of the Squares of the parts of the given Binomial, there will remain 3, the Square of the other part; so that the Cubick root found out is $4 + \sqrt{3}$, which will appear by the Proof to be the true Root sought; for $4 + \sqrt{3}$ being multiplied into it self cubically produceth the given Binomial $100 + \sqrt{7803}$. And for the same reason $4 - \sqrt{3}$ is the Cubick root of $100 - \sqrt{7803}$.

Or more briefly, the Proof may be made thus.

To the Cube of 4 the Rational part of the Root found out, viz. to . . .	>	64, that is, bbb
Add the Product of thrice that part multiplied into the Square of the Sord part found out, viz. the Product . . .	>	36, that is, $3bd$
And it makes the sum . . .	>	100, that is, $bbb + 3bd$

Which

Which sum is the same with the Rational part of the given Binomial, and therefore it proves that $4 + \sqrt[3]{3}$ is the Cubick root sought.

In like manner, to extract $\sqrt[3]{(3)}$ out of $44 + \sqrt[3]{1944}$, where the Rational part 44 is less than the other part $\sqrt[3]{1944}$; we may suppose (as before) $bbb + 3bd$ to be 44, and $3bb\sqrt[3]{d} + d$ (that is, $3bb\sqrt[3]{d} + d\sqrt[3]{d}$) to be $\sqrt[3]{1944}$; so that $bbb + 3bd$ more $3bb\sqrt[3]{d} + d$ may design the given Binomial $44 + \sqrt[3]{1944}$, and its Cubick root $b + \sqrt[3]{d}$ the Root sought, whose lesser part may be b ; and the greater $\sqrt[3]{d}$. Then, according to the Rule

To extract $\sqrt[3]{(3)}$ out of . . . $44 + \sqrt[3]{1944}$.

First, from the Square of $\sqrt[3]{1944}$, viz. from . . . 1944

Subtract the Square of 44, . . . 1936

The Remainder is . . . 8

The Cubick root of that Remainder is . . . $2 (= d - bb)$

Which Root 2 is (by Prop. 1.) equal to the difference of the Squares of the parts of the Binomial root sought.

Secondly, find out a Rational number greater than the sum of the parts of the Cubick root sought, with this Caution, that the excess may not be above $\frac{1}{2}$, which may be done, thus, viz.

To the lesser part of the given Binomial, viz. to . . . 44

Add the nearest value in whole numbers of the other . . . 44 or 45

part $\sqrt[3]{1944}$, that is, . . . 44 or 45

So the sum shows that the value in whole numbers of the . . . 88 and 89 .

Whence the Cubick root of the given Binomial is greater than 4, but less than $4\frac{1}{2}$, that the excess of $4\frac{1}{2}$ above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out 2, the true difference of the Squares of the parts of the Cubick root sought, and $4\frac{1}{2}$ a Rational number which doth not exceed the true sum of the same parts above $\frac{1}{2}$; we may by the help of Prop. 3, and 1. find out the parts severally in this manner, viz.

Divide the said . . . 2

By the said . . . $4\frac{1}{2}$

And it gives the Quotient . . . $4\frac{1}{2}$

Which subtracted from the said Divisor $4\frac{1}{2}$, there remains . . . $4\frac{1}{2}$

Which Remainder $4\frac{1}{2}$ doth (by Prop. 3.) exceed the double of the lesser part (which in this Example is the Rational part) of the Cubick root sought, but the excess is less than 1; Therefore $3\frac{1}{2}$ is less than the said double, but $4\frac{1}{2}$ is greater than the same; and consequently because the said lesser part is a Rational whole number, the double thereof must necessarily be 4, to wit, the greatest whole number between $3\frac{1}{2}$ and $4\frac{1}{2}$, and therefore the said part itself is 2: which being found, it is ealie to find the other part; for if to 4 the Square of the said lesser part 2, there be added 2 the Cubick root of the difference of the Squares of the parts of the given Binomial, the sum 6 shall be the Square of the other part. So that the Cubick root found out is $2 + \sqrt[3]{6}$, which will appear to be the true Cubick root sought; for $2 + \sqrt[3]{6}$ multiplied into it self cubically, produceth the given Binomial $44 + \sqrt[3]{1944}$. And for the same reason $\sqrt[3]{6} - 2$ is the Cubick root of $\sqrt[3]{1944} - 44$.

Or more briefly, the Proof may be made thus:

To the Cube of 2, the Rational part of the Root found . . . 8 , that is, bbb

out, viz. to . . . 8 , that is, bbb

Add the Product of thrice that part multiplied into the . . . 36 , that is, $3bd$

Square of the Sord part found out, viz. the Product . . . 36 , that is, $3bd$

And the sum is . . . 44 , that is, $bbb + 3bd$

Which sum is the same with the Rational part of the given Binomial; and therefore it proves that $2 + \sqrt[3]{6}$ is the Cubick root sought.

Lastly, what hath here been shewn concerning the Demonstration of the Extraction of the Cubick Root, may easily be applied to the Extraction of the other Roots before mentioned, so that there is no need of farther discourse in this matter.

CHAP.

CHAP. X.

An Explication of Simon Stevin's General Rule, to extract one Root out of any possible Equation in Numbers, either exactly, or very nearly true.

Equations falling under any of the Forms in the fourteenth and fifteenth Chapters of the first Book of these Elements, are capable (as hath there been shewn) of perfect Resolutions in Numbers, viz. the value of the Root of Roots sought in any of those Equations may be found out and exprest exactly, either by some Rational or Irrational number or numbers; but the perfect Resolution of all manner of Compound Equations in numbers, I have not found in any Author: and since an Exposition of the General Method of *Vita*, the Rules of *Hudde* and others to that purpose, would make a large Treatise, and after all leave the curious Analyst dissatisfied, I shall not clogg these Elements with a tedious discourse upon those difficult Rules, which at the best are exceeding tedious in Operation, and some of them uncertain too, but rather pursue my first Design, which was to explain Fundamentals, and such Rules as are certain and most important in this profound Art. However, I shall lead the industrious Learner a few steps farther in order to his understanding the Resolution of all manner of Compound Equations in numbers; and in this Chapter explain *Simon Stevin's* General Rule, which with the help of the Rules in the following eleventh Chapter, will discover all the Roots of any possible Equation in numbers, either exactly, if they be Rational, or very nearly true if Irrational.

QUEST. I.

If . . . $aaa + 26a = 40188$, what is the number a ?

RESOLUTION.

This Equation not falling under any of the three Forms in Sect. 1. Chap. 15. Book 1. cannot be resolved by any of the Canons in that Chapter, and therefore according to *Simon Stevin's* general Method I search out the number a by trials, thus, viz.

1. I suppose . . . $a = 1$

Thence it follows that . . . $aaa = 1$

And . . . $26a = 26$

Therefore . . . $aaa + 26a = 27$

Which 27 ought to have been 40188, but it's too little; whereby I find that by supposing a to be 1, I did not hit upon the true number a , and therefore I make another trial, in like manner as before, viz.

2. I suppose . . . $a = 10$

Thence it follows that . . . $aaa = 1000$

And . . . $26a = 260$

Therefore . . . $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third trial, viz.

3. I suppose . . . $a = 100$

Thence it follows, that . . . $aaa + 26a = 1002600$

Which 1002600 exceeds the just Result or absolute number 40188 in the latter part of the Equation first propos'd, and therefore the true number a is less than 100; but the second trial shews it to be greater than 10; and therefore the whole number which expresseth the exact, or at least part of the value of a , must necessarily consist of two Characters, and consequently the first (towards the left hand) must be one of these nine, 1; 2; 3; 4; 5; 6; 7; 8; 9; but because by the second Inquiry 10 was found too little, I now make trial with 2 for the first figure of the Root a , viz.

4. I suppose . . . $a = 20$

Thence . . . $aaa + 26a = 8520$

Which Result 8520 being yet less than the just Result 40188, I make trial again, viz.

5. I suppose . . . $a = 30$

Thence . . . $aaa + 26a = 27780$

L I

Which

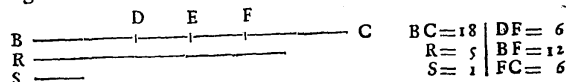
C H A P. VIII.

The second Classis of Examples of the Resolution and Composition of Plane Problems.

IN which Examples, the Resolution ends either in an Equation between the Square of the right line sought, and one or more known Planes; or else in an Analogy whose three first Terms are known Planes, and the fourth gives the Square of the right line sought.

Probl. I.

To cut a given right line into two such parts, that the sum of the Squares of the parts may be to the Square of the difference of the parts in a given Reason. But the first Term of the Reason must exceed the latter.



Suppos.

1. $b = BC$ a right line given to be cut into two parts.

2. $\left\{ \begin{array}{l} r = R \\ s = S \end{array} \right\}$ the Terms of a given Reason.

3. $r \square s$.

Req. to find

4. BF and FC such parts of BC, that $BF + FC = BC$. Also;

5. $\square BF + \square FC . \square : BF - FC :: R . S$.

Resolution.

6. Put a for the difference of the parts sought, viz. $a = BF - FC$.

7. Therefore the Square of the difference of the parts is aa .

8. And from 1° and 6° , the sum of the Squares of the parts (per Theor. 6. Chap. 4.) is $\frac{1}{2}bb + \frac{1}{2}aa$.

9. Therefore according to the tenor of the Problem these must be Proportionals, viz. $r . s :: \frac{1}{2}bb + \frac{1}{2}aa . aa$.

10. Whence, by doubling the Antecedents, this Analogy arises, $2r . s :: bb + aa . aa$.

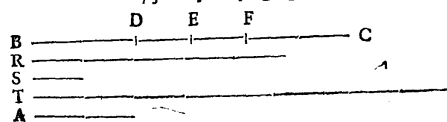
11. Therefore by Division of Reason, $2r - s . s :: bb . aa$. Which last Analogy gives this

C A N O N.

12. As the excess whereby the double of the first Term of the given Reason exceeds the latter Term, is to the latter Term; so is the Square of the line given to be cut into two parts, to the Square of the difference of the parts. Therefore the difference of the parts is given, and consequently the parts are given severally by Theor. 9. Chap. 4.

The reason of the Determination annex'd to the Problem is evident by Theor. 5. Chap. 4. which shews, that if a right line be divided into two unequal parts, the sum of the Squares of the parts is greater than the Square of the difference of the parts, by the double Rectangle of the parts.

The Composition of the foregoing Probl. 1.



Suppos.

Suppos.

13. BC is a right line given to be cut into two parts.

14. R and S are the Terms of a given Reason.

15. $R \square S$.

Req. to find

16. BF and FC such parts of BC, that $BF + FC = BC$. Also;

17. $\square BF + \square FC . \square : BF - FC :: R . S$.

Construction.

18. Find a right line T that may be equal to the excess of $2R$ above S , which is possible to be done, for by Supposition $R \square S$; suppose therefore $T = 2R - S$.

19. By Probl. 11. Chap. 5. let it be made as T to S, so the Square of BC to another Square, whose side suppose to be A, therefore,

$$T . S :: \square BC . \square A.$$

20. Divide BC into two equal parts in E, therefore $EB = EC$.

21. From EC and EB cut off EF and ED, such parts, that each may be equal to $\frac{1}{2}A$; which is possible to be done, if $EB (= EC)$ be greater than $\frac{1}{2}A$. But that EB or EC is greater than $\frac{1}{2}A$, I prove thus;

By Suppos. in 15° , $\dots \dots \dots R \square S$.

And consequently, $\dots \dots \dots 2R \square 2S$.

Therefore by subtracting S from each part, $\dots \dots \dots 2R - S \square S$.

But by Constr. in 18° , $\dots \dots \dots T = 2R - S$.

Therefore from the two last preceding steps, (per Ax. 4. Chap. 2.) $T \square S$.

Therefore from the Analogy in 19° , and from the last preceding step, $BC \square A$.

And consequently, $\dots \dots \dots \frac{1}{2}BC \square \frac{1}{2}A$.

But by Constr. in 20° , $\dots \dots \dots \frac{1}{2}BC = EB = EC$.

Therefore from the two last preceding steps, $\dots \dots \dots EB$ or $EC \square \frac{1}{2}A$.

Which was to be Dem.

22. I say BF and FC are the desired parts of BC. For first, their sum is manifestly equal to BC; and by Constr. in 20° and 21° the difference between the said parts BF and FC, that is, $BF - FC (= BD)$ is equal to DF. But that the sum of the Squares of the parts BF and FC, is to the Square of their difference DF, as R to S, I shall demonstrate by a repetition of the steps of the foregoing Resolution in a backward order.

23. $\dots \dots \dots R . S :: \square BF + \square FC . \square DF$.

Demonstration.

24. By Constr. in 19° , $\dots \dots \dots T . S :: \square BC . \square A$.

25. And by Constr. in 18° and 21° , $2R - S = T$. And $DF = A$.

26. Therefore from 24° and 25° , by exchange of equal quantities, $\dots \dots \dots 2R - S . S :: \square BC . \square DF$.

That is, in 11° , $\dots \dots \dots 2r - s . s :: bb . aa$.

27. Therefore from 26° , by Composition of Reason, $\dots \dots \dots 2R . S :: \square BC + \square DF . \square DF$.

That is, in 10° , $\dots \dots \dots 2r . s :: bb + aa . aa$.

28. Therefore from 27° , by halving the Antecedents, $\dots \dots \dots R . S :: \frac{1}{2}\square BC + \frac{1}{2}\square DF . \square DF$.

That is, in 9° , $\dots \dots \dots r . s :: \frac{1}{2}bb + \frac{1}{2}aa . aa$.

29. By Constr. in 20° and 21° , BC is the sum, and DF the difference of the parts BF and FC, therefore (per Theor. 6. Chap. 4.) $\dots \dots \dots \square BF + \square FC = \frac{1}{2}\square BC + \frac{1}{2}\square DF$.

30. Therefore from 28° and 29° , by exchanging equal quantities, $\dots \dots \dots R . S :: \square BF + \square FC . \square DF$.

Which was to be Demonstr. Therefore that is done which was required by the Problem.

Probl. 11.

Probl. II.

To cut a given right line into two such parts, that the Rectangle of the parts, to the Square of their difference, may have a given Reason.



Suppos.

1. $b = BC$ a right line given to be cut into two parts.

2. $\left\{ \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ the Terms of a given Reason.

Req. to find

3. BF and FC such parts of BC, that $BF \cdot FC = BC$. Also,
4. $\square BF, FC :: BF - FC :: R : S$.

Resolution.

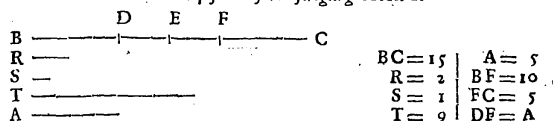
5. Put a for the difference of the parts sought, viz. $a = BF - FC$.
6. Therefore from 1° and 5° , (per Theor. 9. Chap. 4.) the greater part shall be $\frac{1}{2}b + \frac{1}{2}a (= BF)$.
7. And by the same Theorem, the lesser part shall be $\frac{1}{2}b - \frac{1}{2}a (= FC)$.
8. Therefore from 6° and 7° , the Rectangle (or Product) of the parts is $\frac{1}{4}bb - \frac{1}{4}aa$.
9. And from 5° , the Square of the difference of the parts is aa .
10. Therefore from 4° , 8° and 9° , according to the tenor of the Problem, this Analogy ariseth, viz. $r : s :: \frac{1}{4}bb - \frac{1}{4}aa : aa$.
11. Whence, by quadrupling the Antecedents, $4r : s :: bb - aa : aa$.
12. Therefore by Composition of Reason, $4r + s : s :: bb : aa$.
Which last Analogy affords this

CANON.

13. As the summ of the second Term of the given Reason and the quadruple of the first is to the second Term, so is the Square of the line given to be divided into two parts, to the Square of the difference of the parts.

Therefore the difference of the parts is given, and consequently the parts severally, by Theor. 9. Chap. 4.

The Composition of the foregoing Probl. 2.



Suppos.

14. BC is a right line given to be cut into two parts.
15. R and S are the Terms of a given Reason.
Req. to find
16. BF and FC such parts of BC, that $BF \cdot FC = BC$. Also,
17. $\square BF, FC :: BF - FC :: R : S$.
Constrution.
18. Find a right line $T = 4R + S$.
19. By Probl. 11. Chap. 5. let it be made as T to S, so the Square of BC to another Square, whose side suppose to be A, therefore,
 $T : S :: \square BC : \square A$.
20. Divide BC into two equal parts in E, therefore $EC = EB$.
21. From EC and EB cut off EF and ED, such parts, that each may be equal to $\frac{1}{2}A$, which may be done, if $EC (= EB)$ be greater than $\frac{1}{2}A$; but that EC or EB is greater than $\frac{1}{2}A$, I prove thus;

By

By Constrution in 18° , $T \sqsubset S$.
Therefore from 19° , $BC \sqsubset A$.
And consequently, $\frac{1}{2}BC \sqsubset \frac{1}{2}A$.
But by Constr. in 20° , $\frac{1}{2}BC = EC = EB$.
Therefore from the two last preceding steps, EC or $EB \sqsubset \frac{1}{2}A$.

Which was to be Dem.

22. I say BF and FC are such parts of BC as will satisfy the Problem. For first, $BF \cdot FC = BC$, and by Constrution in 20° and 21° , the difference between the said parts BF and FC, that is, $BF - FC (BD)$ is equal to DF. But that the Rectangle of the said parts BF and FC is to the Square of their difference DF as R to S, the following Demonstration, form'd out of the preceding Resolution by a repetition of its steps in a backward order will make manifest.

23. . . . Req. demonstr. . . . $R : S :: \square BF, FC : \square DF$.

Demonstration.

24. By Constr. in 19° , $T : S :: \square BC : \square A$.
25. And by Constr. in 18° and 21° , $4R + S = T$. And $DF = A$.
26. Therefore from 24° and 25° , by ex- $4R + S : S :: \square BC : \square DF$.
change of equal quantities, $4R : S :: \square BC - \square DF : \square DF$.
27. Therefore by Division of Reason, $4R : S :: \frac{1}{4}\square BC - \frac{1}{4}\square DF : \square DF$.
28. And by taking $\frac{1}{4}$ of the Antecedents in 27° , $R : S :: \frac{1}{4}\square BC - \frac{1}{4}\square DF : \square DF$.
29. By Constr. in 20° and 21° , BC is the summ, and DF the difference of the parts BF and FC, therefore by Theor. 7. Chap. 4. $\square BF, FC = \frac{1}{4}\square BC - \frac{1}{4}\square DF$.
30. Therefore from 28° and 29° , by ex- $R : S :: \square BF, FC : \square DF$.
change of equal quantities, . . .
Which was to be Dem. Therefore that is done which was required by the Problem.

A LEMMA, leading to the following Probl. 3.

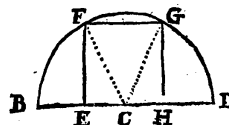
If a Square, or long Square be inscribed in a Semicircle, the Center of the Semicircle is in the middle of the Base of the Square, or long Square.

Suppos.

1. CBFGD a Semicircle, whose Center is C.

2. EFGH is a Square.

3. . . . Req. demonstr. . . . $CE = CH$.



Prepar.

4. Draw the right lines CF and CG.

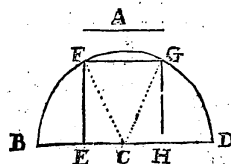
Demonstration.

5. By Defin. 15. Elem. 1. $CF = CG$.
6. And by Suppos. in 2° , $\angle FEC = \angle GCH$.
7. Therefore (per prop. 47. Elem. 1.) $\square EF + \square CE = \square CF = \square CG$.
8. Likewise, $\square HG + \square CH = \square CG = \square CF$.
9. Therefore from 7° and 8° , (per Ax. 1. Chap. 2.) $\square EF + \square CE = \square HG + \square CH$.
10. But by Suppos. $EF = HG$.
11. And consequently, $\square EF = \square HG$.
12. Therefore by subtracting $\square EF$ or $\square HG$ from each part of the Equation in 9° , the remainders will be equal, viz. $\square CE = \square CH$.
13. Therefore $CE = CH$.
Which was to be Dem. The same Demonstration may be made when a long Square is inscribed in a Semicircle.

Probl. III.

Probl. III.

To inscribe a Square in a given Semicircle.



$$\begin{aligned} CB &= CD = 10 \\ EF &= EH = 4.80 \\ CE &= CH = 4.20 \\ EB &= HD = 10 - 4.20 \end{aligned}$$

Suppos.

1. CBFGD is a Semicircle, whose Center is C.

2. $r = CB = CD$ is given.Req. to inscribe $\square EFGH$.

Resolution.

3. Put a for the side of the Square required, viz. $a = EF = EH$.
4. Therefore by the preceding Lemma, $\frac{1}{2}a = CE = CH$.
5. And because (per prop. 47. Elem. 1.) $\square EF + \square CE = \square CF$.
6. Therefore in the letters of the Resolution, $aa + \frac{1}{4}aa = rr$.
7. Therefore, by multiplying the last Equation by 4, $4aa + aa = 4rr$.
8. That is, $5aa = 4rr$.
9. Therefore by taking $\frac{1}{5}$ of the last Equation, $aa = \frac{4}{5}rr$.
10. Therefore, by extracting the square Root out of each part of the last preceding Equation, $a = \sqrt{\frac{4}{5}rr}$.

Hence this

CANON.

11. The square Root of $\frac{4}{5}$ parts of the Square of the Semidiameter is equal to the side of the Square inscribed in the Semicircle.

The Composition of the foregoing Probl. 3.

Suppos.

12. CBFGD is a Semicircle, whose Center is C.

13. $CB = CD$ the Radius is given.Req. to inscribe $\square EFGH$.

Construction.

14. By Probl. 9. Chap. 5. find a mean proportional line A between the given Radius CB and $\frac{1}{2}CB$, therefore $CB : A :: A : \frac{1}{2}CB$.
15. From CB and CD cut off CE and CH , such segments, that as well CE as CH may be equal to $\frac{1}{2}A$, and consequently $EH = A$, which Effect is possible, for by Construction in 14th CB is the greatest of three Proportionals, whereof A is the mean, therefore $CB < A$, and because $CD = CB$, therefore also $CD < A$, and consequently a segment equal to $\frac{1}{2}A$, as CE or CH may be cut off from CB or CD .
16. Make EF and $HG \perp BD$, and draw FG , so is $EFGH$ the Square required to be inscribed, as will be evident by the following Demonstration, form'd out of the preceding Resolution, by a repetition of its steps in a backward (not direct) order.
17. . . . Req. demonstr. . . . $EFGH$ is a \square .

Preparat.

18. Draw the right lines CF and CG .

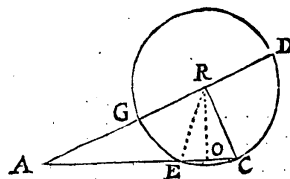
Demonstration.

19. By Constr. in 14th, $CB : A :: A : \frac{1}{2}CB$.
20. And by Constr. in 15th, $EH = A$.
21. Therefore from 19th and 20th, by exchanging equal right lines, $CB : EH :: EH : \frac{1}{2}CB$.
22. And from 21st, (per prop. 17. Elem. 6.) $\square EH = \frac{1}{2} \square CB$.
23. But the quintuples of equal quantities are also equal, therefore from 22nd, $5 \square EH = 4 \square CB$.
24. That

24. That is $4 \square EH + \square EH = 4 \square CB$.
 25. And by taking $\frac{1}{5}$ of all in 24th, $\square EH + \frac{1}{4} \square EH = \square CB = \square CF$.
 26. And because by Constr. in 15th, $CE = \frac{1}{2}EH = CH$.
 27. And consequently, (per Theor. 3. Chap. 4.) $\square CE = \frac{1}{4} \square EH$.
 28. Therefore from 25th and 27th, by exchanging equal quantities, $\square EH + \square CE = \square CF$.
 29. And because by Constr. in 16th, $\angle FEH = \angle GHE$.
 30. And consequently, (per prop. 47. Elem. 1.) $\square EF + \square CE = \square CF$.
 31. Therefore from 28th and 30th, (per Ax. 1. Ch. 2.) $\square EH + \square CE = \square EF + \square CE$.
 32. Therefore from 31st, by subtracting $\square CE$ from each part, $\square EH = \square EF$.
 33. But the sides of equal Squares are also equal, therefore from 32nd, $EH = EF$.
 34. And by arguing in like manner as in the six last preceding steps, it will be manifest that $EH = HG$.
 35. And because from 33rd, 34th and 29th, (per Ax. 1. & prop. 28. Elem. 1.) $EF =$ and $\parallel HG$.
 36. Therefore from 35th, (per prop. 33. Elem. 1.) $EH =$ and $\parallel FG$.
 37. Therefore from 33rd, 34th and 36th, $EFGH$ is equilateral.
 38. And from 29th, and Coroll. prop. 29. Elem. 1. $EFGH$ is right-angled.
 39. Therefore from 37th and 38th, (per def. 29. El. 1.) $EFGH$ is a \square .
- Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. IV.

The Hypotenusal of a right-angled Triangle being given, as also the sum of the leggs containing the right angle, to find the Triangle. But the sum of the leggs must be greater than the Hypotenusal, yet not greater than the right line arising by application of the double Square of the Hypotenusal to the sum of the leggs.



Suppos.

1. ARC is a \triangle right-angled at R .
2. $b = AC$ the Hypotenusal is given.
3. $b = AR + RC$, the sum of the leggs is given.

Req. to find $\triangle ARC$.

Resolution.

4. Supposing the leggs about the right angle to be unequal, to wit, $AR < RC$, put a for their difference, viz. $a = AR - RC = AG$.
5. Therefore from 3rd and 4th, the sum of the Squares of the leggs (per Theor. 6. Chap. 4.) is $\frac{1}{2}bb + \frac{1}{2}aa$.
6. Therefore from 5th and 2nd, (per prop. 47. Elem. 1.) this Equation ariseth, $\frac{1}{2}bb + \frac{1}{2}aa = bb$.
7. And by doubling each part of that Equation, $bb + aa = 2bb$.
8. And by subtracting bb from each part of the last Equation, $aa = 2bb - bb$.
9. Therefore by extracting the square Root out of each part of the last preceding Equation, $a = \sqrt{2bb - bb}$.

Hence this

M m

CANON.

CANON.

10. The difference of the legs about the right angle is equal to the Square Root of the excess whereby the double Square of the Hypotenusal exceeds the Square of the sum of the legs.

Therefore the difference of the legs is given, and consequently by the given sum and difference of the legs, the legs shall be given severally, *per Theor. 9. Chap. 4.*

11. But in order to the Geometrical Effect of the Problem propounded, the truth and reason of the Determination annex'd to it must be made manifest. First then, the reason of the first part of the Determination, to wit, that the right line given for the sum of the legs about the right angle must be longer than the line given for the Hypotenusal, is evident by *prop. 22. Elem. 1.* which shews that the sum of every two sides of a plain Triangle is greater (or longer) than the third. The latter part of the Determination is discovered by the Canon, which requires that bb be subtracted from $2hh$, and therefore bb must not exceed $2hh$, and consequently, by dividing as well bb as $2hh$ by b , 'tis manifest that b must not be greater than $\frac{2hh}{b}$; that is, the right

line given for the sum of the legs about the right angle, must not be longer than the right line arising by the Application of the double Square of the Hypotenusal to the sum of the legs. The truth of this Determination will more fully appear by the following

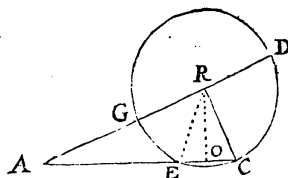
THEOREM.

12. In a right-angled plain Triangle, the sum of the legs about the right angle is sometimes less than the right line arising by the Application of the double Square of the Hypotenusal to the sum of the legs, and sometimes equal to, but never greater than the said right line.

The legs about the right angle are either unequal, or else equal between themselves; I shall begin with the first Case.

Suppos. in Case 1.

13. $\angle ARC$ is a Δ .
14. $\angle ARC$ is \perp .
15. $AR = RC$.
16. $RCGD$ is a \circ , whence
17. $AD = AR + RC (RD)$.
18. $AG = AR - RC (RG)$.



19. . . . *Req. demonstr.* . . . $AD = \frac{2 \square AC}{AD}$.

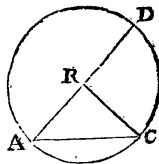
Demonstration.

20. By *Supposition* in 17° , . . . $AD = AR + RC$.
21. And by *Suppos.* in 18° , . . . $AG = AR - RC$.
22. Therefore from 20° and 21° , (*per Theor. 6. Ch. 4.*) $\frac{1}{2}AD + \frac{1}{2}AG = \square AR + \square RC$.
23. And because by *Suppos.* in 14° , $\angle ARC$ is \perp , . . . $\square AC = \square AR + \square RC$.
therefore (*per prop. 47. Elem. 1.*) . . . $\square AC = \square AR + \square RC$.
24. Therefore from 22° and 23° , (*per Ax. 1. Ch. 2.*) $\frac{1}{2}AD + \frac{1}{2}AG = \square AC$.
25. And by doubling the last Equation, . . . $\square AD + \square AG = 2 \square AC$.
26. Therefore from 25° , . . . $\square AD = 2 \square AC$.
27. Therefore by Application of each part to AD , $AD = \frac{2 \square AC}{AD}$.
Which was *Case 1.* to be Dem.

Suppos. in Case 2.

28. $\angle ARC$ is a Δ .
29. $\angle ARC$ is \perp .
30. $AR = RC = RD$.
31. $AD = AR + RC$.

32. . . . *Req. demonstr.* . . . $AD = \frac{2 \square AC}{AD}$.



Demon.

Demonstration.

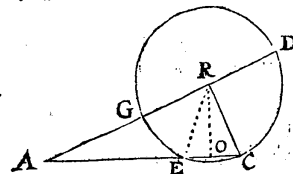
33. Because by *Suppos.* in 30° and 31° , . . . $AR = RC = \frac{1}{2}AD$.
34. Therefore their Squares are also equal, *viz.* . . . $\square AR = \frac{1}{4} \square AD$.
35. Likewise from 33° , . . . $\square RC = \frac{1}{4} \square AD$.
36. The sum of the Equations in 34° and 35° , gives, . . . $\square AR + \square RC = \frac{1}{2} \square AD$.
37. And because by *Suppos.* $\angle ARC$ is \perp , therefore . . . $\square AR + \square RC = \square AC$.
(*per prop. 47. Elem. 1.*) . . . $\frac{1}{2} \square AD = \square AC$.
38. Therefore from 36° and 37° , (*per Ax. 1. Chap. 2.*) . . . $\square AD = 2 \square AC$.
39. And by doubling the last Equation, . . . $AD = \frac{2 \square AC}{AD}$.
40. Therefore by Application of each part to AD , . . . $AD = \frac{2 \square AC}{AD}$.

Which was *Case 2.* to be Demonstr.

41. Now because in every right-angled Triangle, the sides about the right angle are either unequal or equal between themselves, and it hath been demonstrated, that when the said sides are unequal, their sum is less than the right line arising by the Application of the double Square of the Hypotenusal to the said sum; but when the said sides are equal to one another, their sum is equal to the said right line; it is evident that the sum of the sides about the right angle can never be greater than the right line arising by the said Application. Therefore the truth of the Theorem is manifest, and consequently the Hypotenusal and the sum of the legs about the right angle must be given with due Caution, according to the import of the Determination annex'd to the Problem, that its Solution may be possible.

The Composition of the foregoing Probl. 4.

H _____
B _____
F _____



Suppos.

42. H = the Hypotenusal of a right-angled Triangle is given.
43. B = AD the sum of the legs about the right angle is given.
44. $AD < H$, but AD not $< \frac{2 \square H}{AD}$. (*Determination.*)
Req. to find the Triangle.

Construction.

45. By the *Determination* in 44° AD is not greater than $\frac{2 \square H}{AD}$, suppose then it be granted, or discovered by H and B given in numbers, that AD is less than $\frac{2 \square H}{AD}$, and consequently, (by multiplying each part by AD), that $\square AD < 2 \square H$, then it evidently follows, that 'tis possible (*per Probl. 4. Chap. 5.*) to find out a right line F , such, that its Square shall be equal to $2 \square H - \square AD$, suppose therefore
 $F = \sqrt{2 \square H - \square AD}$.
46. From AD cut off $AG = F$, which may be done, for that AD is greater than F , I prove thus;
By *Suppos.* in 44° , . . . $AD < H$.
Therefore . . . $\square AD < \square H$.
And by doubling each part, . . . $2 \square AD < 2 \square H$.
And by subtracting $\square AD$ from each part, . . . $\square AD < \square H - \square AD$.
But by *Constr.* in 45° , . . . $\square F = 2 \square H - \square AD$.
Therefore from the two last preceding steps, (*per Ax. 3. Ch. 2.*) $\square AD < \square F$.
Therefore . . . $AD < F$.
Which was to be Dem.

47. Divide GD into two equal parts in R , therefore $RG = RD$.
Mm 2

48. Make

48. Make $RC \perp AR$, also $RC = RD$ or RG , and draw AC .
 49. I say ARC is the Triangle sought. Now we must shew that it will satisfy the Problem; first then by *Construction* in 48° , $RC \perp AR$, therefore the angle ARC is a right angle; secondly, the sum of the legs AR and RC about the right angle, is equal to $AD (= B)$ the given sum of the legs. It remains only to prove that AC is equal to the given Hypothenusal H ; but that will be made manifest by the following Demonstration, form'd out of the preceding Resolution by a repetition of its steps in a backward (not direct) order.

50. . . . *Req. demonstr.* $AC = H$.

Demonstration.

51. By *Constr.* in 45° , $\sqrt{2} \square H - \square AD = F$.
 52. And by *Constr.* in 46° , $AG = F$.
 53. Therefore from 51° and 52°, (per *Ax. 1. Chap. 2.*) $AG = \sqrt{2} \square H - \square AD$.
 54. But the Squares of equal right lines are also equal, $\square AG = 2 \square H - \square AD$, therefore from 53°,
 55. Therefore from 54°, by adding $\square AD$ to each part, $\square AD + \square AG = 2 \square H$.
 56. And by taking the halves of all in 55°, $\frac{1}{2} \square AD + \frac{1}{2} \square AG = \square H$.
 57. And because by *Constr.* in 47° and 48° , AD is the sum, and AG the difference of the parts AR and RC , (or RD), therefore (per *Theor. 6. Chap. 4.*)
 58. Therefore from 56° and 57°, (per *Ax. 1. Chap. 2.*) $\square H = \square AR + \square RC$.
 59. But by *Constr.* in 48° , $\angle ARC$ is \perp , and consequently, (per *prop. 47. Elem. 1.*) $\square AC = \square AR + \square RC$.
 60. Therefore from 58° and 59°, (per *Ax. 1. Chap. 2.*) $\square AC = \square H$.
 61. But the sides of equal Squares are also equal, $AC = H$, therefore

Which was to be Dem. Therefore the Problem is satisfied.

Probl. V.

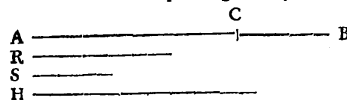
To cut a given right line into two such parts, that the Square of the whole line, to the sum of the Squares of the parts may have a given Reason.

Or thus,

In a right-angled Triangle, the sum of the legs about the right angle being given, as also the Proportion which the Square of the laid sum hath to the Square of the Hypothenusal, to find the Triangle. But the given quantities must be liable to this

Determination.

The first Term of the given Reason must be greater than the latter Term, yet not greater than the double of the latter Term. For in a right-angled Triangle, the Square of the sum of the legs about the right angle is always greater than the Square of the Hypothenusal, but never greater than the double Square of the Hypothenusal, as hath been demonstrated in the preceding *Probl. 4.*



$$\begin{array}{l|l} AB = 34 & H = 26 \\ R = 289 & AC = 24 \\ S = 169 & CB = 10 \end{array}$$

Suppos.

1. AB is a right line given to be cut into two parts.
 2. R and S are the Terms of a given Reason.
 3. $R \leq S$; but R not $\leq 2S$.

Req. to find

4. AC and CB such parts of AB , that $AC + CB = AB$. Also,
 5. $\square AB : \square AC + \square CB :: R : S$.

Con-

Construction.

6. By *Probl. 11. Chap. 5.* let it be made, as R to S , so the Square of AB to another Square, whose side suppose to be H , therefore

$$R : S :: \square AB : \square H.$$

7. Then supposing H to be the Hypothenusal of a right-angled Triangle, and AB the sum of the sides about the right-angle, find out the said sides and Triangle by the foregoing *Probl. 4.* For if R be greater than S , but not greater than $2S$, according to the import of the Determination added to that Problem, 'tis possible to find out such a right-angled Triangle, and then the sides about the right-angle shall be equal to AC and CB , the parts sought by this *Probl. 5.* Therefore,

8. . . . *Req. demonstr.* $R : S :: \square AB : \square AC + \square CB$.

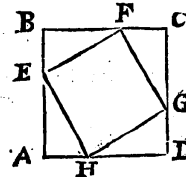
Demonstration.

9. Because by *Constr.* in 6° , $R : S :: \square AB : \square H$.
 10. And by *Constr.* in 7° , $\square AC + \square CB = \square H$.
 11. Therefore from 9° and 10°, by exchanging equal quantities, $R : S :: \square AB : \square AC + \square CB$.

Which was to be Dem. Therefore the Problem is satisfied.

LEMMA, leading to the following *Probl. 6.*

Suppos.



1. $ABCD$ is a Square.
 2. BF and FC are parts of BC .
 3. $CF = BE = AH = DG$, therefore,
 4. $BF = AE = DH = CG$.
 5. EF, FG, GH, HE are right lines.

6. . . . *Req. demonstr.* that $EFGH$ is a Square.

Demonstration.

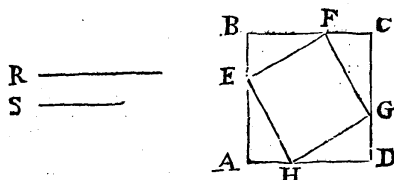
7. Because by *Suppos.* in 3° and 4° , $CF = BE$; and $BF = CG$.
 8. And by *Suppos.* in 1° , $\angle EBF = \angle FCG$.
 9. Therefore from 7° and 8°, (per *prop. 4. El. 1.*) $EF = FG$, and $\angle BEF = \angle GFC$.
 10. And in like manner, $EF = EH = HG = FG$.
 11. Again, because by *Suppos.* in 1° , $\angle EBF$ is \perp .
 12. Therefore (per *Coroll. prop. 32. Elem. 1.*) $\angle BFE + \angle BEF = \perp$.
 13. But it hath been proved in 9°, that $\angle GFC = \angle BEF$.
 14. Therefore from 12° and 13°, (per *Ax. 6. Chap. 2.*) $\angle BFE + \angle GFC = \perp$.
 15. And from 14°, (per *Coroll. prop. 13. Elem. 1.*) $\angle EFG$ is \perp .
 16. And by arguing in like manner: as in the five last preceding steps, it will be evident that $\angle FGH = \perp = \angle GHE = \angle GEF$.
 17. Therefore from 10°, 15° and 16°, (per *Defin. 29. Elem. 1.*) $EFGH$ is a Square.

Which was to be Dem.

Probl. VI.

In a given Square to inscribe another Square whose angular points may lie in the sides of the given Square, and that the Square given to the Square inscribed may be in a given Reason, suppose as R to S . But R must be greater than S , yet not greater than $2S$; as may easily be infer'd from the preceding *Probl. 5.*

R —



$$\begin{aligned} BC &= 34 = BA \\ R &= 289 \\ S &= 169 \\ BF &= 24 \\ FC &= 10 = BE \\ EF &= 26 = FG \end{aligned}$$

Suppos.

1. ABCD is a Square, whose side AB or BC is given.
2. R and S are the Terms of a given Reason.
3. $R \sqsubset S$; but R not $\sqsubset 2S$.

Req. to inscribe

4. $\square EFGH$ in the $\square ABCD$. Also, that
5. $\square ABCD \cdot \square EFGH :: R \cdot S$.

Construction.

6. By the foregoing *Probl.* 5. of this *Chapt.* cut BC the side of the given Square into two such parts in F, that the Square of BC, that is, $\square ABCD$, may be in such proportion to the sum of the Squares of the parts BF, FC, as R to S; which may be done, because by *Supposition* $R \sqsubset S$, yet R not $\sqsubset 2S$, suppose therefore

$$R \cdot S :: \square BC \cdot \square BF + \square FC$$

7. Make BE = CF = AH = DG, and draw the right lines EF, FG, GH and HE; so is EFGH the inscribed Square required. Now we must shew that it will satisfy the Problem, to which end, two things are to be proved, *viz.* First, that EFGH is a Square, and then that the Square ABCD hath such proportion to the Square EFGH as R to S: Therefore,

8. *Reg. demonstr.* $\dots \dots \dots \square EFGH$ is a \square . Also, that $R \cdot S :: \square ABCD \cdot \square EFGH$.

Demonstration.

9. By *Suppos.* in 1° , $\dots \dots \dots \square ABCD$ is a \square .
10. Therefore $BC = BA = AD = DC$.
11. By *Constr.* in 7° , $\dots \dots \dots CF = BE = AH = DG$.
12. Therefore by subtracting the Equations in 11° from those in 10° , there will remain $BF = AE = DH = CG$.
13. Therefore out of $9^\circ, 10^\circ, 11^\circ$ and 12° , by the *Lemma* prefix before this Problem, $\square EFGH$ is a \square .
14. Again by *Constr.* in 6° , $\dots \dots \dots R \cdot S :: \square BC \cdot \square BF + \square FC$.
15. And because by *Suppos.* in 1° , $\dots \dots \dots \angle EBF$ is \perp .
16. And consequently, (per *prop.* 47. *Elem.* 1.) $\square EF = \square BF + \square BE$ ($\square FC$).
17. Therefore from 14° and 16° , by exchanging equal quantities, $R \cdot S :: \square BC$ (or $\square ABCD$) $\cdot \square EF$ (or $\square EFGH$).

Which was to be Demonstr. Therefore the Problem is satisfied.

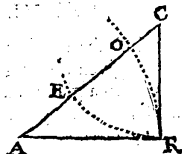
Probl. VII.

In a right-angled Triangle, the difference between the Hypotenusal and each of the sides about the right angle being given, to find the Triangle.

Suppos.

1. ARC is a \triangle right-angled at R.
2. $b = OC = AC - AR$ is given.
3. $d = AE = CA - CR$ is given; whence,
4. $g = d + b$ is given. And,
5. $kk = dd + bb$ is given.

Req. to find $\triangle ARC$.



$$\begin{aligned} AR &= 4 \\ RC &= 3 \\ AC &= 5 \\ OC &= 1 = AC - AR (AO) \\ AE &= 2 = CA - CR (CE) \end{aligned}$$

Resolu-

Resolution.

6. Put a for the excess of the Hypotenusal above the sum of the given differences, *viz.* $a = EO$.
7. Therefore from the premises the Hypotenusal shall be $a + g$ ($= AC$).
8. And the Base $a + d$ ($= AR$).
9. And the Perpendicular $a + b$ ($= CR$).
10. Therefore from 7° the Square of the Hypotenusal is $aa + 2ga + gg$.
11. And from 8° the Square of the Base is $aa + 2da + dd$.
12. And from 9° the Square of the Perpendicular is $aa + 2ba + bb$.
13. Therefore the sum of all in 11° and 12° , gives the sum of the Squares of the Base and Perpendicular, *viz.* $2aa + 2da + 2ba + dd + bb$.
14. That is, (because by *Suppos.* in 5° , $kk = dd + bb$, and from 4° , $2g = 2d + 2b$.) $2aa + 2ga + kk$.
15. Therefore from 10° and 14° , this Equation aritheth, (per *prop.* 47. *Elem.* 1.) $2aa + 2ga + kk = aa + 2ga + gg$.
16. Therefore, by subtracting $aa + 2ga$ from each part of that Equation, this aritheth, $aa + kk = gg$.
17. And by subtracting kk from each part of the last Equation, $aa = gg - kk$.
18. But from 4° , $dd + bb + 2db = gg$.
19. And from 5° , $dd + bb = kk$.
20. And by subtracting the Equation in 19° from that in 18° , this remains, $2db = gg - kk$.
21. Therefore from 17° and 20° , (per *Ax.* 1. *Chap.* 2.) $aa = 2db$.
22. Therefore by extracting the Square Root out of each part of the last Equation, it gives $a = \sqrt{2db}$.
23. Therefore out of $22^\circ, 6^\circ, 7^\circ$ and 4° , the Hypotenusal is given, to wit, $d + b + \sqrt{2db} = AC$.
24. And from $22^\circ, 6^\circ, 8^\circ$ and 3° , the Base is also given, *viz.* $d + \sqrt{2db} = AR$.
25. And from $22^\circ, 6^\circ, 9^\circ$ and 3° , the Perpendicular is also discovered, *viz.* $b + \sqrt{2db} = CR$.

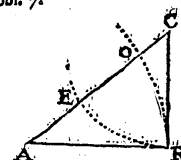
Which three last preceding steps give this

CANON.

26. To the sum of the given differences add the Square Root of their double Rectangle; so shall the sum of that Addition be the Hypotenusal sought. Then add that Square Root to the given differences severally, and these two sums shall be the desired sides about the right angle.

The Composition of Probl. 7.

$$\begin{array}{l|l} B \text{ --- } 1 & AR = 4 \\ D \text{ --- } 2 & RC = 3 \\ M \text{ --- } 2 & AC = 5 \\ E \text{ --- } 3 & \end{array}$$



Suppos.

27. B = the excess whereby the Hypotenusal of a right-angled Triangle exceeds one of the sides about the right angle is given.
28. D = the excess of the Hypotenusal above the other side about the right angle is given also.

Req. to find out the Triangle.

Construction.

29. By *Probl.* 9. *Chap.* 5. find a mean Proportional line, as M, $2D \cdot M :: M \cdot B$, between $2D$ and B , therefore, $\dots \dots \dots$
30. Then

30. Then let a Triangle, as ARC, be made of three right lines equal to these given, to wit, $D+B+M$, $D+M$, $E+M$; which may be done, (per *probl. 22. Elem. 1.*) for the sum of every two of those three lines is manifestly greater than the third; suppose therefore

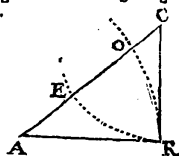
$$\begin{aligned} AC &= D+B+M, \\ AR &= D+M, \\ RC &= B+M. \end{aligned}$$

31. I say ARC is the Triangle required. Now we must shew that it will satisfy the Problem. First then, 'tis manifest that the difference between AC, that is, $D+B+M$, and AR, that is, $D+M$, is equal to the given difference B; also the difference between AC, that is, $D+B+M$, and RC, that is, $B+M$, is manifestly equal to the given difference D. So it remains only to prove that the angle ARC is a right angle, which will be made manifest by the following Demonstration.

Prepar.

32. Make $F=B+M$, therefore
33. From 30° and 32° , $AC=D+F$. Also $RC=F$.
34. *Req. demonstr.* $\angle ARC = \perp$.

Demonstration.

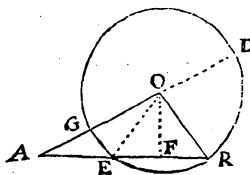


35. Because by *Constr.* in 30° , $AR = D+M$.
36. Therefore, (per *Theor. 2. Chap. 4.*) $\square AR = \square D + \square M + 2\square DM$.
37. By *Constr.* in 32° , $F = B+M$.
38. Therefore from 37° , by drawing ED into each part, (per *prop. 1. Elem. 6.*) $2\square DF = 2\square DB + 2\square DM$.
39. But from the *Constr.* in 29° ; it follows (per *prop. 17. Elem. 6.*) that $\square M = 2\square DB$.
40. Therefore from 38° and 39° , (per *Ax. 6. Chap. 2.*) $2\square DF = \square M + 2\square DM$.
41. Likewise from 36° and 40° , $\square AR = \square D + 2\square DF$.
42. By *Constr.* in 33° , $RC = F$.
43. And consequently, $\square RC = \square F$.
44. Therefore the sum of the Equations in 41° and 43° , gives (per *Ax. 8. Chap. 2.*) $\square AR + \square RC = \square D + 2\square DF + \square F$.
45. By *Constr.* in 33° , $AC = D+F$.
46. And consequently, (per *Theor. 2. Chap. 4.*) $\square AC = \square D + 2\square DF + \square F$.
47. Therefore from 44° and 46° , (per *Ax. 1. Chap. 2.*) $\square AC = \square AR + \square RC$.
48. Therefore, (per *prop. 48. Elem. 1.*) $\angle ARC = \perp$.
Which was to be Dem. Therefore the Problem is satisfied.

Probl. VIII.

The Base, Perpendicular and sum of the legs of a plain Triangle being given severally, to find the Triangle. But the lines given must be subject to the Determination hereafter declared.

Note. There is more than enough given in this Problem, unless it requires a Triangle that hath either unequal acute angles at the Base, in which Case the Perpendicular falls within the Triangle; or else is obtufangled at the Base, in which latter Case the Perpendicular falls without. The following Resolution handles the first Case, but with a little alteration it may be applied to the latter, as will hereafter appear.



$$\begin{array}{l|l} AR = 21 & FR = 6 \\ AO = 17 & OF = 8 \\ RO = 10 & AF = 15 \\ AE = 9 & AG = 7 \\ EF = 6 & AD = 27 \end{array}$$

Preparat.

Preparat.

1. Suppose ARO to be the Triangle sought, having unequal acute angles A and R at the ends of the Base AR, then from the Center O, at the distance of the lesser legg OR, describe the Circle ORGD cutting the greater legg OA in G, so shall AG be the difference of the leggs OA and OR, for $OG = OR$.
2. Produce AO to the Circumference in D, then is AD equal to the sum of the leggs AO and OR, for $OD = OR$.
3. Draw the Semidiameter OE, and let fall $OF \perp ER$, so will OF cut ER into two equal parts in F, (per *prop. 3. Elem. 3.*) These things premised, the Resolution of the Problem propos'd may be formed in manner following.

Suppos.

4. $b = AR$, the Base of $\triangle ARO$ is given.
5. $p = OF$ the Perpendicular is given.
6. $c = AD = AO + OR$, the sum of the leggs is given.

Req. to find the Triangle.

Resolution.

7. Put a for the unknown difference of the leggs, viz. assume $a = AG (= AO - OR)$
8. Then the difference of the segments of the Base made by the falling of the Perpendicular shall be $\frac{ca}{b}$, for, (by

Theor. 2. Probl. 9. Chap. 7.)

9. Therefore from 4° and 8° , (by *Theor. 9. Chap. 4.*) the greater segment of the Base shall be $\frac{1}{2}b + \frac{ca}{2b} (= AF)$

10. And (by the Theorem last mentioned,) the lesser segment shall be $\frac{1}{2}b - \frac{ca}{2b} (= FR)$

11. Therefore the Square of the greater segment (in 9°) shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca$

12. And the Square of the lesser segment (in 10°) shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca$

13. By *prop. 47. Elem. 1.* $\square AF + \square FO = \square AO$

14. Therefore out of 11° , 13° and 5° , the Square of the greater legg AO shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp$

15. And out of 5° and 12° , the Square of the lesser legg OR shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp$

16. Therefore from 14° and 15° , the sum of the Squares of the leggs shall be $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp$

Now because by *Theor. 2. Chap. 4.*

$$\square AO + \square RO + 2\square AO, RO = \square AO + \square OR = \square AD;$$

Therefore to the end the quantities in 16° may be made a compleat Square of AD, let the double Rectangle of the leggs AO and RO be found out in this manner, viz.

17. From 6° and 7° , (by *Theor. 9. Chap. 4.*) the greater legg is $\frac{1}{2}c + \frac{1}{2}a (= AO)$

18. And (by the Theorem last mentioned,) the lesser legg is $\frac{1}{2}c - \frac{1}{2}a (= RO)$

19. Therefore the Rectangle of the leggs is $\frac{1}{4}cc - \frac{1}{4}aa (= \square AO, RO)$

20. And the double Rectangle of the leggs is $\frac{1}{2}cc - \frac{1}{2}aa (= 2\square AO, RO)$

21. Therefore the sum of all in 16° and 20° , gives the Square of the sum of the leggs, viz. $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa (= \square AD)$

N n

22. Which

63. But the sides of proportional Squares are also Proportionals, therefore from 62.^d this Analogy ariseth,

$$\sqrt{\square AD - \square AR} : \sqrt{\square AD - \square AR - 4\square OF} :: AR : AG.$$

That is, in 29^o,

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b : a.$$

Which was Theor. 2. to be Demonstr.

64. Again, because by Theor. 2. in 29^o of Probl. 9. Chapt. 7.

$$AD : AE :: AR : AG.$$

65. Therefore from 63^o and 64^o, (per prop. 11. Elem. 5.)

$$\sqrt{\square AD - \square AR} : \sqrt{\square AD - \square AR - 4\square OF} :: AD : AE.$$

Which was Theor. 3. to be Demonstr.

66. Again, it hath been shewn in 62^o, that

$$\square AR : \square AG :: \square AD - \square AR : \square AD - \square AR - 4\square OF.$$

67. Therefore by converse Reason,

$$\square AR : \square AR - \square AG :: \square AD - \square AR : 4\square OF.$$

Which last Analogy affords

THEOR. 4.

68. As the Square of the Base of any plain Triangle whose legs are unequal, is to the excess whereby the Square of the Base exceeds the Square of the difference of the legs; so is the excess whereby the Square of the sum of the legs exceeds the Square of the Base, to the Square of the double Perpendicular.

Therefore, the Base and legs of any plain Triangle whose legs are unequal, being severally given in numbers, the Perpendicular falling upon that Base within the Triangle, or without upon the Base increased, shall be given also in numbers.

From the said Theor. 4. and prop. 41. Elem. 1. 'tis easie to deduce this following

THEOR. 5.

69. The Rectangle made of these two right lines, to wit, the right line whose Square is equal to the excess whereby a quarter of the Square of the Base of a plain Triangle exceeds a quarter of the Square of the difference of the legs; and the right line whose Square is equal to the excess of a quarter of the Square of the sum of the legs above a quarter of the Square of the Base, shall be equal to the Triangle.

To make this manifest, let the $\triangle ARO$ be taken as before in the Resolution, then

$$70. \dots \text{Req. demonstr. } \square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} = \triangle ARO.$$

Demonstration.

71. By Theor. 4. in 68^o of this Problem,

$$\square AR : \square AR - \square AG :: \square AD - \square AR : 4\square OF.$$

72. And by taking $\frac{1}{4}$ of every Term of that Analogy,

$$\frac{1}{4}\square AR : \frac{1}{4}\square AR - \frac{1}{4}\square AG :: \frac{1}{4}\square AD - \frac{1}{4}\square AR : \square OF.$$

73. But the sides of proportional Squares are also Proportionals, therefore from the last Analogy,

$$\frac{1}{4}\square AR : \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} :: \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} : \square OF.$$

74. Therefore, (per prop. 16. Elem. 6.)

$$\frac{1}{4}\square AR, \square OF = \square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR}.$$

75. But (per prop. 41. Elem. 1.)

$$\frac{1}{4}\square AR, \square OF = \triangle ARO.$$

76. Therefore from 74^o and 75^o, (per Ax. 1. Chapt. 2.)

$$\square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} = \triangle ARO.$$

Which was to be Dem.

Hence the following Canons are deducible, to find out the Area of a plain Triangle Arithmetically, without the help of the Perpendicular, the Base and legs being severally given in numbers, and the legs unequal between themselves.

CANON 1.

77. From a quarter of the Square of the Base subtract a quarter of the Square of the difference of the legs, and reserve the remainder; then from a quarter of the Square of the sum of the legs subtract a quarter of the Square of the Base, and reserve the remainder;

remainder; that done, multiply the first remainder by the second, and extract the Square Root of the Product, so shall that Square Root be the Area of the Triangle.

78. Again, because by Theor. 8. Chapt. 4.

$$\frac{1}{4}\square AR - \frac{1}{4}\square AG = \square \text{ of } \frac{1}{2}\square AR + \frac{1}{2}\square AG \times \frac{1}{2}\square AR - \frac{1}{2}\square AG.$$

79. Likewise by the same Theorem,

$$\frac{1}{4}\square AD - \frac{1}{4}\square AR = \square \text{ of } \frac{1}{2}\square AD + \frac{1}{2}\square AR \times \frac{1}{2}\square AD - \frac{1}{2}\square AR.$$

Therefore from 77^o, 78^o and 79^o, by exchanging equal Factors, there will arise

CANON 2.

80. Multiply these four numbers one into another, to wit,

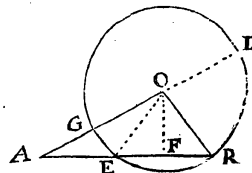
1. The sum of half the Base, and half the difference of the legs;
2. The excess of half the Base above half the difference of the legs;
3. The sum of half the sum of the legs, and half the Base;
4. The excess of half the sum of the legs above half the Base.

Then extract the square Root of the Product made by the continual multiplication of those four numbers, so shall that square Root be the Area of the Triangle.

81. Again, if we suppose $\left\{ \begin{array}{l} B = \text{the Base,} \\ A = \text{the greater leg,} \\ E = \text{the lesser leg} \end{array} \right\}$ of a plain Triangle.

Then the four numbers above mentioned in Canon 2. may be express thus, viz.

$$\begin{array}{l} 1. \left\{ \begin{array}{l} \frac{1}{2}B + \frac{1}{2}A - \frac{1}{2}E, \\ \frac{1}{2}B - \frac{1}{2}A + \frac{1}{2}E, \\ \frac{1}{2}A + \frac{1}{2}E - \frac{1}{2}B, \\ \frac{1}{2}A - \frac{1}{2}E + \frac{1}{2}B. \end{array} \right\} \text{ Or thus, } \left\{ \begin{array}{l} \frac{1}{2}B + \frac{1}{2}A + \frac{1}{2}E - E, \\ \frac{1}{2}B + \frac{1}{2}A + \frac{1}{2}E - A, \\ \frac{1}{2}A + \frac{1}{2}E + \frac{1}{2}B - B, \\ \frac{1}{2}A + \frac{1}{2}E + \frac{1}{2}B - E. \end{array} \right\} \end{array}$$



82. Therefore, if the four numbers last before express, (to wit, those standing on the right hand,) be multiplied one into another continually, the Product shall be equal to the Square of the Area of the Triangle whose three sides are represented by B, A, E. But if those four numbers be well observed, it will be evident that the number third in order is the half sum of the three sides of the Triangle, and the other three numbers are the Remainders arising by the subtraction of the three sides severally from their half sum. Hence therefore ariseth the vulgar Canon, to find out the Area of any plain Triangle whose three sides are severally given in numbers, viz.

CANON 3.

83. From half the sum of the three sides of any plain Triangle subtract the three sides severally; then multiply the said half sum and the three remainders one into another, according to the Rule of Continual Multiplication, and extract the square Root of the last Product, so shall that square Root be the Area of the Triangle.

Divers other Canons might be raised from the premises, to find out the Area of a plain Triangle; but 'tis now time to proceed to the Composition of the Problem in hand, and that its Construction may be possible, the lines given must be subject to this

Determination.

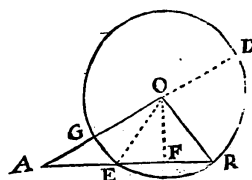
84. c \square $\sqrt{bb - 4pp}$: that is, in words,

The given sum of the legs must be longer than that right line whose Square is equal to the sum of the Square of the Base and the Square of the double of the Perpendicular.

This Determination doth openly shew it self in Theor. 2. in the 34th step of this Problem, and therefore that Theorem having already been demonstrated, the Determination is consequently both true and necessary for limiting the lines given.

The

The Composition of the foregoing Probl. 8.



B _____
P _____
C _____
H _____
I _____
K _____

Suppos.

85. B = the Base of a Triangle is given.
86. P = the Perpendicular is given.
87. C = the sum of the legs is given.
88. $C = \sqrt{B^2 + 4P^2}$ (Determination.)

Req. to make the Triangle.

Construction.

89. By Probl. 4. Chap. 5. find a right line H, such, that its Square may be equal to $\square C - \square B$, which Effect is possible, as is evident by the Determination prescribed in 88°, therefore suppose

$$\square H = \square C - \square B.$$

90. Find likewise a right line I, such, that its Square may be equal to $\square C - \square B - 4\square P$, which Effect the Determination shews to be possible, therefore suppose

$$\square I = \square C - \square B - 4\square P.$$

91. Then by Probl. 8. Chap. 5. let it be made, as the line H to the line I, so the line B (the given Base) to a fourth Proportional, suppose it to be the line K, therefore,

$$\frac{H}{I} = \frac{B}{K}.$$

That is, in 29°, $\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b : a$.
In which Analogy, the first Term H is greater than the second Term I, (as is evident by Constr. in 90° and 91°), therefore the third Term B shall be greater than the fourth K, (per Schol. Prop. 14. Elem. 5.) and consequently $C < K$, for (by the Determination in 88°) $C < B$. Thus far that hath been done which is directed by Theor. 2. in 34° of this Probl. the rest of the Construction follows.

92. Let a Triangle be made of these three right lines, to wit, $B, \frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, which is possible to be done. (per prop. 22. Elem. 1.) if $C < K$, and that the sum of every two of those three lines be longer than the third; but that those lines are so qualified, I prove thus;

First, by what hath been said in 91°, $C < K$, and consequently $\frac{1}{2}C - \frac{1}{2}K$ is equal to some real right line.

Secondly, the sum of B and $\frac{1}{2}C + \frac{1}{2}K$ is manifestly greater than the third line $\frac{1}{2}C - \frac{1}{2}K$.

Thirdly, the sum of the two lines $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$ makes C, which (by the Determination in 88°) is greater than the third line B.

Fourthly, that the sum of B and $\frac{1}{2}C - \frac{1}{2}K$ is greater than the third line $\frac{1}{2}C + \frac{1}{2}K$ may be proved thus;

It hath been shewn in 91°, that $B < K$.
Therefore by adding $\frac{1}{2}C$ to each part, $B + \frac{1}{2}C < K + \frac{1}{2}C$.
Therefore by subtracting $\frac{1}{2}K$ from each part, $B + \frac{1}{2}C - \frac{1}{2}K < K + \frac{1}{2}C$.
Which was to be Dem.

Now since it hath been proved that $\frac{1}{2}C - \frac{1}{2}K$ is equal to some real right line, and that the sum of every two of these three right lines, to wit, $B, \frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, is greater than the third, 'tis possible to make a Triangle of those three lines, (per prop. 22. Elem. 1.) Suppose then it be done, and that the Triangle so made is ARO, (in the preceding Diagram,) having its Base AR equal to the given Base B, and the greater leg AO equal to $\frac{1}{2}C + \frac{1}{2}K$, and the lesser leg RO equal to $\frac{1}{2}C - \frac{1}{2}K$. I say the Triangle ARO

ARO will satisfy the Problem propounded; but to render the Demonstration thereof the more easy to Learners, I shall premise a few things in eight steps next following.

93. If the quantities of the given lines B, P and C be express'd by numbers, it will be easy to discover the kind of the Triangle sought, when the legs are unequal, (as they were supposed to be in the Resolution,) by Theor. 3. in 35° of this Problem; for if the fourth Proportional found out by that Theorem be less than the Base, the Perpendicular falls within the Triangle; if greater, without; if equal to the Base, upon the end of the Base.

Supposing then it be discovered, that the Perpendicular falls upon AR within the Triangle ARO, from the Center O, at the distance of the lesser leg RO, ($= \frac{1}{2}C - \frac{1}{2}K$) describe the Circle ORGD cutting OA in G; then produce AO to the Circumference in D, draw also the Semidiameter OE, and from the Center O let fall OF perpendicular to EB, therefore (per prop. 5. Elem. 3.) $FE = FR$. Then,

94. Because (per defin. 15. Elem. 1.) $OD = OR = OG$.
95. Therefore by adding AO to each part, $AD = AO + OR$.
96. But by Constr. in 92°, $C = AO + OR$.
97. Therefore from 95° and 96°, (per Ax. 1. Chap. 2.) $AD = C$.
98. Again, by Constr. in 92°, $AO = \frac{1}{2}C + \frac{1}{2}K$.
99. Also by Constr. in 92°, $OR = \frac{1}{2}C - \frac{1}{2}K$.
100. Therefore, by subtracting the Equation in 99° from that in 98°, $AC = K$.

101. Now I shall shew that the Triangle ARO, made as before, will satisfy the Problem. First then by Construction in 92°, the Base AR is equal to the given Base B, and it hath been proved in 98°, that $AO + OR = C$ the given sum of the legs. So it remains only to shew, that the Perpendicular OF is equal to the given Perpendicular P; but that is made manifest by the following Demonstration, which is form'd out of the foregoing Resolution, by a repetition of its steps in a backward (not direct) order.

102. Req. demonstr. $OF = P$.

Demonstration.

103. By Constr. in 91°, $\frac{H}{I} = \frac{B}{K}$.
That is, in 29°, $\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b : a$.
104. Therefore, (per prop. 22. Elem. 6.) $\square H : \square I :: \square B : \square K$.

Now that the Terms of the last Analogy may be converted into their equivalent quantities expressible by the lines in the Diagram, these seven Equations next following are to be well observed,

105. By Constr. in 89°, $\square C - \square B = \square H$.
106. And from 97°, $\square AD = \square C$.
107. And from 92°, $\square AR = \square B$.
108. Therefore from 105°, 106°, 107°, $\square AD - \square AR = \square H$.
109. Again, by Constr. in 90°, $\square C - \square B - 4\square P = \square I$.
110. Therefore from 105°, 108° and 109°, $\square AD - \square AR - 4\square P = \square I$.
111. Add from 106°, $\square AG = \square K$.

112. Therefore the Terms of the Analogy in 104° being exchanged for their equivalent quantities in 108°, 109° and 111°, that Analogy will be converted into this, viz.

$$\square AD - \square AR : \square AD - \square AR - 4\square P :: \square AR : \square AG.$$

That is, in 28° $cc - bb : cc - bb - 4pp :: bb : aa$.

113. Therefore by altern and inverse Reason, $\square AR : \square AD - \square AR :: \square AG : \square AD - \square AR - 4\square P$.

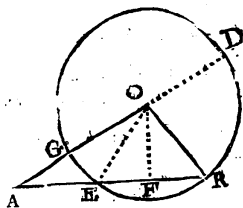
That is, in 27°, $bb : cc - bb :: aa : cc - bb - 4pp$.

Now to return backwards from the 27th to the 25th step of the Resolution, by the lines of the Diagram, some Analogies not express'd in the Resolution must be introduced, (which are infer'd from the Algebraical Fraction $\frac{ccaa - bb aa}{bb}$, as before hath been hinted in 55°) to wit, the four Analogies next following.

Q O

Probl. X.

In a plain Triangle having unequal acute angles at the Base, the Perpendicular, difference of the legs, and difference of the segments of the Base made by the Perpendicular, being severally given, to find the Triangle. But the lines given must be subject to the Determinations hereafter exprest.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

1. Let the Diagram belonging to the preceding Probl. 8. of this Chap. be here repeated, and suppose the $\triangle ARO$ having unequal acute angles A and R at the ends of the Base AR to be the Triangle sought; then respect being had to the Preparatory Construction in the three first steps of the said Probl. 8. the Resolution of this Probl. 10. may be formed thus;

Suppos.

2. $p = OF$ the Perpendicular of $\triangle ARO$ is given.
3. $d = AG$ the difference of the legs is given.
4. $b = AE$ the difference of the segments of the Base is given.

Req. to find the Triangle.

Resolution.

5. It is manifest that if AE the given difference of the segments FA, FR be esteemed the Base of the $\triangle AEO$ obtusangled at E, then AG shall be the difference of the legs AO and EO, as well as of AO and RO, (for $EO = RO = OG$), and OF a common Perpendicular to the two Triangles ARO and AEO; therefore in $\triangle AEO$, the Base AE, the Perpendicular OF, and AG the difference of the legs AO and EO being severally given, the said $\triangle AEO$ shall be given by the foregoing Probl. 9. of this Chap. For, the sum of the legs, to wit, $AD = AO + EO = AO + OR$ shall be given by this following Analogy, (according to Theor. 1. in 17th of the said Probl. 9.) viz.

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: b \cdot AD.$$

6. Then AD and AG the sum and difference of the legs AO and EO, or of AO and RO, being given, the said legs shall be given severally by Theor. 9. Chap. 4.
7. Moreover, forasmuch as AR in reference to $\triangle AEO$ obtusangled at E, is compos'd of the Base AE and ER, ($= 2FE = 2FR$), the said AR, which is also the Base of $\triangle ARO$, shall be given by Theor. 2. in 18th of the preceding Probl. 9. For,

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: d \cdot AR.$$

From the premises tis manifest that the Base and legs of the Triangle sought in this Probl. 10. may be found out by the foregoing Probl. 9. But that there may be a possibility of finding out the Triangle required, the lines given must be liable to these two following Determinations, viz.

Determination 1.

8. The line given for the difference of the segments of the Base made by the Perpendicular falling within the Triangle, must exceed the given difference of the legs, that is, (in the Figure Belonging to this Probl. 10.) $AE > AG$, the truth whereof is manifest by prop. 8. Elem. 3.

9. Again,

9. Again, because by *Supposition* the Triangle sought hath unequal acute angles at the Base, the Perpendicular falls within, and the Base must necessarily exceed the difference of the segments of the Base made by the Perpendicular; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy before exprest in 7th must exceed the given line AE. Hence,

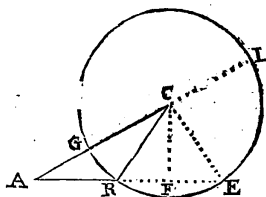
Determination 2.

$$10. \dots \dots \sqrt{4pp + bb - dd} : \sqrt{bb - dd} :: b.$$

Supposing then the given Quantities to be qualified according to the tenour of the Determinations before prescribed, the industrious Learner may easily apply what hath been said in the foregoing 5th, 6th and 7th steps, as well to the Geometrical Effect, as to the Arithmetical Solution of this Probl. 10.

Probl. XI.

In a plain Triangle obtusangled at the Base, the Perpendicular, difference of the legs, and the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle, being severally given, to find the Triangle. But the given lines must be subject to the Determinations hereafter exprest.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Let the Diagram belonging to the foregoing Probl. 9. of this Chap. be here repeated, and suppose the $\triangle ARC$ obtusangled at R, (the end of the Base AR,) to be the Triangle sought; then respect being had to the preparatory Construction in 1st, 2nd and 3rd of Probl. 9. the Resolution of this Probl. 11. may be formed thus;

Suppos.

2. $p = CF$ the Perpendicular of $\triangle ARC$ is given.
3. $d = AG$ the difference of the legs AC and RC is given.
4. $b = AE$ the line compos'd of the Base AR and 2FE, (or 2FR,) is given.

Req. to find the Triangle.

Resolution.

5. It is manifest, that if the given line AE be esteemed the Base of the $\triangle AEC$ having unequal acute angles at A and E, then AG is the difference of the legs AC and EC, as well as of AC and RC, (for $EC = RC$), and CF is a common Perpendicular to the two Triangles AEC and ARC; therefore in $\triangle AEC$, the Base AE, the Perpendicular CF, and AG the difference of the legs AC and EC, (or RC), being given severally, the said $\triangle AEC$ shall be given by Probl. 9. of this Chap. For $AL = AC + EC$, (the sum of the legs of $\triangle AEC$), shall be given by this following Analogy, (according to Theor. 2. in 17th of the said Probl. 9.) viz.

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: b \cdot AL.$$

6. Then AL and AG the sum and difference of the legs AC and EC being severally given, the legs themselves shall be also given severally, by Theor. 9. Chap. 4.
7. Moreover, because AR in reference to the $\triangle AEC$ is the difference of the segments FA and FE, made by the Perpendicular CF, and is also the Base of the $\triangle ARC$ required,

required, the said AR shall be given by *Theor. 2. in Probl. 18. of Probl. 9. of this Chap.* For,

$$\sqrt{bb - dd} : \sqrt{app + bb - dd} :: d : AR.$$

From the premises 'tis evident, that the three sides of the Triangle required by this *Probl. 11.* are discovered by *Probl. 9. of this Chap.* But the lines given must be subject to the following Determinations, that there may be a possibility of finding out a Triangle to satisfy the Problem propounded.

Determination 1.

8. The line given for the summ of the Base and double distance from the foot of the Perpendicular to the obtuse angle, must be longer than the line given for the difference of the legs, that is, $AE < AG$, as may be easily proved; for by *Supposition* $\triangle ARC$ is obtusangled at R, therefore $AE < AR$, and consequently AE much greater than AG, for (*per prop. 8. Elem. 3.*) $AR < AG$.

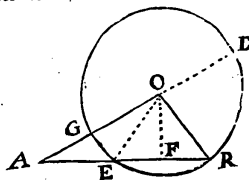
Again, because by *Supposition* the Triangle fought is obtusangled at the Base, the Perpendicular falls without, and the Base shall necessarily be less than the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle, therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy in 7°, must be less than the given line AE. Hence;

Determination 2.

$$\sqrt{app + bb - dd} : x d :: \sqrt{bb - dd} : b.$$

Probl. XII.

In a plain Triangle having unequal acute angles at the Base, the Perpendicular, summ of the legs, and difference of the segments of the Base made by the Perpendicular, being severally given, to find the Triangle. But the lines given must be liable to the Determinations hereafter declared.



Prepar.

AR = 21	FR = 6
AO = 17	OF = 8
OR = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 17

1. Let the Diagram belonging to the preceding *Probl. 8.* of this *Chap.* be here repeated, and suppose the Triangle ARO having unequal acute angles A and R at the ends of the Base AR to be the Triangle fought; then respect being had to the preparatory Construction in the three first steps of the said *Probl. 8.* the Resolution of this *Probl. 12.* may be formed thus;

Suppos.

2. $p = OF$ the Perpendicular of $\triangle ARO$ is given.
3. $c = AD = AO + RO$ the summ of the legs is given.
4. $b = AE = FA - FR$ the difference of the segments of the Base is given.

Req. to find the Triangle.

Resolution.

5. It is evident, that if the given line AE be esteem'd the Base of the $\triangle AEO$ obtusangled at E, then AD is the summ of the legs AO and EO, as well as of AO and RO, for $EO = RO$, and OF is a common Perpendicular to the two Triangles AEO and ARO; therefore in $\triangle AEO$, the Base AE, the Perpendicular OF, and AD the summ of the legs AO and EO being severally given, the Triangle

AEO

AEO shall be given by the foregoing *Probl. 8.* of this *Chap.* For AG, the difference of the legs AO and EO, shall be given by this following Analogy, (according to *Theor. 2. in 14° of Probl. 8.*)

$$\sqrt{cc - bb} : \sqrt{cc - bb - app} :: b : AG.$$

6. Then AD and AG the summ and difference of the legs AO and EO, (or RO,) being given severally, the said legs shall be also given severally, by *Theor. 9 Chap. 4.*
7. Moreover, inasmuch as AR in reference to the $\triangle AEO$ obtusangled at E, is compos'd of the Base AE and ER, ($= 2FE = 2FR$;) the said AR, which is also the Base of $\triangle ARO$ required, shall be given by *Theor. 3. in 35° of Probl. 8.* For,

$$\sqrt{cc - bb} : \sqrt{cc - bb - app} :: c : AR.$$

From the premises 'tis manifest, that the Base and legs of the Triangle fought by this *Probl. 12.* are discovered by the foregoing *Probl. 8.* But that there may be a possibility of finding out the desired Triangle, the given lines must be subject to these two following Determinations, viz.

Determination 1.

$$c < \sqrt{bb + app} : \text{That is,}$$

8. The line given for the summ of the legs must exceed that right line whose Square is equal to the summ of the Square of the given Base and the Square of the double of the given Perpendicular.

This Determination doth openly shew it self in the preceding Analogy in 5°, and hath already been demonstrated in *Probl. 8.* of this Chapter.

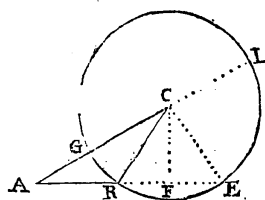
9. Again, because by *Supposition* the Triangle fought hath unequal acute angles at the Base, the Perpendicular falls within, and the Base must necessarily exceed the difference of the segments of the Base made by the Perpendicular, therefore to the end the given lines may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy in 7°, must exceed the given line AE. Hence,

Determination 2.

$$\sqrt{cc - bb - app} : x c :: \sqrt{cc - bb} : b.$$

Probl. XIII.

In a plain Triangle obtusangled at the Base, the Perpendicular, summ of the legs, and the line compos'd of the Base and double distance from the foot of the Perpendicular to the obtuse angle, being given severally, to find the Triangle. But the given lines must be subject to the Determinations hereafter declared.



Preparat.

AR = 9	FE = 6
AC = 17	CF = 8
CR = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 17

1. Let the Diagram belonging to the foregoing *Probl. 9.* of this *Chap.* be here repeated, and suppose the $\triangle ARC$ obtusangled at R, (the end of the Base AR,) to be the Triangle fought; then respect being had to the preparatory Construction in 1°, 2°, and 3° of the said *Probl. 9.* the Resolution of this *Probl. 13.* may be formed thus;

Suppos.

2. $p = CF$ the Perpendicular of $\triangle ARC$ is given.
3. $c = AL = AC + RC$ the summ of the legs is given.

4. $b = AE$

THEOR. 2.

24. In a right-angled Triangle having unequal legs about the right angle, the greater leg is equal to the sum of these two right lines, to wit, the right line whose Square is equal to the Square of half the Hypotenuse together with half the Square of a mean Proportional between the legs, and the right line whose Square is equal to the excess whereby the Square of half the Hypotenuse exceeds half the Square of the said mean. But the lesser leg is equal to the difference of the said two right lines.

Suppos.

25. ABC is a Δ right-angled at C.

26. AC \perp CB.

27. CBGD is a \square ; and ACD is a right line, therefore

28. AD = AC + CB, and AG = AC - CB.

29. M is a right line, such, that AC . M :: M . CB.

Req. demonstr.

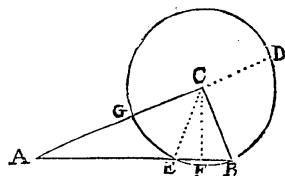
30. Theor. 1. AG = $\sqrt{\square AB - 2 \square M}$:

31. Theor. 2. $\begin{cases} \square AC = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} + \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M} \\ \square CB = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} - \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M} \end{cases}$

Demonstration.

32. By Suppos. in 28°, AG = AC - CB.
 33. Therefore the Square of that Equation, $\square AG = \square AC + \square CB - 2 \square AC, CB$,
 (per Theor. 5. Chap. 4.) gives
 34. By Suppos. in 25° < ACB is \perp , therefore $\square AB = \square AC + \square CB$.
 fore (per prop. 47. Elem. 1.)
 35. Therefore from 33° and 34°, (per Ax. 6. Chap. 2.) $\square AG = \square AB - 2 \square AC, CB$.
 36. From 29°, (per prop. 17. Elem. 6.) $\square M = \square AC, CB$.
 37. And consequently, $2 \square M = 2 \square AC, CB$.
 38. Therefore from 35° and 37°, (per Ax. 6. Chap. 2.) $\square AG = \square AB - 2 \square M$.
 39. But the sides of equal Squares are also equal, therefore from 38°, AG = $\sqrt{\square AB - 2 \square M}$:
 Which was Theor. 1. to be Dem.
 40. Again, because by Suppos. in 28°, AD is the sum, and AG the difference of the legs AC and CB, therefore (per Theor. 7. Chap. 4.)
 41. And because from 29°, (per prop. 17. Elem. 6.) $\square M = \square AC, CB$.
 42. And by taking $\frac{1}{4}$ of all in 38°, $\frac{1}{4} \square AB - \frac{1}{2} \square M = \frac{1}{4} \square AG$.
 43. Therefore from 40°, 41° and 42°, (per Ax. 6. Chap. 2.) $\square \frac{1}{4} AD = \frac{1}{4} \square AB + \frac{1}{2} \square M$.
 44. But the sides of equal Squares are also equal, therefore from 43°, $\frac{1}{4} AD = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M}$:
 45. And for the like reason, 'tis manifest from the Equation in 42°, that $\frac{1}{4} AG = \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M}$:
 46. Therefore, by taking the sum and difference of the Equations in 44° and 45°, there will arise,
 $\frac{1}{4} AD + \frac{1}{4} AG = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} + \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M}$:
 $\frac{1}{4} AD - \frac{1}{4} AG = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} - \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M}$:
 47. And because AD is the sum, and AG the difference of AC and CB, therefore (per Theor. 9. Chap. 4.)
 AC = $\frac{1}{2} AD + \frac{1}{2} AG$; and CB = $\frac{1}{2} AD - \frac{1}{2} AG$.

48. There-



48. Therefore from 46° and 47°, (per Ax. 1. Chap. 2.)
 $\begin{cases} \square AC = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} + \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M} \\ \square CB = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} - \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M} \end{cases}$

Which was Theor. 2. to be Demonstr.

In the next place, to the end the Geometrical Effect of the foregoing Probl. 14. may meet with no obstruction, I shall prove the truth of the Determination annex'd to the Problem, by demonstrating this following

L E M M A.

49. In a right-angled Triangle, if the Square of a mean Proportional between the sides about the right angle, (that is, if the Rectangle of those sides) be applied to the Hypotenuse, the line thence arising shall sometimes be equal to half the Hypotenuse, and sometimes less, but never greater than the said half.

The sides about the right angle are either equal to one another, or else unequal; I shall begin with the first Case.

Suppos. in Case 1.

50. RST is a Δ right-angled at S.

51. RS = ST.

52. M is a right line, such, that

53. RS . M :: M . ST.

54. Req. demonstr. $\frac{\square M}{RT} = \frac{1}{2} RT$.

Demonstration.

55. By Supposition in 51°, RS = ST.
 56. Therefore by drawing ST as a common altitude into each part, $\square RS, ST = \square ST = \square RS$.
 57. Therefore, (per Ax. 8. Chap. 2.) $2 \square RS, ST = \square ST + \square RS$.
 58. And because by Suppos. in 50° < S is \perp , therefore, (per prop. 47. Elem. 1.) $\square RT = \square ST + \square RS$.
 59. Therefore from 57° and 58°, (per Ax. 1. Chap. 2.) $2 \square RS, ST = \square RT$.
 60. And consequently, $\square RS, ST = \frac{1}{2} \square RT$.
 61. But from 53°, (per prop. 17. Elem. 6.) $\square RS, ST = \square M$.
 62. Therefore from 60° and 61°, (per Ax. 1. Chap. 2.) $\square M = \frac{1}{2} \square RT = \square RT, \frac{1}{2} RT$.
 63. Therefore from 62°, by Application of each part to RT, $\frac{\square M}{RT} = \frac{1}{2} RT$.
 Which was Case 1. to be Dem.

Suppos. in Case 2.

64. ABC is a Δ right-angled at C.

65. AC \perp CB.

66. M a right line, such, that AC . M :: M . CB:

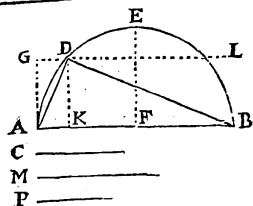
[See the Diagr. in the precedent Page.]

67. Req. demonstr. $\frac{\square M}{AB} = \frac{1}{2} AB$:

Demonstration.

68. By the preceding Theor. 1. before demonstrated 'tis evident that
 69. Therefore, by taking the half of each part, it follows that
 70. Therefore from 69°, by Application of each part to AB, $\frac{\square M}{AB} = \frac{1}{2} AB$,
 Which was Case 2. to be Dem.

71. Now because in every right-angled Triangle, the sides about the right angle are either equal or unequal between themselves, and it hath been demonstrated, that when the said sides are equal to one another, the right line arising by the Square of a mean



$$\begin{array}{lcl}
 AB = 169 & C = \sqrt{5070} \\
 BD = 156 & M = \sqrt{10140} \\
 AD = 65 & P = 60 \\
 DK = 60 & \\
 BK = 144 & \\
 KA = 25 &
 \end{array}$$

Resolution.

4. Because (by *prop. 41. Elem. 1.*) the Rectangle of BD into DA is equal to the double Area of $\triangle ABD$, therefore from 3° the said Rectangle or double Area is $2cc$.
5. And out of 2° and 3° , (by *Theor. 1.* in 23° of the foregoing *Probl. 14.* of this *Chapt.*) the Square of the difference of the legs about the right angle shall be $bb - 4cc$.
6. And by adding $8cc$, (to wit, four Rectangles of the legs,) to $bb - 4cc$, (that is, the Square of the difference of the legs,) the sum of that Addition gives (per *Theor. 7. Chap. 4.*) the Square of the sum of the legs, to wit, $bb + 4cc$.
7. And because (by *Theor. 3. Chap. 4.*) a quarter of the Square of any whole right line is equal to the Square of the half; therefore from 6° the Square of half the sum of the legs shall be $\frac{1}{4}bb + cc$.
8. And consequently from 7° , half the sum of the legs is $\sqrt{\frac{1}{4}bb + cc}$.
9. And from 5° , (by *Theor. 3. Chap. 4.*) the Square of half the difference of the legs is $\frac{1}{4}bb - cc$.
10. And consequently from 9° , half the difference of the legs is $\sqrt{\frac{1}{4}bb - cc}$.
11. Therefore from 8° and 10° , (by *Theor. 9. Chap. 4.*) the legs shall be given severally, viz.

$$BD = \sqrt{\frac{1}{4}bb + cc} + \sqrt{\frac{1}{4}bb - cc}$$

$$DA = \sqrt{\frac{1}{4}bb + cc} - \sqrt{\frac{1}{4}bb - cc}$$

From 6° and 5° arith

THEOR. 1.

12. In every right-angled Triangle having unequal sides about the right angle, the Square of the sum of those sides is equal to the Square of the Hypotenuse together with the quadruple of the Area: But the Square of the difference of the same sides is equal to the excess whereby the Square of the Hypotenuse exceeds the quadruple of the Area. The Equations in 11° give

THEOR. 2.

13. In every right-angled Triangle having unequal sides about the right angle, if to and from the Square of half the Hypotenuse, the Area be added and subtracted severally, and out of the sum and remainder severally the Square Root be extracted, the sum and difference of those square Roots shall be equal to the sides about the right angle.

The truth of the Determination annex'd to this *Probl. 15.* hath already been demonstrated in the preceding *Probl. 14.* and the reason thereof will appear in the following Construction.

Suppos.

14. AB = the Hypotenuse of a right-angled Triangle is given.
15. C is a right line given, whose Square is equal to the Area of that Triangle.

$$16. \frac{2 \square C}{AB} \text{ not } \leq \frac{1}{2} AB.$$

Req. to find the Triangle.

Constr.

Construction.

17. This Problem might be effected according to the direction of the foregoing Theorem in 13° , but more compendiously thus; First, by *Probl. 2. Chap. 5.* find a right line M, such, that its Square may be equal to $2 \square C$, therefore

$$M = \sqrt{2 \square C}.$$

18. Then by *Probl. 7. Chap. 5.* let it be made as AB to M, so M to a third proportional line, suppose it to be the line P, therefore

$$AB : M :: M : P.$$

19. Upon AB describe the Semicircle FADB.

20. Make $AG \perp AB$; also $AG = P$; and $GL \parallel AB$, which Parallel GL shall necessarily either touch the Semicircle FADB, or cut the same; for by *Suppos.* in 16° , $\frac{2 \square C}{AB} \text{ not } \leq \frac{1}{2} AB$, and by *Constr.* in 17° and 18° , P is equal to $\frac{2 \square C}{AB}$; therefore P is not greater than $\frac{1}{2} AB$, (= the Semidiameter FE.) But GL was before drawn parallel to AB at the distance of the right line P, (= AG,) and therefore the said Parallel shall either touch the Semicircle in E, or else cut the same. Supposing then the Parallel GL to cut the Semicircle in D, draw the right lines AD and DB, so shall $\triangle ADB$ be the right-angled Triangle required. But now we must shew that it will satisfy the Problem.

21. First then by *Construction* in 19° , AB the Base of the Triangle ADB is that which in 14° was prescribed for the Hypotenuse of the right-angled Triangle sought; secondly, by *Constr.* in 19° and 20° the angle ADB is in the Semicircle FADB, and therefore 'tis a right angle, (per *prop. 31. Elem. 3.*) thirdly and lastly, that the Area of the right-angled Triangle ADB is equal to the Square of the given right line C, the following Demonstration will make manifest.

Prepar.

22. From the point D in the Circumference, let fall DK perpendicular to the Diameter AB.

$$23. \dots \text{Req. demonstr.} \dots \triangle ADB = \square C.$$

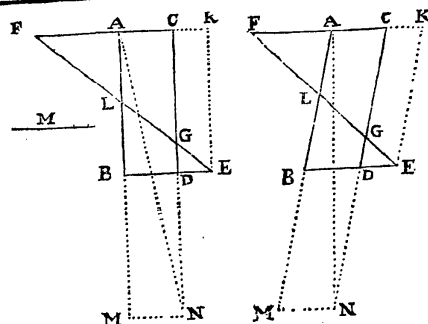
Demonstration.

24. By *Constr.* in 18° , $AB : M :: M : P$.
25. And from the *Constr.* in 20° and 22° , (per *prop. 34. Elem. 1.*) $P = AG = DK$.
26. Therefore from 24° and 25° , by taking DK instead of P, $AB : M :: M : DK$.
27. Therefore from 26° , (per *prop. 17. Elem. 6.*) $\square AB, DK = \square M$.
28. But by *Constr.* in 17° , $\square C = \square M$.
29. Therefore from 27° and 28° , (per *Ax. 1. Chap. 2.*) $\square AB, DK = \square C$.
30. And because by *Constr.* $\angle ADB$ is \perp , therefore (per *prop. 41. Elem. 1.*) $\square AB, DK = 2 \triangle ADB$.
31. Therefore from 29° and 30° , (per *Ax. 1. Chap. 2.*) $2 \triangle ADB = \square C$.
32. Therefore from 31° , (per *Ax. 9. Chap. 2.*) $\triangle ADB = \square C$.
Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. XVI. (Prop. 164. Lib. 7. Pappi.)

A Parallelogram BACD being given by Position, from a given point E in BD produced, to draw a right line EF to concur with CA produced in F, so as to make the Triangle FCG equal to the given Parallelogram BACD.

Suppos.



$AB = 75$
 $BD = 32$
 $DE = 20$
 $CK = 20$
 $FA = 48$
 $EK = 75$
 $FK = 100$
 $CG = 60$
 $GD = 15$
 $AL = 36$
 $LB = 39$
 $EF = 125$

in Fig. 1.

Suppos.

1. $BACD$ is a Parallelogram given by Position.
2. E is a point given in BD produced.
3. $b = AC = BD$ is given.
4. $a = CD = AB$ is given.
5. $c = DE$ is given.

Req. to find

6. AF a right line to be added to CA in a direct line, that EF being drawn, it may make $\triangle FCG = BACD$.

Prepar.

7. By the given point E draw $EK \parallel DC$, and to concur with AC produced in K , whence $CK = DE$, and $EK = DC$.
8. Let CD be continued to N , so, that $DN = DC$, whence $CN = 2DC$, and therefore $\triangle ACN = \square AC, CD = BACD$, (per prop. 41. Elem. 1.)

Resolution.

9. Suppose that done which is required, and put $a = FA$.

10. Then because by *Constr.* in 7° $EK \parallel DC$, the Triangles FCG and FKE are equiangular, (per prop. 29. Elem. 1.) therefore (per prop. 4. Elem. 6.) these are Proportional, viz.

$$\frac{FK}{a+b+c} \cdot \frac{FC}{a+b} :: \frac{KE}{a+b} \cdot \frac{CG}{d} \cdot \frac{da+db}{a+b+c}$$

11. By *Constr.* in 8° ,

$$\triangle ACN = BACD.$$

12. And the Problem requires

$$\triangle FCG = BACD.$$

13. Therefore from 11° and 12° , (per Ax. 1. Chap. 2.)

$$\triangle ACN = \triangle FCG.$$

14. And because those equal Triangles ACN and FCG have a common angle FCN , the sides about that angle shall be reciprocally proportional, (per prop. 17. Elem. 6.) therefore

$$\frac{FC}{a+b} \cdot \frac{AC}{b} :: \frac{CN}{2d} \cdot \frac{CG}{da+db}$$

15. Therefore, by halving the Antecedents in the last Analogy,

$$\frac{1}{2}a + \frac{1}{2}b \cdot b :: d \cdot \frac{da+db}{a+b+c}$$

16. But it hath been shewn above in 10° that

$$\frac{a+b+c}{a+b} \cdot \frac{a+b}{d} :: \frac{da+db}{a+b+c}$$

17. Therefore from 15° and 16° , (per prop. 11. Elem. 5.)

$$\frac{a+b+c}{a+b} \cdot \frac{a+b}{d} :: \frac{1}{2}a + \frac{1}{2}b \cdot b.$$

18. And by doubling the two latter Terms, their Reason is not alter'd, therefore

$$\frac{a+b+c}{a+b} \cdot \frac{a+b}{d} :: a+b \cdot 2b.$$

19. Therefore from the last Analogy by Division of Reason,

$$c \cdot a+b :: a-b \cdot 2b.$$

20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes,

$$aa - bb = 2bc.$$

21. There-

21. Therefore by adding bb to each part, $aa = bb + 2bc$ ($= \square FA$.)
22. Therefore, by extracting the Square Root out of each part, $a = \sqrt{bb + 2bc} = FA$.
23. Again, from 3° and 5° , respect being had to the Diagram, $BE (= BD + DE) = b + c$.
24. Therefore by squaring each part, $\square BE = bb + cc + 2bc$.
25. And by subtracting $\square DE = cc$ from each part, $\square BE - \square DE = bb + 2bc$.
26. But from 21° , $\square FA = bb + 2bc$.
27. Therefore from 25° and 26° , (per Ax. 1. Chap. 2.) $\square FA = \square BE - \square DE$.
28. Therefore by extracting the Square Root out of each part, $FA = \sqrt{\square BE - \square DE}$.

The Equations in 21° and 27° do afford this

THEOREM.

29. If $FC \parallel BE$, and $AB \parallel CD$, and $\triangle FCG = BACD$, then the Square of FA is equal to the Square of BD together with twice the Rectangle of BD into DE . Moreover, the Square of FA is equal to the excess by which the Square of BE exceeds the Square of DE .

Therefore BD and DE being given severally, FA shall be given also, and consequently EF may be drawn to solve the Problem propounded.

But to manifest the truth of the said Theorem, I shall form a Demonstration thereof by a repetition of the steps of the preceding Resolution in a direct order, to which end, let respect be had to the Diagram, *Suppos.* and *Prepar.* at the beginning of the Problem.

30. . . . *Req. demonstr.* . . . $\square FA = \square BD + 2\square BD, DE = \square BE - \square DE$.

Demonstration.

31. Forasmuch as $\triangle FKE$ and $\triangle FCG$ are equiangular, (for by *Constr.* in 7° , $EK \parallel CD$), therefore (per prop. 4. Elem. 6.) $FK \cdot FC :: KE \cdot CG$.
32. By *Constr.* in 8° , $\triangle ACN = BACD$.
33. And by *Suppos.* in 29° , $\triangle FCG = BACD$.
34. Therefore from 32° and 33° , (per Ax. 1. Chap. 2.) $\triangle ACN = \triangle FCG$.
35. And because $\angle FCN$ is common to those equal Triangles ACN and FCG , therefore, (per prop. 17. Elem. 6.) $FC \cdot AC :: CN$ (or $2CD$) $\cdot CG$.
36. Therefore from 35° , by halving the Antecedents, $\frac{1}{2}FC \cdot AC :: CD$ (or KE) $\cdot CG$.
37. But it hath been shewn in 31° , that $FK \cdot FC :: KE \cdot CG$.
38. Therefore from 36° and 37° , (per prop. 11. Elem. 5.) $FK \cdot FC :: \frac{1}{2}FC \cdot AC$.
39. And from 38° , by doubling the two latter Terms, $FK \cdot FC :: FC$ (or $FA + AC$) $\cdot 2AC$.
40. And from 39° , by Division of Reason, $CK \cdot FC :: FA - AC \cdot 2AC$.
41. That is, (as is evident by the Diagram), $DE \cdot FA + AC :: FA - AC \cdot 2BD$.
42. Therefore from 41° , (per prop. 16. Elem. 6.) $\square FA + AC \cdot 2 = \square BD, DE$.
43. But by *Theor. 8. Chap. 4.* $\square FA + AC \cdot 2 = \square FA - AC \cdot 2 = \square FA - \square AC$ ($\square BD$).
44. Therefore from 42° and 43° , (per Ax. 1. Chap. 2.) $\square FA - \square BD = 2 \square BD, DE$.
45. Therefore from 44° , by adding $\square BD$ to each part, $\square FA = \square BD + 2 \square BD, DE$.

Which was to be Dem.

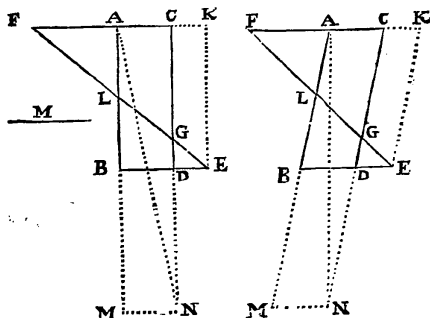
46. Again, because by *Supposition* in 29° and by the Diagram, $BE = BD + DE$.
47. Therefore by squaring each part of that Equation, (per *Theor. 2. Chap. 4.*) $\square BE = \square BD + \square DE + 2 \square BD, DE$.

Qq

48. And

48. And from 47° , by subtracting $\square DE$ from each part, $\square BE - \square DE = \square BD + 2\square BD, DE$.
 49. Therefore from 45° and 48° , (per *Ax. 1. Chap. 2.*) $\square FA = \square BE - \square DE$.
 Which was also to be Dem. Therefore the truth of the preceding Theorem is manifest.

The Composition of the foregoing Probl. 16.



Suppos.

50. BACD is a Parallelogram given by Position.
 51. E is a point given in BD continued.
 52. . . . Reg. to draw EF a right line, such, that $\triangle FCG = BACD$.

Construction.

53. By *Probl. 9. Chap. 5.* find a mean proportional line M between BD and $BD + 2DE$, therefore $BD \cdot M :: M \cdot BD + 2DE$.
 54. Produce CA to such a point F, that AF may be equal to the line M, (to wit, the mean Proportional found out in 53), then draw a right line from E to F, so shall the Triangle FCG be equal to the Rectangle BACD, as was required; the truth whereof will evidently appear by the following Demonstration, form'd out of the foregoing Resolution by a repetition of its steps in a backward (not direct) order. But by way of Preparation, draw EK || and = DC; also make $CN = 2CD$; draw AN, and produce FC to K.
 55. . . . Reg. demonstr. . . . $\triangle FCG = BACD$.

Demonstration.

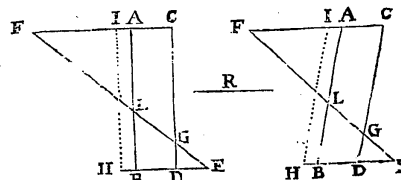
56. By *Constr.* in 53° and 54° , $BD \cdot M$ (or FA) $:: M \cdot BD + 2DE$.
 57. Therefore (per *prop. 17. Elem. 6.*) $\square FA = \square BD + 2\square BD, DE$.
 58. Therefore from 57° , by subtracting $\square BD$ from each part, $\square FA - \square BD = 2\square BD, DE$.
 59. But by *Theor. 8. Chap. 4.* $\square FA - \square AC$ (or $\square BD$) $= \square \{FA + AC, FA - AC\}$.
 60. Therefore from 58° and 59° , $\square \{FA + AC, FA - AC\} = 2\square BD, DE$.
 61. Therefore from 60° , (per *prop. 14. Elem. 6.*) $DE(CK) \cdot FA + AC :: FA - AC \cdot 2BD(2AC)$.
 62. That is, as is evident by the Diagram, $CK \cdot FC :: FA - AC \cdot 2AC$.
 63. Therefore from 62° , (by *Comp. of Reason.*) $FK \cdot FC :: FC(FA + AC) \cdot 2AC$.
 64. And from 63° , by halving the two latter Terms, . . . $FK \cdot FC :: \frac{1}{2}FC \cdot AC$.

65. But

65. But because $\triangle FKE$ and $\triangle FCG$ are like, (for $EK \parallel DC$), therefore, (per *prop. 4. Elem. 6.*) $FK \cdot FC :: KE(CD) \cdot CG$.
 66. Therefore from 64° and 65° , (per *prop. 11. Elem. 5.*) $\frac{1}{2}FC \cdot AC :: CD \cdot CG$.
 67. And from 66° , by doubling the Antecedents, $FC \cdot AC :: CN(2CD) \cdot CG$.
 68. And because $\angle FCN$ is common to $\triangle FCG$ and $\triangle ACN$, and it appears in 67° , that the sides about that common angle are reciprocally proportional, therefore (per *prop. 15. Elem. 6.*) $\triangle FCG = \triangle ACN$.
 69. But by *Constr.* in 8° , $\triangle BACD = \triangle ACN$.
 70. Therefore from 68° and 69° , (by *Ax. 1. Ch 2.*) $\triangle FCG = BACD$.
 Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. XVII.

A Parallelogram BACD being given by Position, from a given point E in BD produced, to draw a right line EF to meet with CA produced in F, that the Triangle FCG may have a given Reason to the Parallelogram BACD, suppose as HD to BD.



Construction.

1. By the point H draw $HI \parallel BA$ or DC (per *prop. 31. Elem. 1.*) then by the last preceding Problem draw a right line EF, so as to make the Triangle FCG equal to the Parallelogram HICD, so shall $\triangle FCG$ be to BACD as HD to BD, which was required; the truth whereof will be manifest by the following Demonstration.
 2. . . . Reg. demonstr. . . . $\triangle FCG \cdot BACD :: HD \cdot BD$.

Demonstration.

3. Because (per *prop. 1. Elem. 6.*) $HICD \cdot BACD :: HD \cdot BD$.
 4. And by *Constr.* in 1° , $\triangle FCG = HICD$.
 5. Therefore from 3° and 4° , $\triangle FCG \cdot BACD :: HD \cdot BD$.
 Which was to be Dem.

6. After the same manner, from the given point E a right line may be drawn so as to make the Triangle FCG equal to a given Space, suppose the Square of the right line R, by making the Parallelogram HICD equal to the Square of R, and $\triangle FCG = HICD$: For,

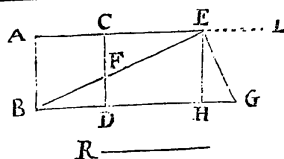
If by *Construction* $HICD = \square R$,
 And by *Constr.* $HICD = \triangle FCG$.
 Then it follows (per *Ax. 1.*) that . . . $\triangle FCG = \square R$.

Probl. XVIII. (*Prop. 71. Lib. 7. Pappi.*)

A Square BACD (whose side is BD or DC) being given, to draw a right line from the angle B, as BE, that may so cut the side DC, and concurr with the side AC produced towards L, that FE may be equal to a given right line R.

Qq 2

A — C



$$\begin{aligned} BD &= HE = 60 \\ R &= FE = 91 \\ BF &= EG = 65 \\ DG &= 109 \\ DF &= HG = 25 \\ FC &= 35 \\ DH &= CE = 84 \end{aligned}$$

Prepar.

- Suppose that done which is required, viz. that BE is a right line so drawn from the angle B that it cuts DC in F, and concurs with AC produced in E, and makes FE equal to the given right line R.
- Make EG perpendicular to BE, and let BD be continued until it concurs with EG in G, and from E let fall EH perpendicular to BG, whence it follows (per prop. 8, & 2. Elem. 6.) that
 - $\triangle BEG$
 - $\triangle BHE$
 - $\triangle GHE$
 - $\triangle BDF$
 are like (that is, equiangular) right-angled Triangles, and therefore the sides about the equal angles are Proportionals, (per prop. 4. Elem. 6.)
- And because the right-angled Triangles BDF and GHE are like, and the side BD (= DC) in the one, is equal to the side HE in the other, and the angle BFD opposite to BD is equal to the angle G opposite to HE, therefore the remaining sides of $\triangle BDF$ shall be also equal to the remaining sides of $\triangle GHE$, viz. each side to its correspondent side, (per prop. 26. Elem. 1.) therefore

$$BF = GE, \text{ and } DF = HG.$$

These things being premised, the Resolution of the Problem may be formed thus:

Suppos.

- $b = BD = DC = HE$ is given.
- $d = FE = R$ is given.

Resolution.

- For DG put a , viz. suppose . . . $a = DG$.
- And for BF (= GE) put e , viz. suppose . . . $e = BF = GE$.
- Then from 4° and 6° . . . $b + a = BG$.
- And from 7° and 5° . . . $e + d = BE$.
- The Square of the Equation in 7° gives . . . $ee = \square BF$.
- And the Square of the Equation in 8° gives . . . $bb + 2ba + aa = \square BG$.
- And the Square of the Equation in 9° gives . . . $ee + 2ed + dd = \square BE$.
- Now because by Constr. in 2°, the Triangle BEG is right-angled at E, and from 3°, $\square BF = \square GE$, therefore from 10°, 11°, 12°, (per prop. 47. Elem. 1.) this Equation arith, viz.

$$\square BG = \square BE + \square GE (\square BF)$$

$$bb + 2ba + aa = ee + 2ed + dd + ee.$$

- And because from 2° . . . $\triangle BEG$ and $\triangle BHE$ are like.
- Therefore (per prop. 4. Elem. 6.) . . . $BG : GE :: BE : EH$.
- That is, in the letters of the Resolution, . . . $b + a : e :: e + d : b$.
- Which Analogy being reduced to an Equation, gives . . . $bb + ba = ee + ed$.
- And by subtracting the Equation in 17° from that in 13°, this remains, . . . $ba + aa = ee + ed + dd$.
- And if instead of $ee + ed$ in the last preceding Equation, there be taken $bb + ba$, which in 17° appears to be equal to $ee + ed$, then the Equation in 18° will be reduced to this . . . $ba + aa = bb + ba + dd$.
- Whence, by subtracting ba from each part, there remains . . . $aa = bb + dd$.
- Therefore by extracting the Square Root out of each part of the last Equation, it gives . . . $a = \sqrt{bb + dd} = DG$.

Hence

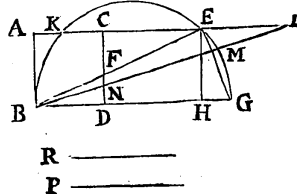
Hence this

THEOREM.

- The right line DG is equal to that right line whose Square is equal to the sum of the Squares of BD and FE. Therefore if BD (or DC) and FE be given severally, then DG is given also, by the help whereof the Problem may be effected.

This Theorem is demonstrated in Prop. 71. of the 7th Book of Pappus's Mathematical Collections, and the truth thereof is also manifest by the foregoing Resolution, wherein the Argumentation is clearly Geometrical as well as Algebraical, and therefore there is no need of any further Demonstration of the said Theorem.

The Composition of the foregoing Probl. 18.



$$\begin{aligned} BD &= HE = 60 \\ R &= FE = 91 \\ BF &= EG = 65 \\ DG &= 109 \\ DF &= HG = 25 \\ FC &= 35 \\ DH &= CE = 84 \end{aligned}$$

Suppos.

- BACD is a Square given, whose side is BD or DC.
- R is a right line given.
- Req. To draw a right line from the angle B, as BE, that may so cut the side DC, and concur with the side AC produced towards L, that FE may be equal to a given right line R.

Construction.

- By Probl. 2. Chap. 5. find a right line P, such, that } its Square may be equal to $\square BD + \square R$, therefore } $P = \sqrt{\square BD + \square R}$.
- To BD add the line P, so, that BD and P may } $DG = P$, and $BG = BD + P$.
- make a straight line, as BDG, therefore . . .
- Upon BG describe the Semicircle BKEG.
- Let AC be continued towards L, so shall the line produced cut the Semicircle BKEG, I say cut it, not lye without it; for $\frac{1}{2}$ BG is greater than BD or BA, as may be proved thus:
 - Because by Constr. in 27° . . . $DG = P$.
 - And by Constr. in 26° . . . $P < BD$.
 - Therefore (per Ax. 4. Chap. 2.) . . . $DG < BD$.
 - And by adding BD to each part, . . . $BD + DG < 2BD$.
 - But by Constr. in 27° . . . $BD + DG = BG$.
 - Therefore (per Ax. 3. Chap. 2.) . . . $BG < 2BD$.
 - And consequently, . . . $\frac{1}{2} BG < BD$ or BA.

Which was to be proved. And therefore ACE, which is parallel to BG at the distance of BA shall necessarily cut the Semicircle BKEG in two points, as in K and E.

- Lastly, draw the right line BE, so shall FE be equal to the given right line R, as was required. But that $FE = R$, I demonstrate thus;
- Req. demonstr. . . $FE = R$.

Demonstration.

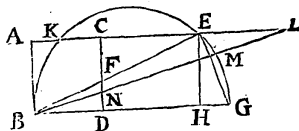
- By Constr. in 26° . . . $\square BD + \square R = \square P$.
- And from the Constr. in 27° . . . $\square DG = \square P$.
- Therefore (per Ax. 1. Chap. 2.) . . . $\square DG = \square BD + \square R$.
- But by the Theor. in 22° of this Problem, . . . $\square DG = \square BD + \square FE$.
- Therefore from 34° and 35°, (per Ax. 1. Chap. 2.) } $\square BD + \square FE = \square BD + \square R$.
- And from 36°, by subtracting $\square BD$ from each part, } $\square FE = \square R$.
- Therefore, . . . $FE = R$.

Which was to be Dem.

39. But

39. But that FE is the only right line that can be found equal to the given right line R , and in such a Position as the Problem requires, I shall demonstrate in the next place, to which end, let some other line besides BE , as BL , be drawn from B to concur with AC produced beyond E in L : Now if BL doth effect the Problem, then NL must be equal to FE ; but NL is greater than FE , as will be made manifest by the following Demonstration.

40. *Req. demonstr.* $NL < FE$.



R ———
P ———

Demonstration.

- | | |
|---|--|
| 41. Because by <i>Suppos.</i> | $\angle BAL$ is \perp . |
| 42. Therefore (per prop. 47. Elem. 1.) | $\square BL = \square BA + \square AL$. |
| 43. And in like manner, | $\square BE = \square BA + \square AE$. |
| 44. By <i>Suppos.</i> in 39°, $AL < AE$, and consequently, | $\square AL < \square AE$. |
| 45. Therefore from 42°, 43° and 44°, (per Ax. 4. Chap. 2.) | $\square BL < \square BE$. |
| 46. Therefore from 45°, | $BL < BE$. |
| 47. Again, because by <i>Suppos.</i> | $\angle BDC$ is \perp . |
| 48. Therefore (per prop. 47. Elem. 1.) | $\square BF = \square BD + \square DF$. |
| 49. And in like manner, | $\square BN = \square BD + \square DN$. |
| 50. And because $DF < DN$, and consequently, | $\square DF < \square DN$. |
| 51. Therefore from 48°, 49° and 50°, | $\square BF < \square BN$. |
| 52. Therefore from 51°, | $BF < BN$. |
| 53. And consequently, | $BN > BF$. |
| 54. And because it hath been shewn in 46°, that | $BL < BE$. |
| 55. Therefore from 52°, 53°, 54°, (per Ax. 16. Chap. 2.) | $BL - BN < BE - BF$. |
| 56. That is, (as is evident by the Diagram), | $NL < FE$. |

Which was to be Dem. And therefore BL will not effect the Problem propounded. The like Demonstration will hold good in comparing BE to any other right line that shall be drawn from B to cut AC , and to concur with AC produced.

57. Here the Learner may observe, that in resolving a Problem by the Algebraick Art, there may often-times be found out various Equations so constituted, that every one of them may be capable of solving the Problem, but the simplest of those Equations is to be preferred before the rest, and chiefly to be aimed at, though for the most part 'tis much harder to be discovered than those more compounded. As, in the foregoing *Probl. 18*, among various Equations that may be found out to solve the same, that in 21° at the end of the preceding Resolution is the simplest. But who would think, that the way to solve that Problem is to search out the quantity of the line DG , and not rather of one of these lines, to wit, BE , BF , AE , CE , DF ? for by any one of these lines, from the consideration of the like right-angled Triangles BAE , FCE , BDF we may come to an Equation more easily than by the line DG , but the Geometrical Construction of such Equation will be much harder than that of the Equation whereby DG is before discovered: And because the Equation resulting upon the search of any of the said five lines, to wit, BE , BF , AE , CE , DF falls under a higher Form than any of the Equations expounded in this Book, I shall refer the more curious Reader for satisfaction concerning the same, to *Pag. 82, 83, 84 of Renatus des Cartes's Geometry*, set forth by *Fran. van Schooten* in 1659. yet I shall here shew how the quantities of those five lines before mentioned are also deducible from the preceding Resolution.

First then, the same things being supposed as before, draw EG , and make $EH \perp BG$; then let the Equations in the preceding 17th and 21st steps be here repeated, viz.

58. It

58. It hath been shewn in 17°, that $bb + ba = ee + ed$.
 59. And in 21°, that $a = \sqrt{bb + dd}$:
 60. Therefore if $\sqrt{bb + dd}$: instead of a be drawn into b , the Equation in 58° will be reduced to this, $bb + b\sqrt{bb + dd} = ee + ed$. viz.
 61. Which last Equation may be reduced into these three Proportionals, viz. $e + d, \sqrt{bb + b\sqrt{bb + dd}}, e + d$.
 62. Of which three Proportionals, the mean, to wit, $\sqrt{bb + b\sqrt{bb + dd}}$: is given, as also d the difference of the extremes $e + d$ and e , therefore the extremes shall be given severally, by the *Theor.* in 24° of *Probl. 12. Chap. 5. viz.*

$$\sqrt{\frac{1}{2}dd + bb + b\sqrt{bb + dd}} : -\frac{1}{2}d = e = BE.$$

$$\sqrt{\frac{1}{2}dd + bb + b\sqrt{bb + dd}} : +\frac{1}{2}d = e + d = BE.$$

63. And because $EH (= DC = DB)$ is a mean Proportional between BH and HG , whose sum is BG , which mean and sum of the extremes are represented in the preceding Resolution by b and $b + a$, whereof b is given in 4°, and a in 21°, for 'tis there found equal to $\sqrt{bb + dd}$: and consequently $b + a = b + \sqrt{bb + dd}$: therefore by the help of the said given mean b , and the said given sum of the extremes, to wit, $b + \sqrt{bb + dd}$: the extremes $BH (= AE)$ and $HG (= DF)$ shall be given severally by the *Theor.* in 21° of *Probl. 13. Chap. 5. viz.*

$$\frac{1}{2}b + \sqrt{\frac{1}{2}bb + \frac{1}{2}dd} : +\sqrt{\frac{1}{2}dd - \frac{1}{2}bb + \frac{1}{2}b\sqrt{bb + dd}} = BH = AE.$$

$$\frac{1}{2}b + \sqrt{\frac{1}{2}bb + \frac{1}{2}dd} : -\sqrt{\frac{1}{2}dd - \frac{1}{2}bb + \frac{1}{2}b\sqrt{bb + dd}} = HG = DF.$$

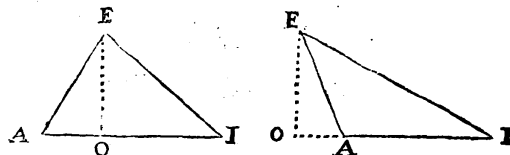
64. And because $CE = DH = BH - BD$, and BH and BD are given as before, therefore $CE (= DH)$ is given also, viz.

$$\sqrt{\frac{1}{2}bb + \frac{1}{2}dd} : +\sqrt{\frac{1}{2}dd - \frac{1}{2}bb + \frac{1}{2}b\sqrt{bb + dd}} = -\frac{1}{2}b = CE.$$

Lastly, for the better illustration of the premisses, I have calculated whole Numbers (placed near the Diagram) to express the Quantities of all the Right lines given and sought in this *Probl. 18*.

A L E M M A, leading to the following *Probl. 19*.

If in any oblique-angled plain Triangle, any one of the three sides be called the Base, and the other two the legs; Then, as the Radius, (or total Sine,) is to the Sine complement of the angle contain'd under the legs; so is the double Rectangle of the legs, to the difference between the sum of the Squares of the legs, and the Square of the Base.



Suppos.

- Let EI be the Base; } of the oblique-angled Triangle AEI .
- AE and AI the legs }
- $\angle A$ (that is, $\angle EAI$) is contain'd under the legs AE , AI .
- $EO \perp AI$.
- R = the Radius, or total Sine.
- $Sc. \angle A$ = the Sine complement of the Angle A , (or $\angle EAI$) that is, the Sine of the angle AEO .

Req. demonstr.

7. { If $\angle A$ be acute, then, $R \cdot Sc. \angle A :: 2 \square AE, AI \cdot \square AE + \square AI - \square EI$.
 { If $\angle A$ be obtuse, then, $R \cdot Sc. \angle A :: 2 \square AE, AI \cdot \square EI - \square AE - \square AI$.

Demonstr.

Demonstration.

8. By a vulgar Axiom in the Doctrine of plain Triangles, $R. Sc. < A :: AE : AO.$
9. Therefore by taking the common altitude AI, $R. Sc. < A :: \square AE, AI : \square AO, AI.$
10. And by doubling the two latter Terms, $R. Sc. < A :: 2\square AE, AI : 2\square AO, AI.$
11. And because by Supposition in Case 1. $< A$ is acute, therefore (per prop. 13. Elem. 2.) $\square AE + \square AI - \square EI = 2\square AO, AI.$
12. Therefore, from 10° and 11° , by exchanging equal quantities, $R. Sc. < A :: 2\square AE, AI : \square AE + \square AI - \square EI.$ Which was to be Dem.
13. But when $< A$ is obtuse, then (per prop. 12. Elem. 2.) $\square EI - \square AE - \square AI = 2\square AO, AI.$
14. Therefore from 10° and 13° , by exchanging equal quantities, $R. Sc. < A :: 2\square AE, AI : \square EI - \square AE - \square AI.$ Which was also to be Dem. Therefore the truth of the Lemma is manifest. Hence this

COROLLARY.

15. If in an oblique-angled plain Triangle the three sides be given severally, the angles shall also be given severally, without the help of the Perpendicular; for if the side opposite to an angle sought be called the Base, and the other two sides the legs; then as the double Rectangle of the legs is to the difference between the sum of the Squares of the legs and the Square of the Base; so is the Radius to the Sine complement of the angle opposite to the Base. Which angle is acute when the Square of the Base is less than the sum of the Squares of the legs; but obtuse when greater.

An Example in Numbers, where the sum of the Squares of the legs exceeds the Square of the Base.

Suppos. in $\triangle AEC.$

16. $AC = 7$, the Base is given.
 17. $AE = 8$,
 18. $EC = 3$, } the legs are given.
 Req. to find $< E.$

Solution Arithmetical.

19. By the preceding Corollary; As 48 the double Rectangle of the legs, AE, EC , is to 24 the excess of the sum of the Squares of the legs above the Square of the Base AC ; So is 100000 the Radius, to 50000 the Sine of 30° . degrees, whose complement 60° . degrees is the measure of the angle E sought.

An Example in Numbers, where the Square of the Base exceeds the sum of the Squares of the legs.

Suppos. in $\triangle ABC.$

20. $AC = 7$, the Base is given.
 21. $AB = 5$,
 22. $BC = 3$, } the legs are given.
 Req. to find $< ABC.$

Solution Arithmetical.

23. By the preceding Corollary; As 30, the double Rectangle of the legs, A, B, BC , is to 15, the excess of the Square of the Base above the sum of the Squares of the legs; So is 100000 the Radius, to 50000 the Sine of 30° . degrees, whose complement 60° . degrees subtracted from 180° . degrees, leaves 120° . degrees for the angle ABC sought.

Note. Because in this second Example the Square of the Base exceeds the sum of the Squares of the legs, the angle sought is obtuse; and therefore the complement of the angle relating to the Sine which is the fourth Proportional of the before-mentioned Analogy, being subtracted from 180° . degrees, leaves the angle sought. But when the sum of

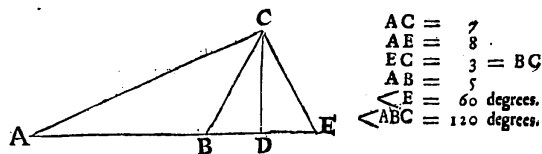
the Squares of the legs exceeds the Square of the Base, then the complement it self of the angle relating to the said Sine (or fourth Proportional) is the angle sought; as in the first Example.

Probl. XIX.

The Base (that is, any side) of a plain Triangle being given, as also the angle opposite to the Base, and the sum of the sides (or legs) containing that angle, to find the Triangle. But the given sum of the legs must exceed the given Base, (per prop. 22. Elem. 1.)

Note. Because in this Problem the given angle is not of the same kind with the right lines given, a right line is to be found out, by the help of that angle, which may stand instead of the angle: To which end, let a Circle be described at any distance, and make an angle at the Center equal to the given angle, then from one end of that arch of the Circumference which is the measure of the said angle at the Center, let fall a Perpendicular upon a Diameter drawn to the other end of the said arch, so is the said Perpendicular the Sine of the given angle, and the segment of the Diameter between the foot of the same Perpendicular and the Center of the Circle is the Sine-complement of the given angle. Now instead of the given angle the said Sine-complement may be taken, by the help whereof, and of the Radius (or Semidiameter) of the said Circle, the Resolution of the Problem propounded may be formed in manner following.

The Problem hath three Cases, for the given angle opposite to the Base given is either right, or acute, or obtuse, the first of those Cases hath already been solved in Probl. 4. of this Chap. I shall therefore begin with the second Case, which supposeth the given angle to be acute, and the legs containing that angle to be unequal.



$$\begin{aligned} AC &= 7 \\ AE &= 8 \\ EC &= 3 = BG \\ AB &= 5 \\ E &= 60^\circ \text{ degrees.} \\ \angle ABC &= 120^\circ \text{ degrees.} \end{aligned}$$

Suppos.

1. $b = AC$ the Base of $\triangle ACE$ is given.
 2. $c = AE + EC$ the sum of the legs is given.
 3. $< E$ opposite to the Base AC is acute, and given.
 4. $r =$ the Radius (or total Sine) is given.
 5. $d =$ the Sine-complement of $< E$ is given.

Req. to find out the Triangle.

Resolution.

6. Suppose ACE to be the \triangle sought, and put a for the difference of the legs AE, EC ; viz. assume $a = AE - EC.$
7. Therefore from 2° and 6° , (per Theor. 9. Chap. 4.) the greater leg shall be $\frac{1}{2}c + \frac{1}{2}a = AE.$
8. And (by the same Theor.) the lesser leg shall be $\frac{1}{2}c - \frac{1}{2}a = EC.$
9. Therefore the double Product of the legs is $\frac{1}{2}cc - \frac{1}{4}aa.$
10. And the sum of the Squares of the legs is $\frac{1}{2}cc + \frac{1}{2}aa.$
11. And because the given angle E is acute, the sum of the Squares of the legs exceeds the Square of the Base, (per prop. 13. Elem. 2.) therefore $\frac{1}{2}cc + \frac{1}{2}aa$ exceeds bb , and the excess it self is $\frac{1}{2}cc - \frac{1}{4}aa.$
12. And from $4^\circ, 5^\circ, 9^\circ$ and 11° , this Analogy is manifest, (by the Lemma prefix before this Probl.) viz.

- $r : d :: \frac{1}{2}cc - \frac{1}{4}aa : \frac{1}{2}cc + \frac{1}{2}aa - bb.$
13. Therefore from that Analogy, by Composition of Reason converse,

$$r + d : r :: cc - bb : \frac{1}{2}cc - \frac{1}{4}aa.$$

R r

14. And

MO + OP = C the line prefcribed for the fumm of the leggs; it remains only to prove that the angle MOP is equal to the given angle E; but that will be made manifest by the following Demonstration, which is formed out of the steps of the foregoing Resolution, by returning backwards from the 14th step to the 1st.

Prepar.

See the last preceding Diagram.

34. From the Center O, at the distance of OP, describe the Circle OPRNS, and produce MO to N in the Circumference; therefore MN = MO + OP (ON) = C, and RM = OM - OP (OR) = G.
35. . . Req. demonstr. < MOP = < E.

Demonstration.

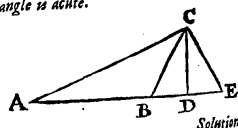
36. By Constr. in 30°. H . R :: □ K . □ F.
37. Therefore from 36°, 28°, 29°. . . R + D . R :: 2□C - 2□B . □C - □G.
38. And because by Constr. in 33°. . . r + d . r :: 2cc - 2bb . cc - aa.
39. Therefore from 37° and 38°. . . MN = C . MP = B . MR = G.
40. And by halving the two latter Terms, R + D . R :: 2□MN - 2□MP . □MN - □MR.
41. Therefore from 40°, by Division of Reason, D . R :: 1/2 □MN + 1/2 □MR - □MP . 1/2 □MN - 1/2 □MR.
42. Therefore inverly, R . D :: 1/2 □MN - 1/2 □MR . 1/2 □MN + 1/2 □MR - □MP.
43. And because by Theor. 7. . . R . D :: 1/2 □OM, OP . 1/2 □MN - 1/2 □MR.
44. And by Theor. 6. Chap. 4. . . □OM + □OP = 1/2 □MN + 1/2 □MR.
45. Therefore from 42°, 43°, 44°. . . R . D :: 2□OM, OP . □OM + □OP - □MP.
46. By the last Term of the last preceding Analogy 'tis evident that . . . □OM + □OP < □MP.
47. Therefore in Δ MOP, (per prop. 13. Elem. 2.) . . . < MOP is acute.
48. Therefore in Δ MOP, (by the Lemma prefix before this Probl. . . Rad. Sin.comp. < MOP :: 1/2 □OM, OP . 1/2 □OM + □OP - □MP.
49. Therefore from 45° and 48°. . . R . D :: Rad. Sin.comp. < MOP.
50. But by Suppos. in 26° and 27°. . . R . D :: Rad. Sin.comp. < E.
51. Therefore from 49° and 50°. . . Rad. Sin.comp. < MOP :: Rad. Sin.comp. < E.
52. Therefore, (per prop. 14. El. 5.) . . Sin.comp. < MOP = Sin.comp. < E.
53. Therefore from 51° and 52°. . . < MOP = < E. Which was to be Dem.
After the same manner, the third Case of Probl. 19. (viz. when the given angle is obtuse,) may be Geometrically effected and demonstrated.

Examples in Numbers, to illustrate the foregoing Resolution of Probl. 19.

Example 1. where the given angle is acute.

Suppos. in Δ AEC.

54. . . AC = 7, the Base is given.
55. AE + EC = 11, the fumm of the leggs is given.
56. . . < E = 60 degrees is given.
Req. to find AE and EC severally.



Solution

Solution Arithmetical.

57. Suppose the Radius of a Circle to be 100000.
58. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees, is 50000.
59. Therefore the fumm of the Radius and that Sine-complement is . . . 150000.
60. Then (by Canon 1. in the preceding 15th step,) As the said fumm 150000 is to the Radius 100000; So is 144, (the excess whereby 242 the double Square of 11 the given fumm of the leggs exceeds 98 the double Square of the given Base 7,) to a fourth Proportional 96, which subtracted from 121 the Square of the given fumm of the leggs, leaves 25, whose square Root 5 is the difference of the leggs: Therefore (per Theor. 9. Chap. 4.) the leggs themselves, to wit, AE and EC shall be 8 and 3.

Example 2. where the given Angle is obtuse.

Suppos. in Δ ABC.

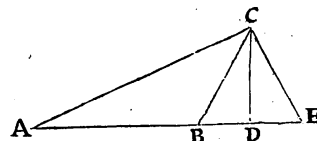
61. . . AC = 7, the Base is given.
62. AB + BC = 8, the fumm of the leggs is given.
63. . . < ABC = 120 degrees is given.
Req. to find AB and BC severally.

Solution Arithmetical.

64. Suppose the Radius of a Circle to be 100000.
65. Then the Sine-complement of the given angle ABC 120. degrees, that is, the Sine of 30. degrees, is 50000.
66. Therefore the excess of the Radius above the Sine-complement is . . . 50000.
67. Then (by Canon 2. in the preceding 20th step,) As the said excess 50000 is to the Radius 100000; So is 30 (the excess whereby 128 the double Square of 8 the given fumm of the leggs exceeds 98 the double Square of the given Base 7,) to a fourth Proportional 60; which subtracted from 64 the Square of the given fumm of the leggs, leaves 4, whose square Root 2 is the difference of the leggs sought: Therefore (per Theor. 9. Chap. 4.) the leggs themselves, to wit, AB and BC shall be 5 and 3.

Probl. XX.

The Base of a plain Triangle being given, as also the angle opposite to the Base, and the difference of the sides (or leggs) containing that angle, to find the Triangle. But the given Base must be greater than the given difference.



$$\begin{aligned} AC &= 7 \\ AE &= 8 \\ EC &= 3 = BC. \\ AB &= 5 \\ \angle E &= 60 \text{ degrees.} \\ \angle ABC &= 120 \text{ degrees.} \end{aligned}$$

Suppos.

1. b = AC the Base of Δ ACE is given.
2. c = AE - EC the difference of the leggs is given.
3. < E opposite to the Base AC is acute, and given.
4. r = the Radius (or total Sine) is given.
5. d = the Sine-complement of < E is given.
Req. to find AE and EC severally.

Resolution.

6. For the fumm of the leggs AE and EC, put a, viz. suppose . . . a = AE + EC.
7. Then out of 1°, 2°, 4°, 5° and 6°, (by the Theor. in 21° of the foregoing Probl. 19.) this following Analogy will arise, r + d . r :: 2aa - 2bb . aa - cc.
8. Therefore

8. Therefore by doubling the Consequents
(*per Schol. Clavii in prop. 22. Elem. 5.*)
9. And by halving each of the two latter Terms,
10. And inverly,
11. Therefore by Conversion and Inversion of
Reason,
- Hence

CANON 1.

12. When the given angle is acute, let it be made, As the excels of the Radius above the Sine-complement of that angle, is to the double Radius; So the excels of the Square of the given Base above the Square of the given difference of the legs, to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the legs, and the Square Root of the sum shall be the sum of the legs sought. Lastly, the sum, as also the difference of the legs being given, the legs shall be given severally, by *Theor. 9. Chap. 4.*
13. But when the given angle is obtuse, then by the *Theorem* in 21st of the preceding *Probl. 20.* this Analogy will arise,
14. Therefore by doubling the Consequents,
15. And by halving each of the two latter Terms,
16. Therefore inverly,
17. Therefore by Conversion and Inversion of
Reason,
- Hence

CANON 2.

18. When the given angle is obtuse, let it be made, As the sum of the Radius and Sine-complement of that angle, is to the double Radius; So the excels of the Square of the given Base above the Square of the given difference of the legs to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the legs, and the Square Root of the sum shall be the sum of the legs sought. Lastly, the sum and difference of the legs being given, the legs shall be given severally, by *Theor. 9. Chap. 4.*

From the preceding Canons in 12th and 18th this following Theorem is deducible, and easy to be demonstrated by the steps of the Resolution in a direct order.

THEOREM.

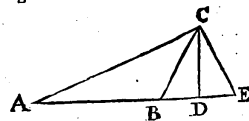
19. If any one of the three sides of an oblique-angled plain Triangle be called the Base, and the other two sides (or legs) be unequal; Then the excels of the Square of the Base above the Square of the difference of the legs, shall be to the excels of the Square of the sum of the legs above the Square of their difference; As the excels of the Radius above the Sine-complement of the angle opposite to the Base, is to the double Radius, when the said angle is acute; but as the sum of the Radius and the Sine-complement is to the double Radius, when the said angle is obtuse.

The Composition of this *Probl. 20.* may be made like that of the preceding *Probl. 19.* But waving the Composition, I shall illustrate the Resolution by Examples in Numbers.

Example 1. where the given angle is acute.

Suppos. in $\triangle AEC$.

20. . . AC = 7 the Base is given.
21. AE - EC = 5 the difference of the legs is given.
22. . . $\angle E = 60$. degrees is given.
- Req. to find AE and EC severally.



Solution Arithmetical.

23. Suppose the Radius of a Circle to be . . . 100000.
24. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees is . . . 50000.
25. Therefore the excels of the Radius above the Sine-complement is . . . 50000.
26. The

26. Then by *Canon 1.* (in the preceding 12th step,) As the said excels 50000 is to the double Radius 200000; So is 24, (the excels whereby 49 the Square of the Base exceeds 25 the Square of the difference of the legs,) to a fourth Proportional 96, which increased with 25 the Square of the difference of the legs, makes 121, whose Square Root 11 is the sum of the legs sought. Therefore (by *Theor. 9. Chap. 4.*) the legs themselves, to wit, AE and EC shall be 8 and 3.

Example 2. where the given angle is obtuse.

Suppos. in $\triangle ABC$.

27. . . AC = 7 the Base is given.
28. AB - BC = 2 the difference of the legs is given.
29. . . $\angle ABC = 120$. degrees is given.
- Req. to find AB and BC severally.

Solution Arithmetical.

30. Suppose the Radius of a Circle to be . . . 100000.
31. Then the Sine-complement of the given angle ABC, 120. degrees, that is, the Sine of 30 degrees is . . . 50000.
32. Therefore the sum of the Radius and that Sine-complement is . . . 150000.
33. Then (by *Canon 2.* in the preceding 18th step,) As the said sum 150000 is to the double Radius 200000; So is 45, (the excels of 49 the Square of the Base above 4 the Square of the difference of the legs,) to a fourth Proportional 60, which increased with 4 the Square of the difference of the legs, makes 64, whose Square Root 8 is the sum of the legs sought. Therefore (by *Theor. 9. Chap. 4.*) the legs themselves, to wit, AB and BC shall be 5 and 3.

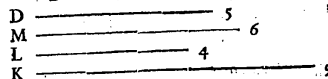
CHAPTER IX.

The third Classis of Examples of the Resolution and Composition of Plane Problems.

IN which Examples, the Resolution ends in an Analogy, wherein the Mean of three proportional right lines is given, as also the Difference or else the Summ of the Extremes, to find the Extremes severally.

Probl. I.

To find two right lines, such, that their difference may be equal to a right line given; and that the Rectangle made of the lines found out may be equal to a given Space.



Suppos.

1. $d = D$ the difference of two right lines is given.
2. $m = M$ a right line given, whose Square is equal to a given Space.
- Req. to find
3. L and K two right lines, such, that $K - L = D$. Also, that
4. $\square KL = \square M$.

Resolution.

5. Put a for the lesser of the two right lines sought, viz. sup-
pose $a = L$.
6. Therefore from 1st and 5th the greater right line sought shall be $a + d (= K)$.
7. And from 5th and 6th, the Rectangle (or Product of their
multiplication) shall be $a(a + d)$.

8. But

8. But the said Rectangle must be equal to the given Space, } $aa + da = mm$.
 supposed to be equal to the Square of m , therefore
 9. Which Equation may be resolved into these Proportionals, } $a + d : m :: m : a$.
viz.
 10. Of which three Proportionals $a + d$, m and a , the mean m is given, as also d the
 difference of the extremes $a + d$ and a , therefore the extremes severally, which are
 the right lines sought by this Problem, shall be given also, *per* *Probl. 12. Chap. 5.* whence
 also, (respect being had to the Theorem in 24th of the same *Probl.*) there will arise this
 following

C A N O N.

11. } $\sqrt{\frac{1}{2}dd + mm} + \frac{1}{2}d = K$, } the lines sought.
 } $\sqrt{\frac{1}{2}dd + mm} - \frac{1}{2}d = L$, }

That is, in words,

To the Square of half the given difference add the Square equal to the given Space, and
 extract the Square Root of the sum: Then adding half the said difference to the said Square
 Root, this sum shall be equal to the greater of the two right lines sought; but subtracting
 the said half difference from the said Square Root, the remainder shall be equal to the lesser
 line sought.

This Canon may be also Synthetically infer'd from the given lines, by the help of
Theor. 7. and *9. Chap. 4.* For,

12. By considering the things given and sought in 1st, } $\frac{1}{2}dd + mm = \square : \frac{1}{2}K + \frac{1}{2}L$.
 2^o, 3^o and 4^o of this Problem it follows by *Theor. 7.* }
Chap. 4. that
 13. Therefore by extracting the Square Root out of } $\sqrt{\frac{1}{2}dd + mm} = \frac{1}{2}K + \frac{1}{2}L$.
 each part in 12th, } $\frac{1}{2}d = \frac{1}{2}K - \frac{1}{2}L$.
 14. And from 1st and 3^o, }
 15. Therefore (by *Theor. 9. Chap. 4.*) the sum and } $\sqrt{\frac{1}{2}dd + mm} + \frac{1}{2}d = K$.
 difference of the two last preceding Equations gives }
 the two lines sought by this Problem, *viz.* } $\sqrt{\frac{1}{2}dd + mm} - \frac{1}{2}d = L$.

Thus you see the same Canon is discovered as before.

The Composition of *Probl. 1.*

$$\begin{array}{rcl} D & \text{---} & 5 \\ M & \text{---} & 6 \\ L & \text{---} & 4 \\ K & \text{---} & 9 \end{array}$$

Suppos.

16. D a right line equal to the difference of two right lines sought, is given.
 17. M a right line given, whose Square is equal to a given Space.
Req. to find,
 18. Two such right lines, that their difference may be equal to the given difference D, and
 that the Rectangle made of them may be equal to the Square of the given right line M.

Construction.

19. Let the given right-line M be esteemed the mean of three Proportionals, and the
 given right line D the difference of the extremes; then by *Probl. 12. Chap. 5.* find the
 extremes, the lesser whereof suppose to be L, therefore the greater shall be equal to
 L + D, and consequently these shall be Proportionals, *viz.*
 $L + D : M :: M : L$.
 20. Make $K = L + D$, whence $K - L = D$.
 21. I say K and L are the two right lines sought, but that they will satisfy the Problem
 propounded, I prove thus: First by *Construction* in the 20th step, the difference of the
 said right lines K and L is equal to the given difference D; so it remains only to prove
 that the Rectangle made of the said right lines K and L is equal to the Square of the
 given right line M, but that is here-under demonstrated by returning backwards from
 the 9th step, (to wit, the last of the Resolution) to the 8th.
 22. *Req. demonstr.* $\square K, L = \square M$.

Demonstr.

Demonstration.

23. By *Constr.* in 19th and 20th, } $K(L + D) : M :: M : L$.
 That is, in 9th, } $a + d : m :: m : a$.
 24. Therefore (*per* 17. *prop. 6. Elem.*) } $\square K, L(\square L + \square D, L) = \square M$.
 That is, in 8th, } $aa + da = mm$.
 Which was to be Demonstr. Therefore that is done which the Problem required.

Probl. II.

To cut a given right line into two such parts, that the Rectangle
 made of the parts may be equal to a Space given. But the side of a
 Square equal to the given Space must not be greater than half the right
 line given.

$$\begin{array}{rcl} A & \text{---} & B \\ & & \text{---} & C \\ M & \text{---} & \end{array} \quad \begin{array}{l} AB = 9 \\ BC = 4 \\ AC = 13 \\ M = 6 \end{array}$$

Suppos.

1. $b = AC$ is a right line given to be cut into two parts.
 2. $m = M$ is a right line given, whose Square is equal to a Figure or Space given.
Req. to find
 3. AB and BC such parts of AC, that $AB + BC = AC$. Also;
 4. $\square AB, BC = \square M$.

Resolution.

5. Put a for either of the parts sought, *viz.* suppose . . . } $a = AB$.
 6. Therefore from 1st and 5th the other part sought shall be } $b - a = BC$.
 7. And from 5th and 6th the Rectangle of the parts is . . . } $ba - aa$.
 8. But the said Rectangle must be equal to the given Space, } $ba - aa = mm$.
 therefore }
 9. Which Equation may be resolved into this Analogy, } $b - a : a :: m : a$.
 10. But that Analogy doth manifestly consist of three Proportionals, whereof the mean m
 is given, as also b the sum of the extremes $b - a$ and a ; therefore the extremes seve-
 rally, (which are the parts required by this *Probl.*) shall be given also, by *Probl. 13.*
Chap. 5. Whence also this

C A N O N.

11. } $\frac{1}{2}b + \sqrt{\frac{1}{4}bb - mm} = AB$,
 } $\frac{1}{2}b - \sqrt{\frac{1}{4}bb - mm} = BC$.

That is, in words,

From the Square of half the right line given to be cut into two parts, subtract the Square
 equal to the given Space, and extract the Square Root of the remainder; then add and
 subtract the said Square Root to and from half the right line given to be cut; so the Sum
 and Remainder shall be the parts required.

This Canon may be also Synthetically infer'd from the given lines, by the help of
Theor. 7. and *9. Chap. 4.* For,

12. By considering the things given and sought in 1st, } $\frac{1}{2}bb - mm = \square : \frac{1}{2}AB - \frac{1}{2}BC$.
 2^o, 3^o and 4^o of this Problem, it follows by }
Theor. 7. Chap. 4. that }
 13. Therefore by extracting the Square Root out of } $\sqrt{\frac{1}{4}bb - mm} = \frac{1}{2}AB - \frac{1}{2}BC$.
 each part in 12th, } $\frac{1}{2}b = \frac{1}{2}AB + \frac{1}{2}BC$.
 14. And from 1st and 3^o, }
 15. Therefore (by *Theor. 9. Chap. 4.*) the sum and } $\frac{1}{2}b + \sqrt{\frac{1}{4}bb - mm} = AB$.
 difference of the two last preceding Equations gives }
 the lines sought by this Problem, *viz.* . . . } $\frac{1}{2}b - \sqrt{\frac{1}{4}bb - mm} = BC$.

Thus you see the same Canon is discovered as before in 11th, and thereby it evidently
 appears, that to the end the given lines may be capable of effecting the Problem propos'd,
 they must be subject to this following

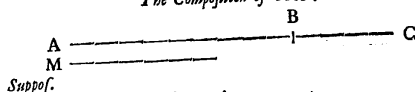
S f

Detr.

Determination.

16. The side of a Square equal to the given Space must not be greater than half the right line given to be cut into two parts. But that this Determination is necessary, I demonstrate thus;

Forasmuch as by the 9th step of the Resolution the right line m is found to be the mean of three Proportionals whereof the sum of the extremes is b , and in 20^o of *Probl. 13. Chap. 5.* it hath been proved that the mean of three Proportionals cannot be greater than half the sum of the extremes, it follows, that m must not be greater than $\frac{1}{2}b$. If therefore m happens to be greater than $\frac{1}{2}b$, the Problem propounded cannot possibly be solved; for then mm cannot be subtracted from $\frac{1}{4}bb$, as the Canon directs. But if m be equal to, or less than $\frac{1}{2}b$, then that which the Canon bids to be done is possible, by which also 'tis easy to perceive, that when $m = \frac{1}{2}b$, then the extreme Proportionals (which are equal to the parts required,) will be equal between themselves, that is, each of them will be equal to $\frac{1}{2}b$, to wit, half the line given to be cut into two parts.

The Composition of *Probl. 2.*

Suppos.

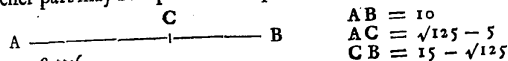
17. AC is a right line given to be cut into two parts.
 18. M is a right line given, whose Square is equal to a Space given.
 19. M not $\leq \frac{1}{2}AC$. (*Determination.*)
Req. to find
 20. AB and BC such parts of AC, that $AB + BC = AC$. Also;
 21. $\square AB, BC = \square M$.

Construction.

22. By *Probl. 14. Chap. 5.* cut the given right line AC into two such parts in B, that the line M may be a mean Proportional between the parts; which effect is possible, for by the Determination in 19^o, M not $\leq \frac{1}{2}AC$; suppose therefore
 $AB \cdot M :: M \cdot BC$.
 23. I say AB and BC are the parts required, for their sum by *Construction* is equal to the given right line AC; and because by *Constr.* also it hath been made, As AB to M, so M to BC, therefore (*per 17. prop. 6. Elem.*) $\square AB, BC = \square M$.
 Which was required to be done.

Probl. III.

To cut a given right line according to the extreme and mean Reason; that is, into two such parts, that the Rectangle made of the whole line and lesser part may be equal to the Square of the greater.



Suppos.

1. $b = AB$ is a right line given.
Req. to find
 2. AC and CB such parts of AB, that $\square AB, CB = \square AC$.

Resolution.

3. Put a for the greater part sought, *viz.* $a = AC$.
 4. Therefore from 1^o and 3^o the lesser part shall be $b - a = CB$.
 5. Therefore from 1^o and 4^o the Rectangle (or Product) of the whole line and lesser part is $bb - ba$.
 6. Which Rectangle, (according to the tenor of the Problem,) must be equal to the Square of the greater part, therefore $bb - ba = aa$.
 7. Therefore, by adding ba to each part of that Equation, $bb = aa + ba$.
 8. Which last Equation may be resolved into this Analogy, $a \cdot b :: b \cdot b + a$.
 9. But

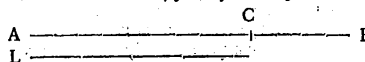
9. But that Analogy doth manifestly consist of three Proportionals, whereof the mean b is given, which is also the difference of the extremes a and $b - a$; therefore the extremes, whereof the lesser is equal to the greater part sought, shall be given also, by *Probl. 12. Chap. 5.* Whence also there will arise this following

CANON.

10. $\sqrt{bb + \frac{1}{4}bb} - \frac{1}{2}b = AC$, the greater segment.

That is, in words,

To the Square of the given line, add the Square of half the same line; then from the square Root of that sum subtract half the given line, and the remainder shall be the greater part sought, which subtracted from the given line leaves the lesser part.

The Composition of *Probl. 3.*

Suppos.

11. Let AB be a right line given to be cut into two such parts, that the Rectangle made of the whole line and the lesser part may be equal to the Square of the greater.

Construction.

12. Let the given line AB be esteemed the mean; as also the difference of the extremes of three Proportionals; then by *Probl. 12. Chap. 5.* find out the extremes severally, the lesser whereof we may suppose to be the right line L, and consequently the greater extreme is $AB - L$, therefore these are Proportionals, *viz.*
 $L \cdot AB :: AB \cdot AB - L$.

13. From AB cut off $AC = L$, which is possible to be done, for by *Construction* in 12^o, $AB \leq L$, because AB is the mean of three Proportionals whose lesser extreme is L. So the given line AB is cut into two parts in C, according to extreme and mean Reason, *viz.* the Rectangle made of the whole line AB and the lesser part CB is equal to the Square of AC the greater part, as will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward order.

14. *Req. demonstr.* $\square AB, CB = \square AC$.

Demonstration.

15. By *Constr.* in 12^o and 13^o, $AC \cdot AB :: AB \cdot AB - AC$.
 That is, in 8^o, $a \cdot b :: b \cdot b - a$.
 16. Therefore (*per prop. 17. Elem. 6.*) $\square AB = \square AC + \square AB, AC$.
 That is, in 7^o, $bb = aa + ba$.
 17. Therefore by subtracting $\square AB, AC$ from each part, $\square AB - \square AB, AC = \square AC$.
 That is, in 6^o, $bb - ba = aa$.
 18. And because (as is evident by the Diagram,) $AB - AC = CB$.
 19. Therefore, by drawing AB as a common altitude into each part of the last Equation, $\square AB - \square AB, AC = \square AB, CB$.
 20. Therefore from 17^o and 19^o, (*per Ax. 1. Chap. 2.*) $\square AB, CB = \square AC$.

Which was to be Demonstr. Therefore the Problem is satisfied.

COROLL. I.

21. From the preceding Resolution, the invention of a right-angled Triangle whose three sides shall be Proportionals is discovered, for if a given right line b be cut according to extreme and mean Reason, and the greater segment be a , 'tis manifest by the 9th step of the foregoing Resolution, that
 $bb = ba + aa$.

22. And because (by *prop. 48. Elem. 1.*) if a Square be equal to two Squares, the sides of those three Squares will constitute a right-angled Triangle, therefore the square Roots
 of

of the three Planes in the preceding Equation in 21° , viz. b , \sqrt{ba} , a shall be the three sides of a right-angled Triangle, and be Proportionals also, for the Rectangle of the extremes is manifestly equal to the Square of the mean. Hence therefore a Canon is discovered to find out a right-angled Triangle, whose three sides shall be Proportionals.

C A N O N.

23. Hyp. = b a right line or number taken at pleasure,
 Base = $\sqrt{bb + \frac{1}{2}bb} = \frac{3}{2}b$.
 Perp. = $\sqrt{b \times \sqrt{bb + \frac{1}{2}bb}} = \frac{1}{2}b$.

That is, in words,

Take any right line (or number) for the Hypotenuse of a right-angled Triangle, and cut it into two parts according to extreme and mean Reason; then the greater part shall be one of the sides about the right angle, and a mean Proportional between the greater part and the Hypotenuse shall be the other side about the right angle; and those three sides shall be Proportionals.

Construction.

$\angle ADB$ is \perp .

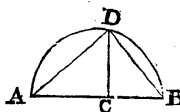
$AB, AD, DB \div \div$.

$AC = DB$.

$AB = 10$

$AD = 7.8615$

$DB = 6.1803$



24. Take any right line, as AB , and (by the foregoing Probl. 3.) cut the same by extreme and mean Reason in C , therefore
 $AB : AC :: AC : CB$.

25. Then upon the line AB describe the Semicircle ADB , and from Centre C draw $CD \perp AB$. Lastly, draw AD and DB , so shall ADB be a right-angled Triangle whose three sides are Proportionals: For first, the $\angle ADB$ being in the Semicircle, is a right angle, (per prop. 31. Elem. 3.) But that the three sides AB, AD, DB are Proportionals, I prove thus;

26. Reg. demonstr. $AB : AD :: AD : DB$.

Demonstration.

27. Because by Constr. in 25° ADB is a Semicircle, and $CD \perp AB$, therefore (per Coroll. prop. 8. Elem. 6.) $AB : DB :: DB : CB$.
 28. And because by Constr. in 24° , $AB : AC :: AC : CB$.
 29. Therefore out of 27° and 28° , (per prop. 17. Elem. 6.) $\square AB, CB = \square DB = \square AC$.
 30. And consequently, $DB = AC$.
 31. And because (by Constr. in 25°) $\triangle ADB$ is right-angled at D ; and $DC \perp AB$, therefore (per prop. 8. Elem. 6.) $AB : AD :: AD : AC$.
 32. Therefore from 30° and 31° , $AB : AD :: AD : DB$.

Which was to be Dem.

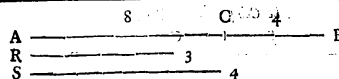
C O R O L L. 2.

33. From the premises the way of solving this following Problem is also deducible; viz. To cut a right line given into three such Proportionals, that the Square of the greatest shall be equal to the Squares of the other two: For if it be made as the sum of the three Proportionals AB, AD, DB (being the sides of a right-angled Triangle, and also Proportionals,) is to every one of them; so the right line given to three others; the Problem will be satisfied.

Probl. IV.

To cut a given right line into two such parts, that the Rectangle made of the whole line and one of the parts, to the Square of the other part may have a given Reason.

A



Suppos.

1. $b = AB$ is a right line given to be cut into two parts.
 2. $\begin{cases} r = R \\ s = S \end{cases}$ are the Terms of a given Reason.

Reg. to find

3. AC and CB such parts of AB ; that $AC \div CB = AB$. Also, that
 4. $\square AB, CB :: AC :: R : S$.

Resolution.

5. Put a for one of the parts sought, viz. $a = AC$.
 6. Therefore from 1 $^\circ$ and 5 $^\circ$ the other part shall be $b - a = CB$.
 7. Therefore the Square of the first part is aa ($= \square AC$).
 8. And the Rectangle of the given line b and latter part is $bb - ba$ ($= \square AB, CB$).
 9. Therefore to answer what the Problem requires, these must be Proportionals, viz. $r : s :: bb - ba : aa$.
 10. Now to avoid an Equation between Solids, let it be made, as r to s , so b to a fourth Proportional, which may be called d , therefore $r : s :: b : d$.
 11. Therefore from 9 $^\circ$ and 10 $^\circ$, (per prop. 11. Elem. 5.) $b : d :: bb - ba : aa$.
 12. And by drawing $b - a$ as a common altitude into b and d severally, this Analogy is manifest, (per prop. 1. Elem. 6.) $b : d :: bb - ba : db - da$.
 13. Therefore from 11 $^\circ$ and 12 $^\circ$, (per prop. 11. Elem. 1.) $bb - ba : aa :: bb - ba : db - da$.
 14. And because the first Term of the last Analogy is equal to the third, the second shall be equal to the fourth, (per prop. 14. Elem. 5.) therefore, $aa = db - da$.
 15. Therefore by adding da to each part of the last Equation, it gives $aa + da = db$.
 16. Which last Equation may be resolved into this Analogy, $a + d : \sqrt{db} :: \sqrt{db} : a$.

In which Analogy (consisting of three Proportionals,) the mean, to wit, \sqrt{db} is given, as also d the difference of the extremes $a + d$ and a , therefore the extremes severally, (the lesser whereof is one of the Parts sought by this Problem,) shall be given also by Probl. 12. Chap. 5. Whence also this

C A N O N.

17. $\sqrt{\frac{1}{2}ad + db} : \frac{1}{2}d = AC$.

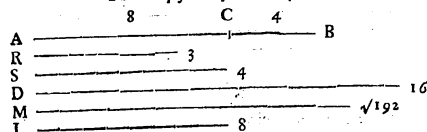
That is, in words,

Let it be made, As R the first Term of the given Reason, to the latter S ; so AB the line given to be cut into two parts, to a fourth Proportional, which may be called D . Then to the Square of half that fourth Proportional, add the Rectangle made of the same Proportional and the given line AB , and from the square Root of the sum subtract half the said fourth Proportional D : The remainder shall be one of the parts sought, which may be called the first. Lastly, the said first part being subtracted from the given line AB , gives also the other part.

18. Note. Whether the first Term of the given Reason be greater or less than the latter, it shall always be; As the first Term is to the latter; so the Rectangle made of the given line and the latter part found out by the Canon, to the Square of the first part.

The

The Composition of Probl. 4.



Suppos.

19. AB is a right line given to be cut into two parts.

20. R and S are the Terms of a given Reason.

Req. to find

21. AC and CB such parts of AB, that $AC + CB = AB$. Also, that22. $\square AB, CB :: \square AC :: R . S$.

Construion.

23. By Probl. 8. Chap. 5. let it be made, as R to S, } $R . S :: AB . D$.
so AB to a fourth, which may be called D, therefore }24. By Probl. 9. Chap. 5. find a mean proportional line, } $AB . M :: M . D$.
as M, between AB and D, therefore }25. Therefore (per prop. 17. Elem. 6.) } $\square M = \square D, AB$.26. Then esteeming the line M to be the mean of three Proportionals, and D the difference of the extremes, find out the extremes, (per Probl. 12. Chap. 5.) the lesser whereof suppose to be L, then consequently the greater is equal to $L + D$, therefore these are Proportionals, viz. $L + D . M :: M . L$.27. Therefore, per Theor. in 24th of Probl. 12. Chap. 5. } $L = \sqrt{\frac{1}{4}\square D + \square M} - \frac{1}{2}D$.28. From AB cut off $AC = L$, which effect is possible, if $AB > L$, but that AB is greater than L, I prove thus,First, it is manifest that } $\square AB + \frac{1}{4}\square D + \square D, AB < \frac{1}{4}\square D + \square D, AB$.And by extracting the square Root out } $AB + \frac{1}{2}D < \sqrt{\frac{1}{4}\square D + \square D, AB}$.of each part, } $AB < \sqrt{\frac{1}{4}\square D + \square D, AB} - \frac{1}{2}D$.And by subtracting $\frac{1}{2}D$ from each part, } $L = \sqrt{\frac{1}{4}\square D + \square D, AB} - \frac{1}{2}D$.But by Constr. in 25th and 27th, } $L = \sqrt{\frac{1}{4}\square D + \square D, AB} - \frac{1}{2}D$.Therefore from the two last preceding } $AB < L$. Which was to be Dem.

steps, (per Ax. 3. Chap. 2.) }

29. I lay the given line AB is cut in C into two parts, according to the tenour of the

Problem propounded. Now we must shew, that the Rectangle made of the whole line

AB and one of the parts, is to the Square of the other part as R to S. But that Analogy

is made manifest by the following Demonstration, formed out of the steps of the pre-

ceding Resolution, by returning backwards from the 16th step to the Analogy in the9th step. Req. demonstr. $R . S :: \square AB, CB . \square AC$.

Demonstration.

31. Because by Constr. in 26th, $L + D . M :: M . L$.32. And by Constr. in 28th, $AC = L$.33. Therefore from 31st and 32nd, by exchange } $AC + D . M :: M . AC$.of equal quantities, $a + d . \sqrt{ab} :: \sqrt{ab} . a$.That is, in 16th, $\square AC + \square D, AC = \square M$.34. Therefore, (per prop. 17. Elem. 6.) } $\square AC + \square D, AC = \square M$.35. And because by Constr. in 24th and 25th, $\square D, AB = \square M$.36. Therefore from 34th and 35th, (per Ax. 1. } $\square AC + \square D, AC = \square D, AB$.Chap. 2.) $aa + da = db$.That is, in 15th, $\square AC = \square D, AB - \square D, AC$.37. Therefore by subtracting $\square D, AC$ from each } $\square AC = \square D, AB - \square D, AC$.part of the Equation in 36th, $aa = db - da$.That is, in 14th, $aa = db - da$. And38. And from 37th, (per prop. 7. } $\square AB - \square AB, AC .$
Elem. 5.) this Analogy is mani- } $\square AC ::$ that is, in 13th, } $bb - ba .$
fest, } $\square AB - \square AB, AC .$ } $ba ::$
 } $\square D, AB - \square D, AC .$ } $db - da .$ 39. And by reason of the common } $AB .$
altitude $AB - AC$, this following } $D ::$ that is, in 12th, } $d ::$
Analogy is manifest, (per pro- } $\square AB - \square AB, AC .$ } $bb - ba .$
pof. 1. Elem. 5.) } $\square D, AB - \square D, AC .$ } $db - da .$ 40. Therefore from 38th and 39th, (per prop. 11. } $AB . D :: \square AB - \square AB, AC . \square AC$.Elem. 5.) } $b . d :: bb - ba . aa$.That is, in 11th, } $AB . D :: R . S$.41. And because by Constr. in 23rd, } $b . d :: r . s$.That is, in 10th, } $R . S :: \square AB - \square AB, AC . \square AC$.42. Therefore from 40th and 41st, (per prop. 11. } $R . S :: \square AB - \square AB, AC . \square AC$.Elem. 5.) } $r . s :: bb - ba . aa$.That is, in 9th, } $CB = AB - AC$.43. And because (per Diagram,) } $\square AB, CB = \square AB - \square AB, AC$.44. And consequently, (per prop. 1. Elem. 6.) } $\square AB, CB = \square AB - \square AB, AC$.by drawing AB into each part, } $R . S :: \square AB, CB . \square AC$.45. Therefore from 42nd and 44th, by ex- } $R . S :: \square AB, CB . \square AC$.changing equal quantities, } $aa = db - da$.

Which was to be Demonstr. Therefore the Problem is satisfied.

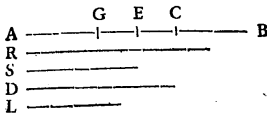
Probl. V.

To cut a given right line into two such parts, that the difference of the Squares of the parts, to the Rectangle made of the same parts, may have a given Reason.

Suppos.

1. $b = AB$ is a right line given to be cut into two parts.2. $r = R$ } are the Terms of a given Reason.3. $s = S$ }

Req. to find

4. AC and CB, such parts of AB, that $AC + CB = AB$. Also, that5. $\square AC - \square CB . \square AC, CB :: R . S$.

Resolution.

6. For the difference of the parts sought put } a .7. Then out of 1st and 6th, (by Theor. 9. Chap. 4.) } $\frac{1}{2}b + \frac{1}{2}a$.the greater part shall be } $\frac{1}{2}b - \frac{1}{2}a$.8. And (by the same Theorem) the lesser part shall be } $\frac{1}{2}b - \frac{1}{2}a$.9. Therefore from 7th and 8th, the Product or Rect- } $\frac{1}{4}bb - \frac{1}{4}aa$.angle of the parts shall be } ba .10. And (from 1st and 6th, per Theor. 8. Chap. 4.) } ba .the difference of the Squares of the parts is } ba .11. Then according to the import of the Problem } $r . s :: ba . \frac{1}{4}bb - \frac{1}{4}aa$.propounded, this Analogy arithmetically, (out of 2nd, 3rd, } $10th and 9th, viz.$ 12. Now to avoid an Equation between Solids, let it } $r . s :: b . d$.be made, as r to s , so b to a fourth Proportional, } $r . s :: b . d$.which may be called d , therefore } $b . d :: ba . \frac{1}{4}bb - \frac{1}{4}aa$.13. Therefore out of 11th and 12th, (per prop. 11. El. 5.) } $b . d :: ba . \frac{1}{4}bb - \frac{1}{4}aa$.

14. And

14. And by drawing a as a common altitude into b and d severally, this following Analogy will be manifest, (per prop. 1. Elem. 6.) viz. $b . d :: ba . da$.
15. Therefore from the 13th and 14th steps, (per prop. 11. Elem. 5.) $ba . \frac{1}{2}bb - \frac{1}{2}aa :: ba . da$.
16. Therefore from the 15th step this Equation ariseth, (per prop. 14. Elem. 5.) viz. $\frac{1}{2}bb - \frac{1}{2}aa = da$.
17. And by multiplying each Term of the Equation in the 16th step by 4, this Equation is produced, viz. $bb - aa = 4da$.
18. Therefore by adding aa to each part, $bb = aa + 4da$.
19. Which Equat. may be resolved into this Analogy, $a + 4d . b :: b . a$.
20. But that Analogy doth manifestly consist of three Proportionals, whereof the mean b is given, as also $4d$ the difference of the extremes $a + 4d$ and a ; therefore the extremes severally, the lesser whereof is the difference of the parts sought by this Problem, shall be given also, by Probl. 12. Chap. 5. whence also, (respect being had to the Theorem in 24^o of the same Problem,) there will arise this

C A N O N.

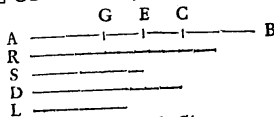
21. . . . $\sqrt{bb + 4dd} - 2d = a$, the difference of the parts sought.

That is, in words,
Let it be made, as the first Term (whether it be the greater or lesser) of the given Reason, is to the second, so the line given to be divided, to a fourth Proportional. Then add the Square of the double of that Proportional to the Square of the line given to be divided, and from the square Root of the sum subtract the double of the said Proportional, so shall the remainder be the difference of the parts sought. Then by the sum and difference of the parts, the parts shall be given severally by Theor. 9. Chap. 4.

The Composition of Probl. 5.

Suppos.

22. AB is a right line given.
23. R and S are the Terms of a given Reason.
24. AC and CB such parts of AB , that $AC + CB = AB$. Also,
25. $\square AC - \square CB . \square AC, CB :: R . S$.



Construction.

26. By Probl. 8. Chap. 5. let it be made, as R to S , so the given line AB to a fourth Proportional, which may be called D , therefore $R . S :: AB . D$.
27. Then esteeming the given line AB to be the mean of three Proportionals, and $4D$ the difference of the extremes, find out the extremes, (per Probl. 12. Chap. 5.) the lesser whereof suppose to be L , then consequently the greater is equal to $L + 4D$, and these are Proportionals, viz.
 $L + 4D . AB :: AB . L$.
- Whence, (per Theor. in 24^o of Probl. 12. Chap. 5.) $L = \sqrt{\square AB + 4 \square D} - 2D$.
- Thus far the Construction hath been made according to the direction of the Canon in 21^o.
28. Divide the given line AB into two equal parts in E , whence $EA = EB = \frac{1}{2}AB$.
29. From EA and EB cut off EG and EC , such parts, that each may be equal to $\frac{1}{2}L$, which is possible to be done, if $\frac{1}{2}AB < \frac{1}{2}L$; but that $\frac{1}{2}AB < \frac{1}{2}L$, I prove thus, First, it is manifest that $\square AB + 4 \square D > \square AB + 4 \square D$. Therefore by extracting the square Root out of each part, $AB + 2D < \sqrt{\square AB + 4 \square D}$.

And

And by subtracting $2D$ from each part, . . . $AB < \sqrt{\square AB + 4 \square D} - 2D$. But by Constr. in 27^o, . . . $L = \sqrt{\square AB + 4 \square D} - 2D$. Therefore, (per Ax. 3. Chap. 2.) . . . $AB < L$; and $\frac{1}{2}AB < \frac{1}{2}L$.

Therefore it is possible to cut off from $\frac{1}{2}AB$, that is, from $EB = EA$, a right line equal to $\frac{1}{2}L$; suppose therefore $EC = EG = \frac{1}{2}L$.

30. I say the given right line AB is cut into two parts in C , as the Problem requires; viz. The difference of the Squares of the parts AC and CB , is to the Rectangle made of the same parts, as R to S ; which Analogy may be demonstrated by a retrograde repetition of the steps of the preceding Resolution, in manner following.

31. . . . Reg. demonstr. . . . $R . S :: \square AC - \square CB . \square AC, CB$.

Demonstration.

32. Forasmuch as by Constr. in 27^o, $L + 4D . AB :: AB . L$. That is, in 19^o, $a + 4d . b :: b . a$.
33. Therefore from 32^o, (per prop. 17. Elem. 6.) $\square AB = \square L + 4 \square DL$. That is, in 18^o, $bb = aa + 4da$.
34. Therefore from 33^o, by subtracting the $\square L$ from each part, $\square AB - \square L = 4 \square DL$. That is, in 17^o, $bb - aa = 4da$.
35. And by taking a quarter of the Equation in 34^o, it gives $\frac{1}{4} \square AB - \frac{1}{4} \square L = \square DL$. That is, in 16^o, $\frac{1}{4}bb - \frac{1}{4}aa = da$.
36. Therefore from 35^o this following Analogy will be manifest, (per 7. prop. 5. Elem.) $\square AB, L . \frac{1}{4} \square AB - \frac{1}{4} \square L :: \square AB, L . \square DL$. That is, in 15^o, $ba . \frac{1}{4}bb - \frac{1}{4}aa :: ba . da$.
37. And by reason of the common altitude L , this following Analogy will be manifest, (per prop. 1. Elem. 6.) $AB . D :: \square AB, L . \square DL$. That is, in 14^o, $b . d :: ba . da$.
38. Therefore from 36^o and 37^o, (per prop. 11. Elem. 5.) $AB . D :: \square AB, L . \frac{1}{4} \square AB - \frac{1}{4} \square L$. That is, in 13^o, $b . d :: ba . \frac{1}{4}bb - \frac{1}{4}aa$.
39. And because by Constr. in 26^o, $AB . D :: R . S$. That is, in 12^o, $b . d :: r . s$.
40. Therefore out of 38^o and 39^o, (per 11. prop. 5. Elem.) $R . S :: \square AB, L . \frac{1}{4} \square AB - \frac{1}{4} \square L$. That is, in 11^o, $r . s :: ba . \frac{1}{4}bb - \frac{1}{4}aa$.
41. And because by Constr. in 29^o, $EC = \frac{1}{2}L = EG$.
42. Therefore by taking $\frac{1}{4}EC$, that is, GC , instead of L in the Analogy in 40^o, this will thence arise, viz. $AB = AC + CB$.
43. And because (per Diagram,) $GC = AC - CB$ (AG.)
44. And by Constr. in 28^o and 29^o, $\square AC - \square CB = \square AB, GC$.
45. Therefore from 43^o and 44^o, (per Theor. 8. Chap. 4.) $AC = \frac{1}{2}AB + \frac{1}{2}GC$.
46. And because by Constr. in 29^o, $CB = \frac{1}{2}AB - \frac{1}{2}GC$.
47. And by (constr. also in 29^o), $\square AC, CB = \frac{1}{4} \square AB - \frac{1}{4} \square GC$.
48. Therefore from 45^o and 47^o, (per Theor. 8. Chap. 4. & Prop. 1. Elem. 6.) $R . S :: \square AC - \square CB . \square AC, CB$.
49. Therefore if instead of the two latter Terms of the Analogy in 42^o, you take their equivalent quantities, to wit, the first parts of the Equations in 45^o and 48^o, this Analogy will arise, viz. . . .

Which was to be demonstr. Therefore the Problem is satisfied.

T t

Probl. VI.

Probl. VI.

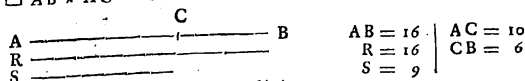
To cut a given right line into two such parts, that the Rectangle made of the whole line and the difference of the parts, to the Square of the lesser part may have a given Reason.

Suppos.

1. $b = AB$ is a right line given to be cut into two parts.
 2. $\begin{cases} r = R \\ s = S \end{cases}$ are the Terms of a given Reason.

Reg. to find

3. AC and CB such parts of AB , that $AC \cdot CB = AB$. Also,
 4. $\square AB \times AC - CB \cdot \square CB :: R \cdot S$.



Resolution.

5. Put a for the greater part sought, viz. $a = AC$.
 6. Therefore out of 1° and 5° , the lesser part shall be $b - a = CB$.
 7. And from 5° and 6° , the difference of the parts is $2a - b$.
 8. And from 1° and 7° , the Rectangle of the given line into the difference of the parts is $2ba - bb$.
 9. And from 6° , the Square of the lesser part is $bb + aa - 2ba$.
 10. Therefore from $1^\circ, 8^\circ$ and 9° , according to the import of the Problem, these must be Proportionals, viz.
 $r \cdot s :: 2ba - bb \cdot bb + aa - 2ba$.
 11. Therefore inversly, $s \cdot r :: bb + aa - 2ba \cdot 2ba - bb$.
 12. And by Composition, $s + r \cdot r :: aa \cdot 2ba - bb$.
 13. And inversly, $r \cdot s + r :: 2ba - bb \cdot aa$.
 14. Now to avoid an Equation between Solids, let it be made, as r to $s + r$; so b to a fourth Proportional, which may be called d , therefore $\dots r \cdot s + r :: b \cdot d$.
 15. Therefore from 13° and 14° , (per prop. 11. Elem. 5.) $\dots b \cdot d :: 2ba - bb \cdot aa$.
 16. And by drawing $2a - b$ as a common altitude into b and d severally, this Analogy is manifest, (per prop. 1. Elem. 6.) $\dots b \cdot d :: 2ba - bb \cdot 2da - db$.
 17. Therefore from 15° and 16° , (per prop. 11. Elem. 5.) $2ba - bb \cdot 2da - db :: 2ba - bb \cdot aa$.
 18. And from 17° , (per prop. 14. Elem. 5.) $2da - db = aa$.
 19. And by adding db to each part of that Equation, $2da = aa + db$.
 20. And by subtracting aa from each part, $2da - aa = db$.
 21. Which last Equation may be resolved into this Analogy, $2d - a \cdot \sqrt{db} :: \sqrt{db} \cdot a$.
 22. But that Analogy doth manifestly consist of three Proportionals, whereof the mean \sqrt{db} is given, as also $2d$ the sum of the extremes $2d - a$ and a ; therefore the extremes severally, the lesser whereof is the greater part sought, shall be given also, by Probl. 13. Ch. 5. And from the premises and the Theorem in 21° of the same Problem there will arise this following $CANON$.

23. $\dots d - \sqrt{db} - db :: AC$, the greater part sought.

That

That is, in words,

Let it be made as R the first Term of the given Reason, to $R + S$ the sum of both Terms, so AB the line given to be cut into two parts, to a fourth Proportional, which may be called D . Then from the Square of that fourth Proportional, subtract the Rectangle made of the said fourth Proportional and the given line AB , and extract the Square Root of the remainder. Lastly, subtract the said Square Root from the said fourth Proportional, and this remainder shall be the greater part sought.

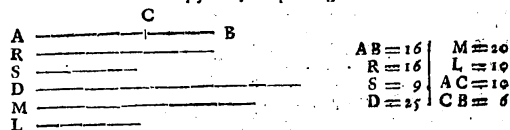
24. Note. Although the Equation found out in 20° may be expounded by either of the two extreme Proportionals mentioned in 22° , yet the lesser of them is only capable of solving the Problem propounded; for the greater extreme is greater than the line given to be cut into two parts, and therefore cannot be equal to either of the parts, which I prove thus,

If \sqrt{db} be the mean of three Proportionals, and $2d$ the sum of the extremes, then (by Probl. 13. Chap. 5.) the greater extreme shall be equal to $d + \sqrt{db} - db$.

Now we must prove that $d + \sqrt{db} - db < b$.
 First, by Constr. in 14° , (and by Schol. prop. 14. Elem. 5.) 'tis evident that $d < b$.

And consequently, by drawing d into each part, $\sqrt{db} < db$.
 Therefore, $\sqrt{db} - db < 0$.
 And consequently, by adding d to each part, $d + \sqrt{db} - db < d$.
 But by Constr. in 14° , $d < b$.
 Therefore (per Ax. 5. Chap. 2.) $d + \sqrt{db} - db < b$.
 Which was to be Dem.

The Composition of the preceding Probl. 6.



Suppos.

25. AB is a right line given to be cut into two parts.
 26. R and S are the Terms of a given Reason.

Reg. to find

27. AC and CB such parts of AB , that $AC \cdot CB = AB$. Also;
 28. $\square AB \times AC - CB \cdot \square CB :: R \cdot S$.

Constr. in.

29. By Probl. 8. Chap. 5. let it be made, as R to $S + R$; so AB to a fourth Proportional, which we may suppose $R \cdot S + R :: AB \cdot D$.
 to be the line D , therefore $AB \cdot M :: M \cdot D$.
 30. By Probl. 9. Chap. 5. find a mean proportional line, as M , between AB and D , therefore $\square AB, D = \square M$.
 31. Therefore it follows from the last Analogy, (per prop. 17. Elem. 6.) that $\square AB, D = \square M$.
 32. Then by Probl. 14. Chap. 5. cut the double of the right line D (before found in 29°) into two such parts, that the line M may be a mean Proportional between them; which Effect is possible, if M be not greater than D ; but that M is less than D , I prove thus.

By the Constr. in 29° 'tis manifest that $D < AB$, and by Constr. in 30° , M is a mean Proportional between AB and D , therefore $M < D$. Therefore 'tis possible to cut $2D$ into two such parts that M may be a mean Proportional between the parts. Suppose then it be done, and that the lesser part is found L , therefore these shall be Proportionals, viz.

$$2D - L \cdot M :: M \cdot L.$$

Tt 2

33. And

33. And consequently, (per Theor. in 21° of Probl. 13. Chap. 5.) $L = D - \sqrt{D^2 - M^2}$
 34. From AB cut off AC = L, which is possible to be done if AB \geq L, but that AB \geq L, I prove thus;
 First, from the Constr. in 29°, $D \leq AB$.
 Therefore, by drawing A B into each part, $\square D, AB \leq \square AB$.
 And by adding $\square D$ to each part, $\square D + \square D, AB \leq \square D + \square AB$.
 And by subtracting $\square D, AB$ from each part, $\square D - \square D, AB \leq \square D - \square AB$.
 And by extracting the square Root out of each part, $\sqrt{\square D - \square D, AB} \leq \sqrt{\square D - \square AB}$.
 And by adding AB to each part, $AB + \sqrt{\square D - \square D, AB} \leq \sqrt{\square D - \square AB}$.
 And by subtracting $\sqrt{\square D - \square D, AB}$ from each part, $AB \leq \sqrt{\square D - \square D, AB}$.
 But it hath been shewn in 31° and 33°, that $L = D - \sqrt{\square D - \square D, AB}$.
 Therefore from the two last preceding steps, $AB \leq L$.
 Which was to be Dem. Therefore 'tis possible to cut off from AB a segment equal to L, as AC.
35. I lay the given right line AB, in the point C is cut into two parts, AC and CB, which will solve the Problem; viz. the Rectangle made of the whole line AB, and the excess of AC above CB, is to the Square of CB, as the line R out to the line S: As will be made manifest by the following Demonstration, form'd out of the foregoing Resolution, by a retrograde repetition of the steps thereof.
36. . . . Req. demonstr. . . . $R \cdot S :: \square AB \times AC - CB \cdot \square CB$.

Demonstration.

37. By Constr. in 32°, $2D - L \cdot M :: M \cdot L$.
 38. And by Constr. in 34°, $AC = L$.
 39. Therefore from 37° and 38°, by exchanging equal quantities, $2D - AC \cdot M :: M \cdot AC$.
 That is, in 21°, (per prop. 17. Elem. 6.) $2d - a \cdot \sqrt{db} :: \sqrt{db} \cdot a$.
 40. Therefore from 39°, (per prop. 17. Elem. 6.) $\square D, AC - \square AC = \square M$.
 41. But by Constr. in 31°, $\square D, AB = \square M$.
 42. Therefore from 40° and 41°, (per Ax. 1. Chap. 2.) $\square D, AC - \square AC = \square D, AB$.
 That is, in 20°, $2da - aa = db$.
 43. And by adding $\square AC$ to each part of the Equation in 42°, $\square D, AC = \square AC + \square D, AB$.
 That is, in 19°, $2da = aa + db$.
 44. And by subtracting $\square D, AB$ from each part in 43°, $\square D, AC - \square D, AB = \square AC$.
 That is, in 18°, $2da - db = aa$.
 45. And from 44°, (per prop. 7. Elem. 5.) this Analogy will be manifest,
 $\square AB, AC - \square AB$
 $\square D, AC - \square D, AB ::$ that is, in 17°, $\frac{2da - db}{aa} :: \frac{2ba - bb}{aa}$
 $\square AB, AC - \square AB$
 $\square AC$
46. And by reason of the common altitude $2AC - AB$, this following Analogy is manifest, (per prop. 1. Elem. 6.)
 $\square AB, AC - \square AB$
 $\square D, AC - \square D, AB ::$ that is, in 16°, $\frac{2ba - bb}{2da - db} :: \frac{2ba - bb}{d}$
 $\square AB$
 D
47. Therefore from 45° and 46°, (per prop. 11. Elem. 5.) $AB \cdot D :: \square AB, AC - \square AB \cdot \square AC$.
 That is, in 15°, $b \cdot d :: \frac{2ba - bb}{aa}$.
 48. But by Construction in 29°, $AB \cdot D :: R \cdot S + R$. That

- That is, in 14°, $b \cdot d :: r \cdot s + r$.
 49. Therefore from 47° and 48°, (per prop. 11. Elem. 5.) $R \cdot S + R :: \square AB, AC - \square AB \cdot \square AC$.
 That is, in 13°, $r \cdot s + r :: \frac{2ba - bb}{aa}$.
 50. Therefore inverly, $S + R \cdot R :: \square AC \cdot \frac{2ba - bb}{aa}$.
 That is, in 12°, $s + r \cdot r :: aa \cdot \frac{2ba - bb}{aa}$.
 51. Therefore by Division of Reason,
 $\square AB + \square AC - \frac{2 \square AB, AC}{\square AB, AC} ::$ that is, in 11°, $\frac{bb + aa - 2ba}{2ba - bb}$
 $\frac{2 \square AB, AC}{\square AB, AC} - \square AB$
52. Therefore inverly,
 $\frac{2 \square AB, AC}{\square AB, AC} - \square AB ::$ that is, in 10°, $\frac{2ba - bb}{bb + aa - 2ba}$
 $\square AB + \square AC - \frac{2 \square AB, AC}{\square AB, AC}$

Now the Scope in the six steps next following is to prove, that $\square AB \times AC - CB$ is equal to $2 \square AB, AC - \square AB$, to wit, the third Term of the Analogy in 11°.

53. By Constr. in 34°, $AB = AC + CB$.
 54. And by adding AC to each part, $AB + AC = 2AC + CB$.
 55. And by subtracting CB from each part in 54°, $AB + AC - CB = 2AC$.
 56. It is evident by the first part of the Equation in 44°, that $2AC = AB$.
 57. Therefore AB may be substituted for each part of the Equation in 55°, and this Equation between two real right lines will remain, viz. $AB - CB = 2AC - AB$.
 58. Therefore by drawing AB as a common altitude into each part of the last Equation, it will produce (by Theor. 1. Chap. 4.) $\square AB \times AC - CB = 2 \square AB, AC - \square AB$.
 Which was to be shewn. It remains to prove that $\square CB$ is equal to $\square AB - \square AC - 2 \square AB, AC$, to wit, the fourth Term of the Analogy in 52°.
 59. By Constr. in 34°, $CB = AB - AC$.
 60. And because the Squares of equal right lines are also equal, therefore $\square CB = \square AB - \square AC - 2 \square AB, AC$.
 from 59°, (per Theor. 5. Chap. 4.)
 Which was to be Demonstr.
61. Lastly, instead of the third and fourth Terms of the Analogy in 52°, their equivalent quantities, to wit, those in the first parts of the Equations in 58° and 60° being taken, this following Analogy ariseth, viz.
 $R \cdot S :: \square AB \times AC - CB \cdot \square CB$.
 Which was Req. demonstr. in 36°. Therefore the Problem is satisfied.

Probl. VII.

To find two right lines that their sum may be equal to a right line given, and that the difference of the Squares of those two lines, to the Square of the lesser of them, may have a given Reason.

This Problem is the same in effect with the preceding sixth, for the difference of the Squares of the two right lines sought by this Problem, is equal to the Rectangle of the sum and difference of the parts sought by the last preceding Problem.

LEMMA,

A L E M M A, leading to the following Probl. 8.

If four right lines be in continual proportion, the sum of the means is a mean Proportional between the sum of the first and second, and the sum of the third and fourth Proportionals.

A	_____	A = 125
B	_____	B = 100
C	_____	C = 80
D	_____	D = 64

Suppos.

1. A, B, C, D \div , viz. A . B :: B . C :: C . D.
2. *Req. demonstr.* A + B . B + C :: B + C . C + D.

Demonstration.

3. By *Suppos.* in 1°, A . B :: B . C .
4. Therefore by Composition of Reason, A + B . B :: B + C . C .
5. And alternately, A + B . B + C :: B . C .
6. Again, by *Suppos.* B . C :: C . D .
7. Therefore by Composition, B + C . C :: C + D . D .
8. And alternately, B + C . C + D :: C . D .
9. Therefore from 6° and 8°, (*per prop.* 11. *Elem.* 5.) B + C . C + D :: B . C .
10. Likewise from 5° and 9°, (*per prop.* 11. *Elem.* 5.) A + B . B + C :: B + C . C + D .

Which was to be Demonstr.

Probl. VIII.

The sum of the extremes, and sum of the means of four right lines in Continual proportion being given severally, to find the Proportionals. But the first sum must be greater than the latter, the reason whereof is manifest, (*per prop.* 25. *Elem.* 5.)

E	_____	F	_____	G	_____	H	_____	I	EF = 16
B	_____								FG = 8
C	_____								GH = 4
									HI = 2

Suppos.

1. EF, FG, GH, HI \div , viz. EF . FG :: FG . GH :: GH . HI.
2. $b = EF + HI$ is given. Also,
3. $c = FG + GH = FH$ is given: Therefore,
4. $d = b + c = EI$ is given.
5. $b = c$. (*Determination.*)
Req. to find EF, FG, GH, HI.

Resolution.

6. By *Suppos.* in 1°, EF, FG, GH, HI \div .
7. Therefore by the *Lemma* prefix before this Problem, EF + FG . FG + GH :: FG + GH . GH + HI.
8. That is, as is evident by the Diagram, EG . FH :: FH . GI.
9. Of which three continual Proportionals the mean FH, that is, c , is given; as also EI, ($= EG + GI$), that is, d , the sum of the extremes. Therefore, by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes shall be given severally, viz.
10. $\frac{1}{2}d + \sqrt{\frac{1}{4}d^2 - ac} = EG = EF + FG$.
11. $\frac{1}{2}d - \sqrt{\frac{1}{4}d^2 - ac} = GI = GH + HI$.
11. Therefore, from 10° and 3° it is manifest, that of these three Proportionals, EF, FG, GH, the sum of the first and second, to wit, EG, is given; also the sum of the second and third, to wit, FH, is given. Likewise of these three Proportionals, FG, GH,

GH, HI, the sum of the first and second, to wit, FH, is given; also GI the sum of the second and third is given; therefore, (according to the Canon in 44° of *Probl.* 5. *Chap.* 7.) FG and GH shall be given severally by these following Analogies, viz.

$$\begin{aligned} 12. \dots & \left\{ \begin{array}{l} c + \frac{1}{2}d + \sqrt{\frac{1}{4}d^2 - ac} : c :: \frac{1}{2}d + \sqrt{\frac{1}{4}d^2 - ac} : FG. \\ c + \frac{1}{2}d - \sqrt{\frac{1}{4}d^2 - ac} : c :: \frac{1}{2}d - \sqrt{\frac{1}{4}d^2 - ac} : GH. \end{array} \right. \end{aligned}$$

Which Analogies, respect being had to the Equations in 10°, and to the Diagram, will give this following

C A N O N.

13. From the Square of half the aggregate of the given sum of the extremes and the given sum of the means, subtract the Square of the sum of the means, and extract the Square Root of the remainder. Then add and subtract that Square Root to and from the said half aggregate, and reserve the sum and remainder. Then it shall be, as the sum reserved together with the sum of the means, is to the sum of the means; so the sum reserved to the greater mean fought. Or, as the remainder reserved together with the sum of the means, is to the sum of the means; so the remainder reserved to the lesser mean. Lastly, the sum before reserved being lessened by the greater mean, gives the greater extreme: or, the remainder reserved being lessened by the lesser mean, gives the lesser extreme.

The Composition of the foregoing Probl. 8.

B	_____	B = 18	K = 36
C	_____	C = 12	L = 18
E	_____	EI = 30	GF = 8
K	_____	EG = 24	GH = 4
L	_____	GI = 6	M = 4
M	_____		

Suppos.

14. B = the sum of the extremes of four right lines in continual proportion is given.
15. C = the sum of the means is given.
16. B \square C. (*Determination.*)
Req. to find the four Proportionals severally.

Construction.

17. Make EI = B + C.
18. Divide EI into two such parts in G, that the line C may be a mean Proportional between the parts; which effect is possible, (*per Probl.* 14. *Chap.* 5.) if C be not greater than $\frac{1}{2}EI$, but that C is less than $\frac{1}{2}EI$, I prove thus;
By the *Determination* in 16°, C \square B.
Therefore by adding C to each part, 2C \square B + C.
But by *Construction* in 17°, EI = B + C.
Therefore, (*per Ax.* 3. *Chap.* 2.) 2C \square EI.
And consequently, C \square $\frac{1}{2}EI$.
Which was to be shewn.

Therefore 'tis possible to cut EI into two such parts, that C shall be a mean Proportional between them. Suppose then EI to be so cut in G, and that EG is greater than GI; therefore,

19. Make K = EG + C; also, L = GI + C.
20. Let it also be made (by *Probl.* 8. *Chap.* 5.) as K to C, so EG to a fourth Proportional GF, therefore, K . C :: EG . GF.
21. Again, let it be made as L to C, so GI to a fourth Proportional GH; therefore, L . C :: GI . GH.
22. Then from EG cut off GF, and from GI cut off GH; (which subtractions are possible, for by *Construction* in 19°, K is greater than C, therefore from the Analogy in 20°, EG is greater than GF: Likewise by *Construction* in 19°, L is greater than C, and consequently from the Analogy in 21°, GI is greater than GH;) so the remainders EF and HI are the extreme Proportionals fought. I say EF, FG, GH and HI are four

four continual Proportionals, which will satisfy the Problem propounded. But to make the truth thereof evident, I shall prove three things, viz. First, that FH the sum of the means FG and GH is equal to the given sum B: Secondly, that the sum of the extremes EF and HI is equal to the given sum B: Thirdly and lastly, that the said EF, FG, GH, HI are in continual proportion in this order, viz.

$$EF : FG :: FG : GH :: GH : HI.$$

First, that FH the sum of the means FG and GH is equal to the given sum C, I prove thus;

Preparat.

23. To the lines K and C find a third Proportional, as M, (per Probl. 7. Chap. 5.) } $K : C :: C : M.$
therefore, }
24. Req. demonstr. FH = C.

Demonstration.

25. By Constr. in 18°, } $EG : C :: C : GI.$
26. Therefore by Composition of Reason, } $EG + C : C :: C + GI : GI.$
27. That is, by exchanging equal right lines } $K : C :: L : GI.$
according to the Construction in 19°, }
28. Therefore alternately, } $K : L :: C : GI.$
29. And by drawing C as a common altitude } $K : L :: \square C : \square C, GI$
into the two latter Terms, }
30. But from the Constr. in 23° and 21° } $\square KM = \square C; \text{ and } \square L, GH = \square C, GI.$
(per prop. 17, & 16. Elem. 6.) }
31. Therefore from 29° and 30°, by ex- } $K : L :: \square K, M : \square L, GH$
change of equal Rectangles, } $K : L :: \square K, M : \square L, M.$
32. But (per prop. 1. Elem. 6.) }
33. That from 31° and 32°, (per prop. 11. } $\square K, M : \square L, GH :: \square K, M : \square L, M.$
Elem. 5.) } $\square L, GH = \square L, M.$
34. Therefore from 33°, (per prop. 14. El. 5.) } $L : M :: L : GH.$
35. Therefore from 34°, (per prop. 14. El. 6.) } $\dots M = GH.$
36. Therefore from 35°, (per prop. 14. El. 5.) } $K : C :: EG : GF.$
37. Again, by Constr. in 20°, } $K : C :: C : M.$
38. And by Constr. in 23°, }
39. Therefore from 37° and 38°, (per } $2K : 2C :: EG + C : GF + M.$
Coroll. Herigon. in prop. 12. Elem. 5.) } $K = EG + C.$
40. And because by Constr. in 19°, } $2K : 2C :: K : GF + M.$
41. Therefore from 39° and 40°, } $2K : 2C :: K : C.$
42. But (per prop. 15. Elem. 5.) }
43. Therefore from 41° and 42°, (per } $K : GF + M :: K : C.$
prop. 11. Elem. 5.) } $GF + M = C.$
44. Therefore from 43°, (per prop. 14. El. 5.) } $GH = M.$
45. But it hath been proved in 36°, that } $GF + GH = C.$
46. Therefore from 44° and 45°, (per Ax. 6. } $GF + GH = FH.$
Chap. 2.) }
47. But 'tis evident by the Diagram, that } $FH = C.$ Which was to be Dem.
48. Therefore from 46° and 47°, (per }
Ax. 1. Chap. 2.) }

Secondly, that the sum of the extremes EF and HI is equal to the given sum B, I demonstrate thus;

49. Req. demonstr. EF + HI = B.

Demonstration.

50. By Constr. in 17°, } $EI = B + C.$
51. And it hath been proved in 48°, that } $FH = C.$
52. Therefore by subtracting C or FH } $EI - FH = B.$
from each part in 50°, }
53. But 'tis evident by the Diagram, that } $EI - FH = EF + HI.$

45. Thge.

54. Therefore from 52° and 53°, (per } $EF + HI = B.$ Which was to be Dem.
Ax. 1. Chap. 2.) }

Thirdly and lastly, that EF, FG, GH, HI are in continual proportion, I demonstrate thus ;

55. Req. demonstr. EF : FG :: FG : GH :: GH : HI.

Demonstration.

56. By Constr. in 20°, } $K : C :: EG : GF.$
57. And by Constr. in 19°, } $EG + C = K$
58. Therefore from 56° and 57°, } $EG + C : C :: EG : GF.$
59. And because it hath been proved in 48°, that } $FH = C.$
60. Therefore from 58° and 59°, } $EG + FH : FH :: EG : GF.$
61. It is manifest by the Diagram, that of the three right lines EF, FG, GH the sum of the first and second is EG, and the sum of the second and third is FH; therefore the last preceding Analogy is qualified in every respect according to the Theorem in 45° of Probl. 5. Chap. 7. whence EF, FG, GH shall be Proportionals, viz.
 $EF : FG :: FG : GH.$

62. Again, by Constr. in 21°, } $L : C :: GI : GH.$
63. And by Constr. in 19°, } $L = GI + C.$
64. Therefore from 62° and 63°, } $GI + C : C :: GI : GH.$
65. But it hath been proved in 48°, that } $FH = C.$
66. Therefore from 64° and 65°, } $GI + FH : FH :: GI : GH.$
67. Therefore from 66°, in like manner as before } $FG : GH :: GH : HI.$
in 61°, (per Theor. in 45° of Probl. 5. Chap. 7.) }
68. And from 61°, } $EF : FG :: FG : GH.$
69. Wherefore from 67° and 68°, (per prop. 11. } $EF : FG :: FG : GH :: GH : HI.$
Elem. 5.) }
Which was to be demonstrated in the last place. Therefore the Problem is satisfied.

LEMMA A, leading to the following Probl. 9.

If four right lines be in continual proportion, the difference of the means is a mean Proportional between the difference of the first and second, and difference of the third and fourth Proportionals.

Suppos.

1. Q, R, S, T \div ; viz. Q . R :: R . S :: S . T.
2. Q \sqsubset R. Whence R \sqsubset S, and S \sqsubset T.

$$\begin{array}{r} Q \\ R \\ S \\ T \end{array} \begin{array}{r} \text{-----} 125 \\ \text{-----} 100 \\ \text{-----} 80 \\ \text{-----} 64 \end{array}$$

3. Req. demonstr. Q - R . R - S :: R - S . S - T.

Demonstration.

4. By Suppos. in 1°, } $Q : R :: R : S.$
5. Also by Suppos. in 2°, } $Q - R : R :: R - S : S.$
6. Therefore by Division of Reason, } $Q - R : R :: R - S : S.$
7. And alternately, } $Q - R : R - S :: R : S.$
8. Again, by Suppos. } $R : S :: S : T.$
9. Therefore by Division of Reason, } $R - S : S :: S - T : T.$
10. And alternately, } $R - S : S - T :: S : T.$
11. Therefore from 8° and 10°, (per prop. 11. } $R - S : S - T :: R : S.$
Elem. 5.) }
12. Likewise from 7° and 11°, per prop. 11. El. 5. } $Q - R : R - S :: R - S : S - T.$
Which was to be Demonstr.

Probl. IX.

The difference of the extremes, and difference of the means of four right lines in Continual proportion being given severally, to find the

V u

Proportio

Proportionals. But the given difference of the extremes must be greater than the triple difference of the means.

Q	_____	125
R	_____	100
S	_____	80
T	_____	64

Suppos.

1. $Q, R, S, T \div \div$; viz. $Q : R :: R : S :: S : T$.
2. $Q \sqsubset R$, whence $R \sqsubset S$; and $S \sqsubset T$.
3. $b = Q - T$ is given.
4. $c = R - S$ is given.
5. $d = b - c$ is given.
6. $b \sqsubset 3c$. (*Determination.*)

Req. to find Q, R, S, T .

Resolution.

7. By *Suppos.* in 1° and 2° , . . . $Q, R, S, T \div \div$; also $Q \sqsubset R$.
8. Therefore by the *Lemma* prefix before this } $Q - R : R - S :: R - S : S - T$.
9. Of which three continual Proportionals in 8° the mean $R - S$, that is, c , is given; as also $d (= Q + S - T - R)$, the sum of the extremes $Q - R$ and $S - T$; therefore by the Theorem in 1° of *Probl. 13. Chap. 5.* the extremes shall be given severally, viz.

$$\begin{cases} \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} = Q - R. \\ \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} = S - T. \end{cases}$$
10.
11. Therefore from $4^\circ, 5^\circ$ and 10° 'tis manifest, that of these three Proportionals Q, R, S , the difference of the first and second, to wit, $Q - R$ is given, also $R - S$ the difference of the second and third is given. Likewise of these three Proportionals R, S, T , the difference of the first and second, to wit, $R - S$ is given, also $S - T$ the difference of the second and third is given; therefore, (according to the Canon in $13.$ of *Probl. 6. Chap. 7.*) R and S shall be given severally by these following Analogies, viz.

$$\begin{cases} \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} = c : c :: \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} : R. \\ c - \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} : c :: \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} : S. \end{cases}$$
12.

Which Analogies, respect being had to the Equations in 10° , do afford this

C A N O N.

13. From the Square of half the excess by which the given difference of the extremes exceeds the given difference of the means, subtract the Square of the given difference of the means, and extract the Square Root of the remainder; then add and subtract that Square Root to and from the said half-excess, and reserve the Summ and Remainder. Then it shall be, as the excess of the Summ reserved above the given difference of the means, is to the difference of the means; so is the Summ reserved to the greater mean sought. Or, as the excess of the difference of the means above the Remainder reserved, is to the difference of the means; so is the Remainder reserved to the lesser mean sought. Lastly, if the Summ reserved be added to the greater mean it gives the greater extreme, and if the Remainder reserved be subtracted from the lesser mean it gives the lesser extreme.

But to the end there may be a possibility of effecting the Problem propounded, the given lines must be liable to this

Determination.

14. The given difference of the extremes must be greater than the triple of the given difference of the means.
- The truth of this Determination will be made manifest by the following Theorem, and that 'tis necessary, will be evident in 28° of the following Construction of the Problem.

T H E O R E M.

15. If four right lines be in continual proportion, the difference of the extremes is greater than the triple difference of the means.

Suppos.

Suppos.

16. $Q, R, S, T \div \div$; viz. $Q : R :: R : S :: S : T$.
17. $Q \sqsubset R$. Whence $R \sqsubset S$; and $S \sqsubset T$.
18. . . . *Req. demonstr.* . . . $Q - T \sqsubset 3R - 3S$.

Demonstration.

19. By *Suppos.* in 16° and 17° , . . . $Q, R, S, T \div \div$; also, $Q \sqsubset R$.
20. Therefore by the *Lemma* prefix before this } $Q - R : R - S :: R - S : S - T$.
21. But if four quantities be Proportionals, the sum of the extremes is greater than the sum of the means, (*per prop. 25. Elem. 5.*) therefore from 20° , . . . $Q + S - R - T \sqsubset 2R - 2S$.
22. And by adding R to each part in 21° , . . . $Q + S - T \sqsubset 3R - 2S$.
23. Wherefore by subtracting S from each part in 22° , . . . $Q - T \sqsubset 3R - 3S$.

Which was to be Demonstr.

The Composition of the foregoing Probl. 9.

B	_____	B = 61	L = 4
C	_____ G	C = 20	R = 100
E	_____ I	E I = 41	S = 80
K	_____	E G = 25	Q = 125
L	_____	G I = 16	T = 64
R	_____	K = 5	
S	_____		
Q	_____		
T	_____		

Suppos.

24. B = the difference of the extremes of four right lines in continual proportion is given.
25. C = the difference of the means is given.
26. $B \sqsubset 3C$. (*Determination*)

Req. to find the Proportionals severally.

Construction.

27. Make $E I = B - C$.
28. Divide $E I$ into two such parts in G that C may be a mean Proportional between the parts, which may be done, (*per Probl. 14. Chap. 5.*) if C be not greater than $\frac{1}{2} E I$. But that C is less than $\frac{1}{2} E I$, I prove thus;

By the Determination in 26° , . . . $3C \sqsubset B$.

Therefore by subtracting C from each part, . . . $2C \sqsubset B - C$.

But by *Constr.* in 27° , . . . $E I = B - C$.

Therefore, (*per Ax. 3. Chap. 2.*) . . . $2C \sqsubset E I$.

And consequently, . . . $C \sqsubset \frac{1}{2} E I$.

Which was to be shewn. Therefore 'tis possible (*per Probl. 14. Chap. 5.*) to cut $E I$ into two such parts that C may be a mean Proportional between them. Suppose then that $E I$ is so cut in G , and that $E G$ is the greater part, and $G I$ the lesser; therefore,

$$E G : C :: C : G I.$$

29. Make $K = E G - C$, which is possible to be done, for by *Constr.* in 28° , $E G \sqsubset C$.
30. Make $L = C - G I$, which is possible to be done, for by *Constr.* in 28° , $C \sqsubset G I$.
31. By *Probl. 8. Chap. 5.* let it be made as K to C , so $E G$ to a fourth Proportional, suppose it be found R , therefore

$$K : C :: E G : R.$$

32. Again, let it be made, as L to C , so $G I$ to a fourth Proportional S , therefore

$$L : C :: G I : S.$$

33. Make $Q = E G + R$, whence $Q - R = E G$.

34. Make $T = S - G I$, whence $S - T = G I$; but that $G I$ is less than S , as is implied by this Effecton, I prove thus;

$V u 2$

By

By *Constr.* in 30° , $\dots \dots \dots L = C - GI$.
 Therefore by adding GI to each part, $\dots \dots \dots L + GI = C$.
 Therefore, $\dots \dots \dots L \supset C$.
 Therefore from the Analogy in 32° , (*per Schol. prop. 14. Elem. 5.*) $GI \supset C$.

Which was to be proved.
 35. I say Q, R, S, T (found out by *Constr.* in $33^\circ, 31^\circ, 32^\circ$ and 34°) are the four continual Proportionals sought. But to make it manifest that they will satisfy the Problem, I shall prove three things, *viz.* First, that the difference of the means R and S is equal to the given difference C : Secondly, that the difference of the extremes Q and T is equal to the given difference B : Thirdly and lastly, that the said Q, R, S, T are in continual proportion in this order, *viz.* $Q : R :: R : S :: S : T$.

First then, that the difference of the means R and S is equal to the given difference C , I demonstrate thus:

Prepar.

36. By *Probl. 7. Chap. 5.* let it be made as K to C , so C to a third proportional line M , } $K \cdot C :: C \cdot M$.
 therefore $\dots \dots \dots R - S = C$.
Reg. demonstr.

Demonstration.

37. By *Constr.* in 28° , $\dots \dots \dots EG \cdot C :: C \cdot GI$.
 38. And by *Constr.* in 28° , $\dots \dots \dots EG \supset C$.
 39. Therefore from 37 , by Division of Reason, $\dots \dots \dots EG - C \cdot C :: C - GI \cdot GI$.
 40. That is, by exchanging equal right lines, according to the *Constr.* in 29° and 30° , } $K \cdot C :: L \cdot GI$.
 41. Therefore alternly, $\dots \dots \dots K \cdot L :: C \cdot GI$.
 42. And by drawing C as a common altitude into each of the two latter Terms in 41 , } $K : L :: \square C : \square C, GI$.
 43. But from the *Constr.* in 36° and 32° , } $\square K, M = \square C$; and $\square L, S = \square C, GI$.
 44. Therefore from 42 and 43 , by exchange of equal Rectangles, $\dots \dots \dots K \cdot L :: \square K, M \cdot \square L, S$.
 45. But by reason of the common altitude M , this Analogy is manifest, $\dots \dots \dots K \cdot L :: \square K, M \cdot \square L, M$.
 46. Therefore from 44 and 45 , (*per prop. 11. Elem. 5.*) $\dots \dots \dots \square K, M \cdot \square L, S :: \square K, M \cdot \square L, M$.
 47. Therefore from 46 , (*per prop. 14. Elem. 5.*) $\dots \dots \dots \square L, S = \square L, M$.
 48. And from 47 , (*per prop. 14. Elem. 6.*) $\dots \dots \dots L \cdot M :: L \cdot S$.
 49. Therefore from 48 , (*per prop. 14. Elem. 5.*) $\dots \dots \dots M = S$.
 50. Again, by *Constr.* in 36° , $\dots \dots \dots K \cdot C :: EG \cdot R$.
 51. And by *Constr.* in 36° , $\dots \dots \dots K \cdot C :: C \cdot M$.
 52. Therefore from 50 and 51 , (*per prop. 11. Elem. 5.*) $\dots \dots \dots EG \cdot R :: C \cdot M$.
 53. Therefore alternly, $\dots \dots \dots EG \cdot C :: R \cdot M$.
 54. Therefore from 38 and 53 , by Division of Reason, $\dots \dots \dots EG - C \cdot C :: R - M \cdot M$.
 55. And because by *Constr.* in 29° , $\dots \dots \dots K = EG - C$.
 56. Therefore from 54 and 55 , $\dots \dots \dots K \cdot C :: R - M \cdot M$.
 57. But by *Constr.* in 36° , $\dots \dots \dots K \cdot C :: C \cdot M$.
 58. Therefore from 56 and 57 , (*per prop. 11. Elem. 5.*) $\dots \dots \dots R - M \cdot M :: C \cdot M$.
 59. Therefore from 58 , (*per prop. 14. Elem. 5.*) $\dots \dots \dots R - M = C$.
 60. But it hath been proved in 49 , that $\dots \dots \dots M = S$.
 61. Therefore from 59 and 60 , (*per Ax. 6. Chap. 2.*) $\dots \dots \dots R - S = C$. Which was to be Dem.

Secondly, that the difference of the extremes Q and T is equal to the given difference B , I demonstrate thus:

62. . . . *Reg. demonstr.* $Q - T = B$. *Demonstr.*

Demonstration.

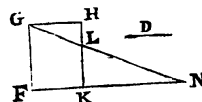
63. By *Constr.* in 33° , $\dots \dots \dots Q = EG + R$.
 64. And by *Constr.* in 34° , $\dots \dots \dots T = S - GI$.
 65. And from 61 , $\dots \dots \dots R \supset S$.
 66. Therefore from $63, 64$ and 65 , $\dots \dots \dots Q \supset T$; and $EG + R \supset S - GI$.
 67. And by subtracting the Equation in 64 from the Equation in 63 , $\dots \dots \dots Q - T = EG + R + GI - S$.
 68. But by *Constr.* in 28° , $\dots \dots \dots EI = EG + GI$.
 69. Therefore from 67 and 68 , (*per Ax. 6. Chap. 2.*) $\dots \dots \dots Q - T = EI + R - S$.
 70. Again, by *Constr.* in 27° , $\dots \dots \dots B - C = EI$.
 71. And it hath been proved in 61 , that $\dots \dots \dots C = R - S$.
 72. Therefore the sum of the Equations in 70 and 71 gives $\dots \dots \dots B = EI + R - S$.
 73. Therefore from 69 and 72 , (*per Ax. 1. Chap. 2.*) $\dots \dots \dots Q - T = B$. Which was to be Dem.
 Thirdly and lastly, that Q, R, S, T are in continual proportion I demonstrate thus:
 74. . . . *Reg. demonstr.* $Q : R :: R : S :: S : T$.

Demonstration.

75. By *Constr.* in 31° , $\dots \dots \dots K \cdot C :: EG \cdot R$.
 76. And by *Constr.* in 29° and 33° , $\dots \dots \dots EG - C = K$; and $Q - R = EG$.
 77. And it hath been proved in 61 , that $\dots \dots \dots C = R - S$.
 78. Therefore from $75, 76$ and 77 , by exchanging equal right lines, $\dots \dots \dots Q - R - R - S \cdot R - S :: Q - R \cdot R$.
 79. Therefore from the last preceding Analogy, by the *Theor.* in 14° of *Probl. 6. Chap. 7.* $\dots \dots \dots Q \cdot R :: R \cdot S$.
 80. Again, by *Constr.* in 32° , $\dots \dots \dots L \cdot GI :: C \cdot S$.
 81. And by *Constr.* in 30° and 34° , $\dots \dots \dots C - GI = L$; and $S - T = GI$.
 82. And it hath been proved in 61 , that $\dots \dots \dots R - S = C$.
 83. Therefore from $80, 81$ and 82 , by exchanging equal right lines, $\dots \dots \dots R - S - S - T \cdot S - T :: R - S \cdot S$.
 84. Therefore from the last preceding Analogy by the *Theor.* in 14° of *Probl. 6. Chap. 7.* $\dots \dots \dots R \cdot S :: S \cdot T$.
 85. Wherefore from 79 and 84 , (*per prop. 11. Elem. 5.*) $\dots \dots \dots Q : R :: R : S :: S : T$.
 Which was to be demonstrated in the last place. Therefore the Problem is satisfied.

Probl. X.

A Rectangle $FGHK$ being given by Position, to draw a right line GN from G one of the angles opposite to the Base FK , to cut the Base produced, suppose in N , so as to make the Triangle KLN (lying without the Rectangle) equal to a given Space; suppose the Square of the right line D .



$FK = 12$	$KL = 10$
$FG = 15$	$GN = 39$
$\square D = 120$	$LN = 26$
$KN = 24$	$GL = 13$

Suppos.

1. $FGHK$ is a \square given.
2. $b = FK$ or GH is given.
3. $c = EG$ or KH is given.
4. $d = D$ the side of a Square given.

Req.

Req. to find

5. KN a right line to be added to FK, so, as FKN may be a strait line, and that GN being drawn it may make $\triangle KLN = \square D$.

Resolution.

6. Put a for the desired increase of the Base FK, viz. $a = KN$.
 7. Then because $\triangle NKL$ and $\triangle NFG$ are equiangular, these sides are Proportionals, (per prop. 4. Elem. 6.) viz.
 $FN : FG :: KN : KL$
 8. That is, in the letters belonging to the Resolution, $a + b : c :: a : \frac{ca}{a+b}$
 9. And because (per prop. 41. Elem. 1.) $\square KN, KL = 2 \triangle KLN$.
 $a \times \frac{ca}{a+b} = 2dd = 2 \triangle KLN$.
 10. Therefore from $8^\circ, 9^\circ, 4^\circ, 5^\circ$ and 6° ,
 11. Now to avoid an Equation between Solids, let it be made as c to d , so $2d$ to a fourth Proportional, call it t , therefore
 $c : d :: 2d : t$.
 12. Whence, (per prop. 16. Elem. 6.) $ct = 2dd$.
 13. Therefore from 10° and 12° , (per Ax. 1. Ch. 2) $a \times \frac{ca}{a+b} = ct$.
 14. Which Equation may be resolved into these Proportionals, viz.
 $a : t :: c : \frac{ca}{a+b}$
 15. And by drawing $a + b$ as a common Factor into each of the two latter Terms of the last Analogy, this ariseth,
 $a : t :: ca + cb : ca$.
 16. And by casting away the common Factor c , this Analogy ariseth,
 $a : t :: a + b : a$.
 17. And from the last Analogy, by Division of Reason,
 $a - t : t :: b : a$.
 18. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means,
 $aa - ta = tb$.
 19. Which Equation may be resolved into this Analogy,
 $a - t : \sqrt{tb} :: \sqrt{tb} : a$.

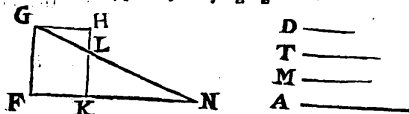
But the last Analogy doth manifestly consist of three Proportionals, whereof the mean, to wit, \sqrt{tb} is given, as also t the difference of the extremes a and $a - t$; therefore the extremes severally, the greater whereof is the desired increase KN, shall be given also, per Probl. 12. Chap. 5, and the Theorem in 24° of the same Probl. gives this following

C A N O N.

20. $\frac{1}{2}t + \sqrt{\frac{1}{2}tt + tb} = KN$. That is, in words,

Let it be made as FG the altitude of the given Rectangle, to D the side of the given Square; so $2D$ the double of the same side, to a fourth Proportional, which may be called T. Then to the Square of half that fourth Proportional T, add the Rectangle of T into FK the Base of the given Rectangle, and extract the square Root of the sum. Lastly, that square Root added to half T, gives KN the desired increase of the Base.

The Composition of the foregoing Probl. 10.



Suppos.

21. FGKH is a \square given.
 22. D is the side of a Square given.

Req. to find

23. KN a right line to be added to FK, so, as FKN may be a strait line, and that GN being drawn, it may make $\triangle KLN = \square D$.

Constr.

Construction.

24. By Probl. 8. Chap. 5. let it be made as FG to D, so $2D$ to a fourth proportional line, suppose it be found T; therefore, $FG : D :: 2D : T$.
 25. By Probl. 9. Chap. 5. find a mean Proportional, as M, between T and FK, therefore $T : M :: M : FK$.
 26. Making M to be the mean of three Proportionals, and T the difference of the extremes, find the extremes severally, (per Probl. 12. Chap. 5.) the greater whereof suppose to be A, then the lesser shall be A - T, therefore $A - T : M :: M : A$.
 That is, in 19° the last step of the Resolution, $a - t : \sqrt{tb} :: \sqrt{tb} : a$.
 27. Produce FK to such a point N, that KN may be equal to the right line A found out in 26° , and draw GN; so shall $\triangle KLN$ be equal to the Square of the given line D, as was required. But that $\triangle KLN = \square D$, the following Demonstration, formed out of the preceding Resolution by a repetition of the steps thereof in a backward (not direct) order will make manifest.
 28. Req. demonstr. $\triangle KLN = \square D$.

Demonstration.

29. By Constr. in 26° and 27° , $KN - T : M :: M : KN$.
 That is, in 19° the last step of the Resolution, $a - t : \sqrt{tb} :: \sqrt{tb} : a$.
 30. Therefore from 29° , (per prop. 17. Elem. 6.) $\square KN - \square T, KN = \square M$.
 31. Likewise from the Constr. in 25° , $\square T, FK = \square M$.
 32. Therefore from 30° and 31° , (per Ax. 1. Chap. 2.) $\square KN - \square T, KN = \square T, FK$.
 That is, in 18° , $aa - ta = tb$.
 33. Therefore from 32° , (per prop. 14. Elem. 6.) $KN - T : T :: FK : KN$.
 That is, in 17° , $a - t : t :: b : a$.
 34. And from 33° , by Composition of Reason, $KN : T :: KN + FK : KN$.
 That is, in 16° , $a : t :: a + b : a$.
 35. And from 34° , by taking in the common altitude FG, $KN : T :: \square FG, KN + \square FG, FK : \square FG, KN$.
 That is, in 15° , $a : t :: ca + cb : ca$.
 36. And because $\triangle NKL$ and $\triangle NFG$ are equiangular, therefore (per prop. 4. Elem. 6.) $FN : FG :: KN : KL$.
 37. And consequently, (per prop. 16. Elem. 6.) $\square FN, KL = \square FG, KN$.
 38. Therefore from 35° and 37° , by exchanging equal Rectangles, $KN : T :: \square FG, KN + \square FG, FK : \square FN, KL$.
 39. And from 38° , by rejecting the common altitude KN + FK, that is, FN, $KN : T :: FG : KL$.
 That is, in 14° , $a : t :: c : \frac{ca}{a+b}$.
 40. Therefore from 39° , (per prop. 16. Elem. 6.) $\square KN, KL = \square FG, T$.
 That is, in 13° , $a \times \frac{ca}{a+b} = ct$.
 41. And because from the Constr. in 24° , (per prop. 16. Elem. 6.) $2 \square D = \square FG, T$.
 42. Therefore from 40° and 41° , (per Ax. 1. Chap. 2.) $\square KN, KL = 2 \square D$.
 That is, in 10° , $a \times \frac{ca}{a+b} = 2dd$.
 43. But (per prop. 41. Elem. 1.) $\square KN, KL = 2 \triangle KLN$.

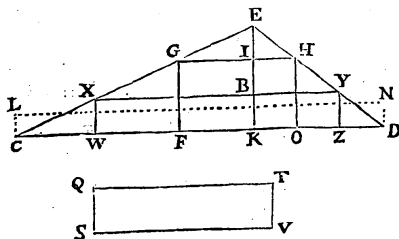
44. There.

44. Therefore from 42° and 43° , (per Ax. 1. Chap. 2.) $\therefore 2 \triangle KLN = 2 \square D$.
 45. Therefore from 44° , (per Ax. 2.1. Chap. 2.) $\therefore \triangle KLN = \square D$.
 Which was to be demonst. Therefore that is done which the Problem required.

Probl. XI.

In a given Triangle to inscribe a Rectangle equal to a given Rectangle. But the right line arising by the Application of the given Rectangle to the Base of the given Triangle, must not exceed a quarter of the Perpendicular falling upon that Base; and consequently the double of the Rectangle must not be greater than the Triangle.

Note. By the Base of the Triangle given in this Problem, is meant such a side as hath not an obtuse angle at either of its ends within the Triangle; for 'tis easie to conceive, that if the Triangle be obtusangled at the Base, a Rectangle cannot be inscribed within the Triangle, so, as that the Base of a Rectangle may be a segment of the Base of the Triangle, and all the angular points of the Rectangle lye in the sides of the Triangle.



Suppos.

1. CDE is a \triangle given.
2. $b = CD$ the Base is given, and neither $\angle C$ nor $\angle D$ is obtuse.
3. $p = EK$ the Perpendicular is given.
4. $\square ST$, and the sides thereof, to wit, SV and SQ are given.
5. $r = \frac{\square ST}{CD} = CL$ or DN is given; whence, $\square CN = \square ST$.

Req. to inscribe

6. $FGHO$ a \square within the $\triangle CDE$, with condition that
7. $\square FGHO$ may be equal to $\square ST$.

Resolution.

8. Put a for the altitude of the Rectangle required, viz. $a = FG = KI = OH$.
9. Which altitude subtracted from the Perpendicular EK , leaves EI , therefore from 3° and 8° , $p - a = EI$.
10. It is manifest by the Lemma prefixt before Probl. 11. Chap. 7. that $EK \cdot EI :: CD \cdot GH$.
11. Therefore (from 3° , 9° and 2° ;) in the letters belonging to the Resolution, $p \cdot p - a :: b \cdot \frac{bp - ba}{p}$.
12. And because according to the import of the Problem, $\square FG, GH = \square ST$, or $\square CD, CL$.
13. Therefore (from 8° , 11° , 2° and 5° ;) in the letters of the Resolution, $a \times \frac{bp - ba}{p} = br$.
14. Which last Equation is reducible to this Analogy, (per prop. 14. Elem. 6.) $a \cdot r :: b \cdot \frac{bp - ba}{p}$.
15. Therefore from 11° and 14° , (per prop. 11. Elem. 5.) $p \cdot p - a :: a \cdot r$.

16. Which

16. Which last Analogy gives this Equation, (per prop. 16. Elem. 6.) $pa - aa = pr$.
 17. And that Equation may be resolved into these Proportions, (per prop. 14. Elem. 6.) $p - a :: \sqrt{pr} :: \sqrt{pr} : a$.
 Of which three Proportionals the mean, to wit, \sqrt{pr} is given, as also p the sum of the extremes, therefore the extremes severally, (to wit, KI and EI , or EB and KB , the segments of the Perpendicular EK ;) shall be given also, (by Probl. 13. Chap. 5.) and the Theorem in 21° of the same Probl. 13. gives this following

C A N O N.

18. $\therefore \frac{1}{2}p + \sqrt{\frac{1}{2}pp - pr} = KI (= EB)$ } viz. in words,
 $\frac{1}{2}p - \sqrt{\frac{1}{2}pp - pr} = KB (= EI)$ }

First, let it be made as the Base of the given Triangle, to one of the sides of the given Rectangle; so the other side of the same Rectangle, to a fourth Proportional, which may be called r . Secondly, from the Square of half the Perpendicular falling upon the said Base, subtract the Rectangle made of the said Perpendicular and fourth Proportional r . Thirdly, extract the square Root of the remainder. Fourthly, add and subtract the said square Root to and from the said half Perpendicular; the sum and remainder shall be segments of the Perpendicular, either of which may be taken for the altitude of the Rectangle required to be inscribed. Lastly, the Base of the Rectangle sought is equal to the right line arising by the Application of the given Rectangle to the altitude before found.

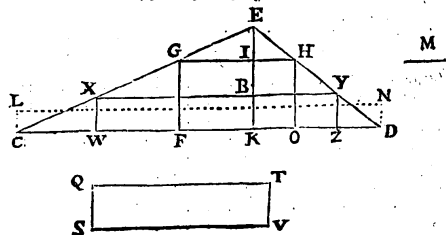
This Canon may be propounded in the form of a Theorem, which may easily be demonstrated by a repetition of the steps of the preceding Resolution in a direct (not retrograde) order, but taking the truth of the Canon for granted, I shall proceed to the Composition of the Problem, for the effecting whereof, 'tis necessary that the given quantities be qualified according to the tenour of this

Determination.

19. The right line arising by the Application of the given Rectangle, to the Base of the given Triangle, must not exceed a quarter of the Perpendicular falling upon that Base; and consequently, the double of the Rectangle must not be greater than the Triangle.

Which Determination shews it self openly in the Canon, where it appears, that to the end there may be a possibility of subtracting pr from $\frac{1}{2}pp$, 'tis necessary that pr not $< \frac{1}{2}pp$. And consequently, by dividing each part by p , r not $< \frac{1}{2}p$. And by doubling each part, $2r$ not $< p$. And by drawing b into each part, $2rb$ not $< \frac{1}{2}pb$. But from 2° and 5° , $2rb = 2\square ST$. And by prop. 41. Elem. 1. $\frac{1}{2}pb = \triangle CDE$. Therefore from the three last preceding steps, by exchanging equal quantities, $2\square ST$ not $< \triangle CDE$. Therefore from the premises, both the rite and truth of the Determination are manifest.

The Composition of the foregoing Probl. 11.



Suppos.

20. CDE is a \triangle given.
21. CD the Base is given, and neither $\angle C$ nor $\angle D$ is obtuse.

22. EK

22. EK the Perpendicular is given.
 23. $\square ST$, and the sides thereof, to wit, SV and SQ are given.
 24. $CL = DN = \frac{\square ST}{CD}$ is given, (per Probl. 8. Chap. 5.)
 25. CL not $\perp \frac{1}{2} EK$. (Determination.)
 Req. to inscribe
 26. $\square FGH O$ in $\triangle CDE$, so, that $\square FGH O = \square ST$.

Construction.

27. By Probl. 9. Chap. 5. find a mean Proportional, as M, between EK and CL, therefore,
 $EK \cdot M :: M \cdot CL$.
 28. By Probl. 14. Chap. 5. cut EK into two such parts in I, that M (before found) may be a mean Proportional between the parts, which effection is possible if M be not greater than $\frac{1}{2} EK$; but that M is greater than $\frac{1}{2} EK$, I prove thus;
 By the Determination in 25°, CL not $\perp \frac{1}{2} EK$.
 Therefore by drawing EK into each part, $\square EK, CL$ not $\perp \frac{1}{2} \square EK$.
 But from the Constr. in 27°, (per prop. 17. Elem. 6.) $\square EK, CL = \square M$.
 Therefore from the two last preceding steps, (per Ax. 4.) $\square M$ not $\perp \frac{1}{2} \square EK$.
 Chap. 2.) M not $\perp \frac{1}{2} EK$.
 Therefore by extracting the square Root out of each part, M not $\perp \frac{1}{2} EK$.
 Which was to be proved. Therefore 'tis possible to cut EK into two such parts, that M shall be a mean Proportional between them; suppose then it be done, (per Probl. 14. Chap. 5.) and that the parts are EI and KI, therefore $EI \cdot KI = EK \cdot M$. Also,
 EI (or $EK - KI$). $M :: M \cdot KI$.

That is, in 17°, $p - a :: \sqrt{pr} :: \sqrt{pr} \cdot a$.

29. Then set either of the said parts of EK, suppose the greater part, from K to I; and by the point I, draw GIH parallel to CD. Lastly, from the points G and H let fall GF and HO Perpendiculars to the Base CD, so shall FGH O be the inscribed Rectangle required. But to make it manifest that the said Rectangle will satisfy the Problem, two things are to be proved, viz. First, that all the angles of the quadrilateral Figure FGH O are right angles; and then that the said Rectangle is equal to the given Rectangle SQIV.
 30. . . . Req. demonstr. . . . FGH O is a \square .

Demonstration.

31. By Constr. in 29°, $GH \parallel FO$.
 32. Also by Constr. in 29°, GF and HO are $\perp FO$.
 33. Therefore, (per defin. 10. Elem. 1.) $\angle GFO = \angle HOH$.
 34. Therefore from 31°, 32°, 33° (per prop. 29. Elem. 1.) $FGHO$ is \square .

Which was to be Demonstr.

It remains to prove that $\square FG, GH$ (that is, $\square FH$) = $\square ST$; but that equality will be manifest by the following Demonstration, form'd out of the preceding Resolution by a retrograde repetition of the steps thereof.

35. . . . Req. demonstr. . . . $\square FG, GH$ (that is, $\square FH$) = $\square ST$.

Demonstration.

36. By Constr. in 28°, $EK - KI \cdot M :: M \cdot KI$.
 That is, in 17°, $p - a :: \sqrt{pr} :: \sqrt{pr} \cdot a$.
 37. And from 22°, 29° and 34°, $FG = KI$.
 38. Therefore from 36° and 37°, $EK - FG \cdot M :: M \cdot FG$.
 39. And from 38°, (per prop. 17. Elem. 6.) $\square EK, FG = \square FG = \square M$.
 40. Likewise from the Constr. in 27°, $\square EK, CL = \square M$.
 41. Therefore from 39° and 40°, (per Ax. 1.) $\square EK, FG = \square FG = \square EK, CL$.
 Chap. 2.) $p \cdot a - a \cdot a = pr$.
 That is, in 16°, $EK \cdot EK - FG :: FG \cdot CL$.
 42. Therefore from 41°, (per prop. 14. Elem. 6.) $EK \cdot EK - FG :: FG \cdot CL$.
 That is, in 15°, $p \cdot p - a :: a :: a \cdot a$.
 43. By Constr. in 29°, $GH \parallel CD$.

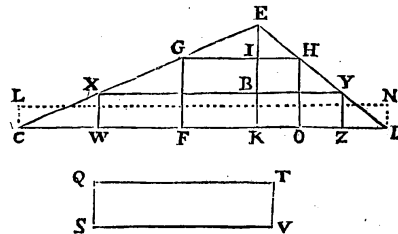
44. There-

44. Therefore by the Lemma prefix before $\square EK \cdot EK - FG :: CD \cdot GH$.
 Probl. 11. Chap. 7. EI
 That is, in 11°, $p \cdot p - a :: b :: \frac{bp - ba}{p}$.
 45. Therefore from 42° and 44°, (per prop. 11.) $FG \cdot CL :: CD \cdot GH$.
 Elem. 5.) $\square FG, GH = \square CD, CL$.
 46. And from 45°, (per prop. 16. Elem. 6.) $\square FG, GH = \square CD, CL$.
 47. But from the Constr. in 24°, $\square ST = \square CD, CL$.
 48. Therefore from 46° and 47°, (per Ax. 1.) $\square FG, GH$ (that is, $\square FH$) = $\square ST$.
 Chap. 2.)

Which was to be Demonstr. Therefore that is done which the Problem required.

49. Note. If KB be made equal to EI, then shall EB be equal to IK, (by reason of the common intersegment IB,) and consequently EK is cut in B as well as in I, according to the import of the preceding Construction in 28°. Therefore if by the point B a parallel be drawn to the Base CD, as XBY, and from the points X and Y, perpendiculars be let fall upon CD, as XW and YZ, the inscribed $\square WY$, that is, $WXYZ$ shall be also equal to the given Rectangle ST, that is, $SQIV$, and the Demonstration may be formed as before, by taking KB or WX instead of KI. So two Rectangles are inscribed in the given $\triangle CDE$, each of which is equal to the given Rectangle $SQIV$.

Examples in Numbers to illustrate the preceding Resolution of Probl. 11.



Suppos.

50. $CD = 168$ the Base }
 51. $CE = 117$ } the legs } of $\triangle CDE$ are given severally.
 52. $DE = 75$ }
 53. $SV = 84$ } the sides of $\square ST$, therefore $\square ST = 1680$.
 54. $SQ = 20$ }
 55. $\square FH = \square ST = 1680$.
 56. $EK \perp CD$.

Req. to find in Numbers,

57. FG or HO , }
 58. GH or FO , } the sides of $\square FH$.

Solution Arithmetical.

59. $EK = 45$, found out by the three sides of $\triangle CDE$ given in 50°, 51°, 52°, by the help of Theor. 4. in 68° of Probl. 8. Chap. 8.

60. $KI = 30$ } found out by the Canon in 18° of this Problem.
 61. $IE = 15$ }

62. $FG = 30 = KI$, found out in 60°.

63. $GH = 56 = FO$, given from 55° and 62°. For $\frac{1680}{30} = 56$.

The Proof.

64. $\square FG, GH = 1680 = \square SV, SQ$, (= $\square ST$). Also,

65. $\square EI$ (or $EK - KI$). $GH :: EK \cdot CD$.

66. Therefore by the converse of the Lemma prefix before Probl. 11. Chap. 7.
 $GH \parallel CD$. Also G and H are in CE and DE.

X 2 2

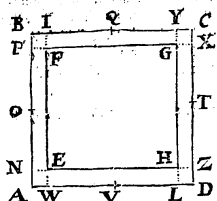
Another

Another Example.

67. Again, the same $\triangle CDE$ and its sides being given in numbers as before in 50° , 51° and 52° , you will find (by the like Operation as in Example 1.) $XW = 15 = YZ$, and $XY = 112 = WZ$, whence the Area of $\square XZ$ is 1680, which is the same with the Area of $\square ST$ prescribed in Example 1. And that the Rectangle XZ or $WXYZ$ is inscribed in $\triangle CDE$, may be proved in like manner as before in 65° and 66° .

Probl. XII.

Within a given Rectangle to make a Rectangle, with this condition, that there may be an equal parallel distance between their sides; and that the Space lying between the sides of both the Rectangles may be to the inscribed Rectangle in a given Reason.



$$\begin{aligned} BC &= 440 \\ BA &= 400 \\ R &= 2 \\ S &= 9 \\ FG &= 400 \\ FE &= 360 \\ FI &= 20 = GX \end{aligned}$$

Suppos.

1. $ABCD$ is a \square given.
2. $b = BC = AD$ is given.
3. $c = BA = CD$ is given.
4. $b < c$.
5. $d = b - c = BC - BA$ is given.
6. r and s the Terms of a given Reason.

Req. to make

7. $\square EFGH$ within the $\square ABCD$ in such manner, that
8. $FI = GX = HL = EN$. Also, that
9. $\square BC, BA = \square FG, FE$. $\square FG, FE :: r . s$.

Prepar.

10. By viewing the Diagram, and reflecting upon what is given and required, it will be evident that $BC = FG + 2GX$ (2 FI)
11. Likewise, $BA = FE + 2GX$ (2 FI)
12. And by subtracting the Equation in 11 from that in 10, this remains, viz. $BC - BA = FG - FE (= d)$
13. Whence 'tis manifest that the difference between the length and breadth of the Rectangle required to be inscribed is given; for 'tis equal to the difference between the length and breadth of the given $\square ABCD$.

Resolution.

14. Put a for the shorter side of the required Rectangle $EFGH$, viz.
15. Therefore from 5° , 13° and 14° , the longer side shall be $a + d (= FG)$.
16. Therefore from 14° and 15° , the Area of $\square EFGH$ is equal to $aa + da (= \square EFGH)$.
17. And from 2° and 3° the Area of the given Rectangle is $bc (= \square ABCD)$.
18. And by subtracting the Area in 16 from that in 17, there will remain $bc - aa - da (= BFGCHDEA)$.
19. Therefore from 9° , 18° and 16° , according to the tenour of the Problem, this Analogy arises, viz. $r . s :: bc - aa - da . aa + da$.

20. There-

20. Therefore from 19° , by Composition of Reason, $r + s . s :: bc . aa + da$.
21. Now to avoid an Equation between Solids, let it be made as $r + s$ to s , so b to a fourth Proportional, call it f , therefore $r + s . s :: b . f$.
22. Therefore from 20° and 21° (per prop. 11. El. 5.) $b . f :: bc . aa + da$.
23. And this Analogy, by reason of the common Factor c is evident, (per prop. 1. Elem. 6.) viz. $b . f :: bc . fc$.
24. Therefore from 22° and 23° , (per prop. 11. Elem. 5.) $bc . aa + da :: bc . fc$.
25. Therefore from 24° , (per prop. 14. Elem. 5.) $aa + da = fc$.
26. Which Equation may be resolved into these Proportionals, viz. $a . \sqrt{fc} :: \sqrt{fc} . a + d$.

Of which three Proportionals the mean, to wit, \sqrt{fc} is given, as also d the difference of the extremes $a + d$ and a , therefore per Probl. 12. Chap. 5. the extremes shall be given severally, (which are the sides of the Rectangle required to be inscribed,) and the Theorem in 24° of the said Probl. 12. gives this following

$$CANON.$$

$$27. \dots \left\{ \begin{array}{l} \sqrt{\frac{1}{2}dd + fc} : -\frac{1}{2}d = EF. \\ \sqrt{\frac{1}{2}dd + fc} : +\frac{1}{2}d = FG. \end{array} \right\} \text{viz. in words,}$$

Make $r + s$ the sum of the Terms of the given Reason the first of four Proportionals, the latter of those Terms the second Proportional, BC or AD the longer side of the given Rectangle the third Proportional, and to those three find a fourth, which may be called f . Then to the Square of half the difference between the length and breadth of the given Rectangle, add the Rectangle made of the said fourth Proportional and the said breadth. Then to and from the Square Root of that sum, add and subtract the said half difference, so shall the sum and remainder made by that addition and subtraction be the desired length and breadth of the Rectangle to be inscribed, which length or breadth subtracted from the length or breadth of the given Rectangle, the half of the remainder is the parallel distance between the sides of both the said Rectangles.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 12.

Suppos.

28. $BC = 440$ } the sides of the given Rectangle $ABCD$.
29. $BA = 400$ }
30. $R = 2$ } the Terms of the given Reason.
31. $S = 9$ }

Req. to make

32. $\square EFGH$ within the $\square ABCD$, in such manner, that
33. $FI = GX = HL = EN$. Also,
34. $\square BC, BA = \square FG, FE$. $\square FG, FE :: R . S :: 2 . 9$.

Solution Arithmetical.

35. $\square BC, BA = 176000$, from 28° and 29° .
36. $FG \dots \dots \dots 400$, } found out by the Canon in 27° .
37. $FE \dots \dots \dots 360$, }
38. $\square FG, FE = 144000$, from 36° and 37° .
39. $\square BC, BA = \square FG, FE = 32000$, from 35° and 38° .

The Proof.

40. $R . S :: \square BC, BA = \square FG, FE$. $\square FG, FE$.
41. $2 . 9 :: 32000$.
42. $FI = 20 = GX = HL = EN$ the parallel distance.

Another way of resolving the preceding Probl. 12.

43. The same things being given and required as before, let a be put for the side of a Square equal to the inscribed Rectangle, therefore $aa = \square EFGH$.
44. From 2° and 3° the Area of the given Rectangle is bc .
45. Therefore the difference of those Rectangles is $bc - aa$.

46. There-

46. Therefore according to the tenour of the Problem } $r \cdot s :: bc - aa \cdot aa$.
 this Analogy aritheth, viz.
 47. Whence, by Composition of Reason, this Analogy
 aritheth, which gives the Area of the Rectangle to be } $r \cdot s :: bc \cdot aa$.
 inscribed,
 From the last Analogy aritheth

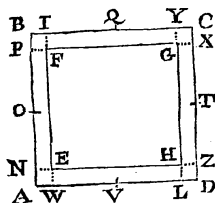
CANON 2.

As the sum of the Terms of the given Reason is to the latter Term, so is the Area of the given Rectangle to the Area of the inscribed Rectangle; therefore the Area of the inscribed Rectangle is given also. Then the Area of the inscribed Rectangle being given, as also the difference of the sides, (for this difference, as before hath been shewn in 13^o, is equal to the difference of the sides of the given Rectangle,) the sides shall be given severally by *Probl. 1.* of this Chapter. And lastly, the length of the inscribed Rectangle being subtracted from the length of the given Rectangle, or the breadth from the breadth, the half of the remainder is the parallel distance between the sides of both the Rectangles.

This Canon may be exemplified by the numbers given in the preceding *Examp. 1.* And in regard the Composition of this Problem according to either of the said ways of Resolution will not be difficult to him that understands the preceding Problems of this Chapter, I shall wave the Composition, and leave it as an exercise to the industrious Learner.

Probl. XIII.

A Nobleman having made choice of a plot of ground for the making of a Garden of pleasure, gives direction to a Surveyor to trace out a Rectangle, or long-Square, whose length and breadth shall be equal to two given right lines BC and BA. Also to make another long-Square within the former, in such manner, that there may be an equal parallel distance between the sides of both the said long-Squares. Moreover, the Nobleman's design is, that the space lying between the sides of both the long-Squares shall be sunk perpendicularly, to make a Mote or Ditch whose depth shall be equal to a given right line S, and the breadth thereof such, that the earth digged out of the intended Ditch being layd upon the said interieur long-Square as a Base, may be capable of raising a rectangular Mount whole altitude shall be equal to a given right line R. The Question is, to find out the length and breadth of the interieur long-Square, as also the breadth of the Ditch, that is, the parallel distance between the sides of both the long-Squares.



Suppos.

1. $BC = AD$, } the sides of $\square ABCD$ are severally given.
2. $BA = CD$, }
3. R = a right line given for the height of the desired Mount.
4. S = a right line given for the depth of the desired Mote or Ditch.

Req. to find

5. FG , or EH , the length of the interieur $\square EFGH$.
6. EF , or HG , the breadth of the said $\square EFGH$.

7. $FI =$

7. $FI = GX = HL = EN$ the parallel distance.
8. $R \times \square EFGH = S \times \text{Plane } BFGCDHEA$.

Construction.

9. By the preceding *Probl. 12.* let a Rectangle or long-Square be made within the given $\square ABCD$, in such manner, that there may be an equal parallel distance between their sides, and that the Space lying between the sides of both Rectangles may have such proportion to the inscribed Rectangle, as the given right line R , (prescribed for the height of the Mount,) hath to the given right line S , (prescribed for the depth of the Ditch.) Now suppose that by the said 12th Problem the $\square EFGH$ is so made within the $\square ABCD$, that the sides of the one keep an equal parallel distance to the sides of the other, viz. $FI = GX = HL = EN$; and that as R is to S , so the interval or Plane $BFGCDHEA$, to the $\square EFGH$. Then it will be manifest (*per prop. 34. Elem. 11.*) that $R \times \square EFGH$ (which is equal to the Solidity of the Mount,) is equal to $S \times \text{Plane } BFGCDHEA$, (which is equal to the Solidity of the Ditch;) as was required.

The quantities of the length and breadth of the inscribed Rectangle (or Base of the Mount,) as also of the parallel distance (or breadth of the Ditch) may be found out in numbers by either of the Canons of the preceding *Probl. 12.* and for the greater evincence, I shall here add

An Example in Numbers, to illustrate the preceding Construction of *Probl. 13.*

Suppos.

10. $BC = 440$ } the sides of the given $\square ABCD$.
11. $BA = 400$ }
12. $R = 2$ } the given altitude of the Mount to be raised perpendicularly upon $\square EFGH$.
13. $S = 9$ } the given depth of the Ditch $BFGCDHEA$.

Req. to find out in Numbers,

14. FG and FE the sides of $\square EFGH$. Also,
15. $FI = GX = HL = EN$ the parallel distance; with condition also, that
16. $R \times \square EFGH$ may be equal to $S \times \text{Plane } BFGCDHEA$.

Solution Arithmetical.

17. $FG = 400$ } found out by the quantities given in 10^o, 11^o, 12^o, 13^o, according
18. $FE = 360$ } to the preceding Construction in 9^o of this *Probl. 13.*
19. $FI = 20$ }

The Proof.

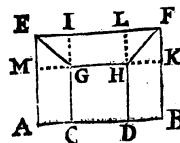
20. $\square EFGH$, $FE = 144000$, the Area of $\square EFGH$, viz. the Base of the Mount.
21. $R \times \square EFGH$, $FE = 288000$, the Solid content of the Mount.
22. $\square BCBA - \square EFGH$, $FE = 32000$, the Area of $BFGCDHEA$.
23. $S \times \text{Area of } BFGCDHEA = 288000$, the Solidity of the Ditch.
24. $R \times \square EFGH$, $FE = S \times \text{Plane } BFGCDHEA = 288000$, as was required.

Probl. XIV.

Within a given Rectangle $AEFB$ to make a Rectangle $CGHD$, with this condition, that after the right lines EG and HF are drawn, the Spaces $CGHD$, $EGHF$, $HDBF$ and $GCAE$ may be equal to one another, and consequently every one of them equal to a fourth part of the given Rectangle $AEFB$.

Suppos.

1. $AEFB$ is a \square given.
2. $IE = LF = AC = DB$.
3. $EM = FK = HL = IG$.
4. $IL = GH = CD$.
5. $g = AB = EF$ is given.
6. $f = AE = BF$ is given.



Req.

Req. to make

$$7. \square CGHD = EGHF = HDBF = GCAE = \frac{1}{4} \square AEFB.$$

Resolution.

8. Put

9. Then because $EF = IL = IE + LF$, 'tis manifest from 2° , 5° and 8° , that

10. And because by *Suppos.* in 1° $IE = LF$, the half of the Equation in 9° gives

11. And the summ of the Equations in 8° and 10° gives

12. Then supposing $\square CGHD$ to be equal to $\frac{1}{4} \square AEFB$, that is, $\frac{1}{4}fg$; let $\frac{1}{4}fg$ be divided by a , that is, GH , and the Quotient gives

13. And because $BF = BK$, by subtracting $\frac{1}{4}fg$ from f , that is, BK from BF ,

there will remain

14. Now the Problem requires that

15. That is, in the letters belonging to the Resolution, (as appears by the 11^{th} and 13^{th} steps,)

16. Which last Equation may be resolved into these Proportionals, viz.

17. And by doubling the two first Terms of that Analogy, this aritheth, viz.

18. Whence by Conversion of Reason,

19. And by drawing a into each of the two later Terms of the last preceding Analogy,

20. And by dividing each of the two later Terms of the Analogy in 19° by f , this aritheth, viz.

21. Whence, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation aritheth, viz.

22. And by subtracting $\frac{1}{4}ga$ from each part of the Equation in 21° , this aritheth, viz.

23. Which last Equation may be resolved into these Proportionals, viz.

24. But of those three continual Proportionals, the mean, to wit, $\frac{1}{4}g$ is given, as also $\frac{1}{4}g$ the difference of the extremes $a + \frac{1}{4}g$ and a , therefore the extremes shall be given severally, (per *Probl. 12. Chap. 5.*) the lesser of which shall be equal to the desired line CD , ($= GH = IL$), represented by a in the precedent Resolution of this Problem: And the Theorem in 24° of *Probl. 12. Chap. 5.* gives this following

C A N O N.

$$25. \dots a = \sqrt{\frac{1}{4}fg} - \frac{1}{4}g = CD = GH = IL$$

That is, in words,

To the Square of one eighth part of the Base (that is, either of the sides) of the given Rectangle, add a quarter of the Square of the same Base, and from the square Root of the sum subtract one eighth part of the said Base, the remainder shall be that side of the required Rectangle which is a segment of the Base of the given Rectangle. Whence the rest of the lines in the Diagram belonging to this *Probl. 14.* shall be given also.

26. It is evident, that no Term of any Analogy or Equation in the foregoing Resolution exceeds the dimensions of a Square, and therefore the forming of the Composition of this Problem by a retrograde repetition of the steps of the Resolution will not be difficult to him that understands the Resolutions and Compositions of the precedent Problems of this

this Chapter: waving therefore the Geometrical Effectation and Demonstration, I shall apply the Canon before exprest to the Arithmetical Solution of the Problem propounded.

An Example in Numbers, to illustrate the preceding Resolution of *Probl. 14.*

Suppos.

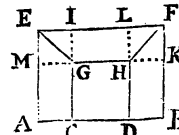
27. $AEFB$ is a \square , whose Base is AB , and altitude AE .

28. $AB = 10 = EF$ is given.

29. $AE = 6 = BF$ is given.

30. $EM = FK = HL = IG$.

31. $IE = LF = AC = DB$.



Req. to find in numbers,

32. The quantities of the lines CD ($= IL$), IE ($= LF$), HD ($= BK$), FK ($= HL$), with this condition, that the Area of every one of these four Spaces, to wit, $\square CGHD$, $EGHF$, $HDBF$ and $GCAE$ may be equal to a quarter of the Area of the given Rectangle $AEFB$, viz.

$$\square CGHD = EGHF = HDBF = GCAE = 15 = \frac{1}{4} \square AEFB.$$

Solution Arithmetical.

33. From 28° , by the Canon in 25° , you will find $\sqrt{\frac{1}{4} \cdot 10 \cdot 6} - \frac{1}{4} \cdot 10 = CD = GH = IL$.

34. And by subtracting $\sqrt{\frac{1}{4} \cdot 10 \cdot 6} - \frac{1}{4} \cdot 10$ from 10 , that is, IL from EF , there will remain

35. And because $IE = LF$, the half of $\frac{10}{2} - \frac{10}{4}$ gives $\frac{10}{4} - \frac{10}{8} = IE = LF = KH$.

36. And the summ of the numbers in 33° and 35° makes $\frac{10}{4} + \frac{10}{8} = IF = EL$.

37. Then by dividing 15 the Area of $\square CGHD$, that is, $\frac{1}{4}$ of 60 the Area of $\square AEFB$, by $\frac{10}{4} - \frac{10}{8}$, that is, CD , the Quotient gives

38. And by subtracting $\frac{10}{4} + \frac{10}{8}$ from 6 , that is, BK from BF , there will remain

So in the six last preceding steps the quantities of all the lines sought by *Probl. 14.* are found out in numbers; but that they will satisfy the condition prescribed in 32° , will be evident by

The Proof.

39. The Product of $\frac{10}{4} + \frac{10}{8}$ into $\frac{10}{4} - \frac{10}{8}$ is 15 . That is, CD into DH is $\square CGHD$.

40. The Product of $\frac{10}{4} + \frac{10}{8}$ into $\frac{10}{4} - \frac{10}{8}$ is 15 . That is, IF into FK is $\square EGHF$.

41. The Product of $\frac{10}{4} + \frac{10}{8}$ into $\frac{10}{4} - \frac{10}{8}$ is 15 . That is, BK into KH is $\square HDBF$.

42. The Product of $\frac{10}{4} + \frac{10}{8}$ into $\frac{10}{4} - \frac{10}{8}$ is 15 . That is, $\frac{10}{4}KF$ into KH is $\triangle HKF$.

43. The summ of the Products in 41° and 42° makes $15 = \square HDBK + \triangle HKF = HDBF$.

44. And from 27° , 30° , 31° and 43° , it is evident that $GCAE = HDBF = 15$.

45. Therefore from 39° , 40° , 41° and 42° , 'tis evident that $\square CGHD = EGHF = HDBF = GCAE = 15 = \frac{1}{4} \square AEFB$.

Which was to be done. All which Calculations will be evident to him that understands the Arithmetick of Surd-numbers, handled at large in *Chap. 9. Book II.* of this Treatise.

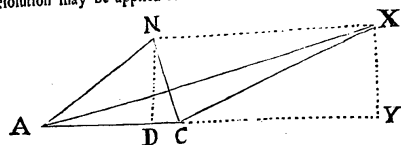
Y y

Probl. XV.

Probl. XV.

The Base, Perpendicular and Proportion of the legs of a plain Triangle being severally given, to find out the Triangle. But the given lines must be subject to the Determination hereafter exprest.

Note. There is more than enough given in this Problem, unless it requires a Triangle that hath either unequal acute angles, or else an obtuse angle at the Base; in the first of those Cases the Perpendicular falls within the Triangle, in the latter without; but the following Resolution may be applied to each Case.



Suppos.

1. $\triangle ACN$ is acute-angled at the ends of the Base AC .
2. $\triangle ACX$ is obtuse-angled at C , the end of the Base AC .
3. $b = AC$ the Base is given.
4. $p = ND = XY$ the Perpendicular is given.
5. r and s are the given Terms of the Proportion of the legs, viz.

$$r : s :: AN : NC :: AX : XC.$$

6. $r : s$. Req. to find the Triangle.

Resolution.

7. Put a for the distance from the foot of the Perpendicular to the remoter end of the Base, viz. suppose $a = DA$ or YA .
8. Therefore from 3^o and 7^o , the distance from the foot of the Perpendicular to the nearer end of the Base is $b - a$, or $a - b$; viz. DC or YC .
9. The Square of which distance is $aa - 2ba + bb (= \square DC$ or $\square YC)$.
10. The Square of the distance in 7^o is $aa (= \square DA$ or $\square YA)$.
11. The Square of the given Perpendicular in 4^o is $pp (= \square ND = \square XY)$.
12. By prop. 47. Elem. 1. $\square DC + \square ND = \square CN$.
13. Likewise, $\square YC + \square YX (= \square ND) = \square CX$.
14. Therefore from 9^o , 11^o , 12^o and 13^o , the Square of the lesser leg, in the letters of the Resolution, is $aa - 2ba + bb + pp (= \square CN$ or $\square CX)$.
15. Again, by prop. 47. Elem. 1. $\square DA + \square ND = \square AN$.
16. Likewise, $\square YA + \square YX (= \square ND) = \square AX$.
17. Therefore from 10^o , 11^o , 15^o and 16^o , the Square of the greater leg is $aa + pp (= \square AN$ or $\square AX)$.
18. And consequently the greater leg is $\sqrt{aa + pp} (= AN$ or $AX)$.
19. And from 14^o , the lesser leg is $\sqrt{aa - 2ba + bb + pp} (= CN$ or $CX)$.
20. Therefore from 5^o , 18^o and 19^o , according to the tenour of the Problem, $r : s :: \sqrt{aa + pp} : \sqrt{aa - 2ba + bb + pp}$.
21. Therefore from 20^o , (per prop. 22. Elem. 6.) $rr : ss :: aa + pp : aa - 2ba + bb + pp$.
22. Now in order to find out an Equation wherein the highest Power of the line a sought may not exceed a Square, to r and s find a third Proportional, which may be called t , therefore, $r : s :: s : t$.

23. There-

23. Therefore from 22^o , (per Coroll. prop. 20. Elem. 6.) $rr : ss :: r : t$.
24. Therefore from 21^o and 23^o , (per prop. 11. Elem. 5.) $r : t :: aa + pp : aa - 2ba + bb + pp$.
25. Therefore from 24^o , by Conversion of Reason, $r : r - t :: aa + pp : 2ba - bb$.
26. Therefore inversely, $r - t : r :: 2ba - bb : aa + pp$.
27. Let it be made as $r - t$ to r , so b to a fourth Proportional, which may be called f , therefore, $r - t : r :: b : f$.
28. Therefore from 26^o and 27^o , (per prop. 11. Elem. 5.) $b : f :: 2ba - bb : aa + pp$.
29. And by drawing $2a - b$ as a common Factor into b and f severally, this Analogy is manifest, (per prop. 1. Elem. 6.) $b : f :: 2ba - bb : 2fa - fb$.
30. Therefore from 28^o and 29^o , (per prop. 11. Elem. 5.) $2ba - bb : aa + pp :: 2ba - bb : 2fa - fb$.
31. Therefore from 30^o , (per prop. 14. Elem. 5.) this Equation aritheth, $2fa - fb = aa + pp$.
32. Whence, by adding fb to each part, $2fa = aa + pp + fb$.
33. And by subtracting aa from each part of the last Equation, this aritheth, $2fa - aa = pp + fb$.
34. Which last preceding Equation may be converted into this Analogy, viz. $2f - a : \sqrt{pp + fb} :: \sqrt{pp + fb} : a$.
35. But that Analogy doth manifestly consist of three continual Proportionals, whereof the mean, to wit, $\sqrt{pp + fb}$ is given, as also $2f$ the sum of the extremes $2f - a$ and a ; therefore the extremes shall be given severally, (by Probl. 13. Chap. 5.) either of which may be taken for the line a sought, viz.

$$a = f + \sqrt{ff - pp - fb} = YA;$$

$$\text{Or, } a = f - \sqrt{ff - pp - fb} = DA.$$

36. From 34^o and 35^o 'tis easie to perceive that $\sqrt{pp + fb}$ cannot be greater than f , for the mean of three Proportionals never exceeds half the sum of the extremes, (as hath been shewn in 20^o of Probl. 13. Chap. 5.) But the said $\sqrt{pp + fb}$ may sometimes be equal to, and sometimes less than f ; to the end therefore there may be a possibility of finding out a Triangle to satisfy the Problem propounded, the given lines must be subject to this following

Determination, . . . $\sqrt{pp + fb}$ not $< f$.

That is, in words,

First, if it be made as r to s , so s to a third Proportional t . Secondly, as the excess of r above t , to r , so the given Base b to a fourth Proportional f . Then the side of a Square equal to the sum of the Square of the given Perpendicular p and the Rectangle of f into b , must not be greater than f , for when the said side happens to be greater than f , 'tis impossible to find a Triangle qualified as the Problem requires, by the help of the given lines r , s , b and p .

This Determination is discovered by the three Proportionals in 34^o , which are rightly infer'd from the preceding Resolution, and since the Resolution is clearly Geometrical as well as Arithmetical, I shall take the truth of the Determination for granted.

37. It hath before been declared in 35^o , that the distance sought, which is represented by a in the Resolution, may be either of the two right lines or extreme Proportionals found out in the said 34^o step; which two right lines will be equal to one another when $\sqrt{pp + fb} = f$, for then each of those lines will be equal to f , (as is evident by the Equations in 35^o ;) in which Case, there can but one Triangle be found out to solve the Problem, and that Triangle will always be obtuse-angled at the Base. But when it happens that $\sqrt{pp + fb} < f$, then the said extreme Proportionals, (to wit, the values of a in 35^o ;) will be unequal between themselves, and in this Case the Problem propounded may be solved by either of those two right lines, or extreme Proportionals, viz.

Y y 2

viz. two different Triangles may be found out wherein these three things will be common; to wit, the Base, the Perpendicular, and the Proportion of the legs; of which Triangles, that which is formed by the help of the greater of the said two right lines, (or extreme Proportionals,) will always be obtuse-angled at the Base; but the other Triangle form'd by the help of the lesser of the said two right lines will sometimes be obtuse-angled at the Base, sometimes acute-angled, and sometimes right angled. Now to discover which of those three kinds of Triangles will happen, I shall give three Rules, which presuppose the quantities of the given lines to be express'd by Numbers.

Rule I.

38. If $\frac{pp}{b} + b < f$; but $\frac{pp}{f} + b \text{ not } < f$; then the lesser value of a in 35° , (that is, $f - \sqrt{ff - pp - fb}$;) is greater than the Base b , and consequently the Triangle form'd by the help of the said lesser value shall be obtuse-angled at the Base.

Rule II.

39. If $\frac{pp}{b} + b > f$; then the lesser value of a in 35° is less than the Base, and consequently the Triangle form'd by the help of the said lesser value shall be acute-angled at the Base.

Rule III.

40. If $\frac{pp}{b} + b = f$; then the lesser value of a in 35° is equal to the Base, and consequently, the Triangle form'd by the help of the said lesser value shall be right-angled at the Base. The truth of Rule 1. may be demonstrated thus;

41. . . . Suppos. in Rule 1. . . . $\frac{pp}{b} + b < f$; but $\frac{pp}{f} + b \text{ not } < f$.
42. . . . Req. demonstr. . . . $f - \sqrt{ff - pp - fb} < b$; as is affirmed in Rule 1.

Demonstration.

43. By Suppos. in 41. . . . $\frac{pp}{b} + b < f$.
44. Therefore by multiplying each part by b , . . . $pp + bb < fb$.
45. And by adding ff to each part in 44, . . . $ff + pp + bb < ff + fb$.
46. And by subtracting pp from each part in 45, (which the Supposition in 41. or the Determination in 36. shews to be possible,) it follows that
47. Likewise by subtracting $2fb$ from each part in 46, . . . $ff + bb - 2fb < ff - pp - fb$.
48. And by extracting the Square Root out of each part in 47, . . . $f - b < \sqrt{ff - pp - fb}$.
49. And by adding b to each part in 48, . . . $f < \sqrt{ff - pp - fb} + b$.
50. Wherefore by subtracting $\sqrt{ff - pp - fb}$ from each part in 49, . . . $f - \sqrt{ff - pp - fb} < b$.
Which was to be Demonstr.

After the same manner the truth of the preceding second and third Rules may be demonstrated; and from the premises the following Canon is deducible, for the Arithmetical Solution of the Problem propounded.

CANON.

51. Let it be made as r the greater Term of the given Reason, (or Proportion,) to s the lesser; so the same s to a third Proportional, which may be called t . Let it also be made as the excess of r above t to r ; so the given Base AC to a fourth Proportional, which may be called f . Then from the Square of f subtract the Square of the given Perpendicular ND (= XY) together with the Rectangle made of f into the Base AC , Perpendicular ND (if any happen, extract the Square Root. That done, add the and out of the remainder, and the sum shall be the distance from the foot of the said Square Root to f before found, and the sum shall be the distance from the remoter end of the Perpendicular falling without the Triangle upon the Base continued to the remoter end of the Base; which distance we may suppose to be AY in the following Fig. 1, 2, and 3.

and 3. Whence $CY = AY - AC$ is given; and consequently, (per prop. 47. Elem. 1.) $CX = \sqrt{CY^2 + XY^2}$ is given, and $AX = \sqrt{AY^2 + XY^2}$ is given also. Therefore $\triangle ACX$, whose angle ACX is obtuse, is given, which I shall call the first of the two Triangles that will solve the Problem propounded.

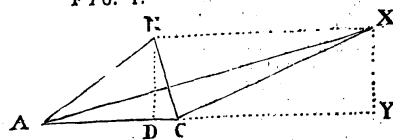
Again, subtract the Square Root before found, from the before mentioned fourth Proportional f , and reserve the remainder. Then observe whether the said remainder be less, greater, or equal to the given Base AC ; if less, then the said remainder shall be equal to AD , to wit, the greater segment of the Base AC made by the falling of the Perpendicular ND within the Triangle ANC in Fig. 1. Whence $AN = \sqrt{AD^2 + ND^2}$ is given. Likewise $CN = \sqrt{CD^2 + ND^2}$ is given, and therefore $\triangle ACN$ is given. Likewise $CN = \sqrt{CD^2 + ND^2}$ is given, which I call the latter of the two Triangles that will solve the Problem. But if the remainder before reserved happens to be greater than the given Base AC , then the said remainder shall be the distance from the foot of the Perpendicular falling without the Triangle to the remoter end of the Base, which distance we may suppose to be AD in Fig. 2. Whence $CD = AD - AC$ is given, and consequently, (per prop. 47. Elem. 1.) $CN = \sqrt{CD^2 + ND^2}$ is given. Likewise $AN = \sqrt{AD^2 + ND^2}$ is given; and therefore in Fig. 2. $\triangle ACN$ obtuse-angled at C is given, which shall be the latter of two Triangles that will solve the Problem. But if the remainder before reserved happens to be equal to the given Base AC ; then the latter of two Triangles that will solve the Problem shall be right-angled at the Base, as the $\triangle ACN$ right-angled at C , in Fig. 3. and consequently, $AN = \sqrt{AC^2 + CN^2}$ is given. Therefore in Fig. 3. $\triangle ACN$ is given also. (per prop. 47. Elem. 1.)

Lastly, when it happens that nothing remains after subtraction is made of the sum of the Square of the given Perpendicular ND and the Rectangle of the given Base AC into the fourth Proportional f , from the Square of the same f ; then f it (self) shall be the distance from the foot of the Perpendicular falling upon the Base continued to the remoter end of the said Base, which distance we may suppose to be AD in $\triangle ADN$ in Fig. 4. Whence $CD = AD - AC$ is given, and consequently, (per prop. 47. Elem. 1.) $CN = \sqrt{CD^2 + ND^2}$ is given. Likewise $AN = \sqrt{AD^2 + ND^2}$ is given. Therefore in Fig. 4. $\triangle ACN$ obtuse-angled at C is given, which is the only Triangle in this Case that will solve the Problem.

TRIANGLES in Numbers, to illustrate the preceding Canon of Probl. 15.

FIG. 1.

52. In this Fig. 1. the Triangles ACX and ACN , the first of which is obtuse-angled at the Base AC , and the latter acute-angled, have one common Base AC , also equal Perpendiculars ND and XY ; and the legs AX, CX of $\triangle ACX$ have the same Proportion one to the other, as the legs AN, CN of $\triangle ACN$.



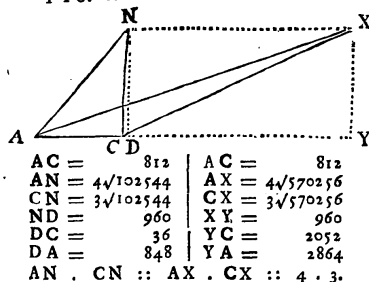
AC = 39	AC = 39
AN = 40	AX = $8\sqrt{153}$
CN = 25	CX = $5\sqrt{153}$
ND = 24	XY = 24
DA = 32	YA = 96
DC = 7	YC = 57

$$AN, CN :: AX, CX :: 8, 5.$$

FIG. 2.

53. In this Fig. 2. the Triangles ACX and ACN, each of which is obtuse-angled at the Base AC, have one common Base AC, also equal Perpendiculars ND and XY, and the legs AX, CX of $\triangle ACX$ have the same Proportion one to the other, as the legs AN, CN of $\triangle ACN$.

FIG. 2.



54. In this Fig. 3. the Triangles ACX and ACN, the first of which is obtuse-angled at the Base AC, and the latter right-angled, have one common Base AC, also equal Perpendiculars ND and XY, and the legs AX, CX of $\triangle ACX$ have the same Proportion one to the other, as the legs AN, CN of $\triangle ACN$.

FIG. 3.

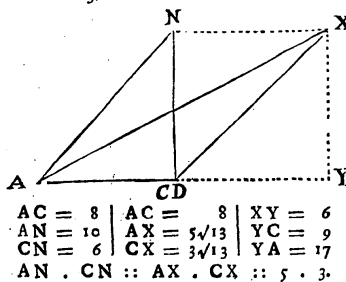
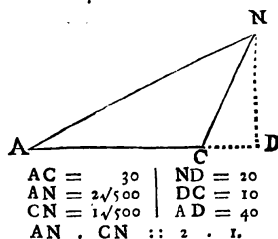


FIG. 4.

55. In this Fig. 4. the Triangle ACN obtuse-angled at C cannot be matcht with any other Triangle that shall have its Base, Perpendicular, and Proportion of the legs, equal to the Base AC, Perpendicular ND, and Proportion of the legs AN, CN of the said $\triangle ACN$.



56. It is prescribed by the preceding Determination in 36°, that $\sqrt{pp+fb}$ must not be greater than f , I shall therefore divide the Composition of this Probl. 15. into two Cases, viz.

Case 1. when $\sqrt{pp+fb} < f$. Case 2. when $\sqrt{pp+fb} = f$.

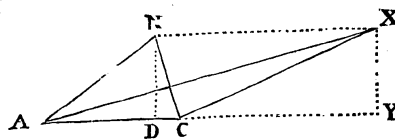
The Composition of Case 1. Probl. 15.

Suppos.

57. B a right line equal to the Base of a plain Triangle is given.
58. P a right line equal to the Perpendicular is given.

59. R and S

59. R and S two right lines expressing the Reason (or Proportion) of the legs of the same Triangle, are given.
60. $R \sqsubset S$.



Req. to find out the Triangle.

B _____
P _____
R _____
S _____
T _____
F _____
M _____
K _____
L _____

Construction.

61. To the given lines R and S find a third Proportional, } $R . S :: S . T$.
(by Probl. 7 Chap. 6.) suppose the line T, therefore }
62. Also (by Probl. 8. Chap. 5.) let it be made as $R - T$ }
to R; so the given Base B to a fourth Proportional line F, } $R - T . R :: B . F$.
therefore }
63. By Probl. 2. Chap. 5. find a right line M, such that its } $\square M = \square P + \square F, B$.
Square may be equal to $\square P + \square B, F$, therefore }
64. By Probl. 14. Chap. 5. divide the double of F into two such parts, that the line M }
may be a mean between them, which Effection is possible, for by Suppos. in Case 1. }
(before express in 56°) the line M (that is, $\sqrt{pp+fb}$) is less than F, suppose }
then that 2 F is cut into two parts, whereof the greater is equal to the line K, and the }
lesser equal to the line L; and that the line M is a mean Proportional between K and L, }
therefore these are Proportionals, viz. }
 $2F - K . M :: M . K$, || $2F - L . M :: M . L$.

Each of which Analogies is correspondent to that in the 34th step of the preceding Resolution, viz.

$$2f - a . \sqrt{pp+fb} :: \sqrt{pp+fb} . a$$

Now by the help of the line K, found out as above, an obtuse-angled plain Triangle to solve the Problem propounded may be made in manner following, viz.

65. Make $AC = B$, (the given Base.)
66. Produce AC to Y, so that AY may be equal to K, which is greater than AC, as may be proved thus;
It is manifest that } $R \sqsubset R - T$.
Therefore from the Analogy in 62°, (per Coroll. of 14. prop. } $F \sqsubset B$.
5. Elem.) } $AC = B$.
But by Constr. in 65°, } $F \sqsubset AC$ (or B).
Therefore (per Ax. 3. Chap. 2.) } $K \sqsubset F$.
And because the greatest of three Proportionals is greater than }
half the sum of the extremes, therefore from 64°, } AY or $K \sqsubset AC$.
Therefore (per Ax. 5. Chap. 2.) }

Which was to be demonstr.

67. Make $YX \perp AY$, also make $YX = P$, the given Perpendicular.
68. Lastly, from A and C (the ends of the Base AC) draw the right lines AX and CX to meet with the top of the Perpendicular YX in X, so the Triangle ACX obtuse-angled at C, (for as before hath been proved in 66°, $AY \sqsubset AC$) shall be

one

one of the two Triangles which in *Case 1.* will satisfy the Problem; which I prove thus,
 69. First, by *Constr.* in 65° the Base AC is equal to the given Base B; secondly,
 by *Confr.* in 67° the line YX is perpendicular to AC continued, and equal to the given
 Perpendicular P. It remains only to prove that the greater leg AX hath such pro-
 portion to the lesser leg CX, as R to S; which Analogy will be made manifest by
 the following Demonstration, formed out of the preceding Resolution by a repetition
 of its steps in a retrograde order, viz. by returning backwards from the end to the
 beginning of the Resolution.

70. . . . Req. demonstr. R . S :: AX . CX.

Demonstration.

71. Forasmuch as by *Confr.* in 64° , $2F - K . M :: M . K$.
 72. And by *Confr.* in 66° , $AY = K$.
 73. Therefore from 71° and 72° , $2F - AY . M :: M . AY$.
 That is, in 34° , the last step of the Resolution, $2f - a . \sqrt{pp} + fb :: \sqrt{pp} + fb . a$.
 74. But from 73° , (per 17. prop. 6. Elem.) $2 \square F, AY - \square AY = \square M$.
 75. And by *Confr.* in 63° , $\square P + \square F, B = \square M$.
 76. Therefore from 74° and 75° , (per 1. Ax.) $2 \square F, AY - \square AY = \square P + \square F, B$.
 Chap. 2.)
 77. And because by *Confr.* in 67° , $YX = P$.
 78. And consequently, $\square YX = \square P$.
 79. And by *Confr.* in 65° , $AC = B$.
 80. Therefore out of 76° , 78° and 79° , $2 \square F, AY - \square AY = \square YX + \square F, AC$.
 That is, in 33° , $2fa - aa = pp + fb$.
 81. And from 80° , by adding $\square AY$ to each part, $2 \square F, AY = \square AY + \square YX + \square F, AC$.
 That is, in 32° , $2fa = aa + pp + fb$.
 82. And by subtracting $\square F, AC$ from each part of the Equation in 81° , $2 \square F, AY - \square F, AC = \square AY + \square YX$.
 That is, in 31° , $2fa - fb = aa + pp$.
 83. And this following Analogy is manifest, (per prop. 7. Elem. 5.) for the first and third
 Proportionals are one and the same, and the second and fourth equal one to the other,
 (as hath before been proved in 81°);

$$\begin{array}{l} 2 \square AC, AY - \square AC \\ \square AY + \square YX :: \text{that is, in } 30^\circ, \\ 2 \square AC, AY - \square AC \\ 2 \square F, AY - \square F, AC \end{array} \left\{ \begin{array}{l} 2ba - bb \\ aa + pp \\ 2ba - bb \\ 2fa - fb \end{array} \right.$$

84. And by reason of the common altitude $2AY - AC$ in the two latter Terms of the
 subsequent Analogy, it will be manifest (per 1. prop. 6. Elem.) that

$$\begin{array}{l} AC \\ F :: \text{that is, in } 29^\circ, \\ 2 \square AC, AY - \square AC \\ 2 \square F, AY - \square F, AC \end{array} \left\{ \begin{array}{l} b \\ f \\ 2ba - bb \\ 2fa - fb \end{array} \right.$$

85. And because the two latter Terms of the Analogy in 84° , are the same, and in the
 same order with the two latter Terms of the Analogy in 83° , therefore from 83° and
 84° (per 1. prop. 5. Elem.) these shall be Proportionals, viz.

$$\begin{array}{l} AC \\ F :: \text{that is, in } 28^\circ, \\ 2 \square AC, AY - \square AC \\ \square AY + \square YX \end{array} \left\{ \begin{array}{l} b \\ f \\ 2ba - bb \\ aa + pp \end{array} \right.$$

86. But from the *Confr.* in 62° and 65° , $AC . F :: R - T . R$
 That is, in 27° , $b . f :: r - t . r$.

87. Therefore from 85° and 86° , (per 11. prop. 5. Elem.) these shall be Proportionals, viz.

$$\begin{array}{l} R - T \\ R :: \text{that is, in } 26^\circ, \\ 2 \square AC, AY - \square AC \\ \square AY + \square YX \end{array} \left\{ \begin{array}{l} r - t \\ r \\ 2ba - bb \\ aa + pp \end{array} \right.$$

88. And

88. And from 87° , by Reason Inverse, these are Proportionals, viz.

$$\begin{array}{l} R - T :: \text{that is, in } 25^\circ, \\ \square AY + \square YX \\ 2 \square AC, AY - \square AC \end{array} \left\{ \begin{array}{l} r - t \\ aa + pp \\ 2ba - bb \end{array} \right.$$

89. And from 88° , by Conversion of Reason, these are Proportionals, viz.

$$\begin{array}{l} R \\ T :: \text{which answer to those in } 24^\circ \\ \square AY + \square YX \\ \square AY - 2 \square AC, AY + \square AC + \square YX \end{array} \left\{ \begin{array}{l} r \\ aa + pp \\ aa - 2ba + bb + pp \end{array} \right.$$

90. And because by *Confr.* in 61° , $R . S :: S . T$.

That is, in 22° , $r . s :: s . t$.

91. Therefore (per Coroll. of 20. prop. 6. Elem.) $R . T :: \square R . \square S$.

That is, in 23° , $r . t :: rr . ss$.

92. Therefore from 89° and 91° , (per 11. prop. 5. Elem.) these shall be Proportionals, viz.

$$\begin{array}{l} \square R \\ \square S :: \text{which answer to those in } 21^\circ, \\ \square AY + \square YX \\ \square AY - 2 \square AC, AY + \square AC + \square YX \end{array} \left\{ \begin{array}{l} rr \\ ss \\ aa + pp \\ aa - 2ba + bb + pp \end{array} \right.$$

93. And because by *Confr.* in 67° , $YX \perp AY$.

94. Therefore (per 47. prop. 1. Elem.) $\square AX = \square AY + \square YX$.

95. Likewise $\square CX = \square CY + \square YX$.

96. Moreover by *Confr.* in 66° , and by the *Diag.* $CY = AY - AC$.

97. And consequently, (per 5. Theor. 4. Chap.) $\square CY = \square AY - 2 \square AC, AY + \square AC$.

That is, in 9° , $\square a - b = aa - 2ba + bb$.

98. Therefore if instead of $\square CY$ in 95° , we take that which in 97° is found equal to

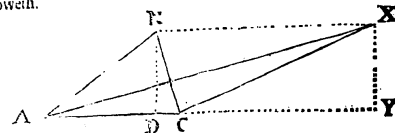
$\square CY$, the Equation in 95° will be reduced to this, viz.

$\square CX = \square AY - 2 \square AC, AY + \square AC + \square YX$.

99. Likewise if instead of the third and fourth Proportionals in 92° , we take those Squares
 which are found equal to them respectively in 94° and 98° , the Analogy in 92° will be
 reduced to this, viz. $\square R . \square S :: \square AX . \square CX$.

100. Wherefore (per prop. 22. Elem. 6.) $R . S :: AX . CX$. Which was to be Dem.

101. Another Triangle to solve the Problem in *Case 1.* before express in 56° , may by
 the help of the lesser Root L before found in 64° , be formed thus, viz. Let the lines
 before given and found out in 57° , 58° , 59° , 60° , 61° , 62° , 63° , 64° , together with the
 Diagram standing between 60° and 61° be here repeated, then will the Construction
 be as followeth.



Req. to find out the Triangle.

B _____
 P _____
 R _____
 S _____
 T _____
 F _____
 M _____
 K _____
 L _____

Z z

Confr.

Construction.

102. Make $AC = B$ (the given Base.)
 103. Upon AC ; continued if need be, make $AD = L$, which lesser Root L , (as before hath been shewn,) will sometimes be greater than the Base; but supposing it be discovered (by Rule 2. in 39th of this Probl.) that L is lesser than B , or AC , cut off from AC a segment equal to L , as AD .
 104. Make $DN \perp AC$ in the point D , also make $DN = P$ the given Perpendicular.
 105. Lastly, from the ends of the Base AC draw the right lines AN and CN to meet with the top of the Perpendicular DN , in N ; so the Triangle ACN acute-angled at A and C , (for by Supposition AD is lesser than AC ;) will satisfy the Problem, as well as the $\triangle ACX$ before found. For first, by Construction in 102 the Base AC is equal to the given Base B ; Secondly, the Perpendicular DN (by Constr. in 104) is equal to the given Perpendicular P ; and by a repetition of the Steps or the Resolution in a backward order, in like manner as before in the preceding Demonstration, saving that L must be used here instead of K , and ND instead of XY ; it may easily be proved that the legs AN and CN are in the given Reason of R to S .
 Moreover, when the lesser Root L is greater than the Base, the Triangle formed by the help of such lesser Root shall be obtuse-angled at the Base, and the Construction and Demonstration in every respect like to that by the greater Root.

But it must be remembered, that when the Perpendicular falls within the Triangle, then the Square of DC is equal to the Square of $AC - AD$; but when it falls without, then the Square of YC is equal to the Square of $YA - AC$: So that before the Construction and Demonstration by the lesser Root be entered upon, it will be requisite to find out the kind of the Triangle, by the help of the three preceding Rules in 38th, 39th, 40th; and when it happens that $r : s :: \sqrt{fb} + pp : p$, then 'tis evident (by 47th prop. 1. Elem.) that the Triangle formed by the lesser Root will be right-angled at the Base, and in such Case there is no need of further proof.

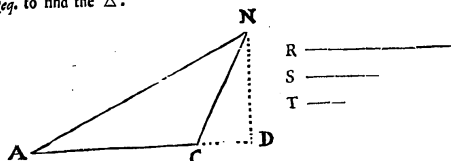
The Composition of Case 2. Probl. 15.

106. Which Case, in the letters belonging to the preceding Resolution presupposeth . . . $f = \sqrt{pp + fb} = a$.
 107. And consequently, by squaring each part, . . . $ff = pp + fb = aa$.
 108. That is, in the lines of the ensuing Constr. and Diagr. $\square AD = \square DN + \square AD, AC$.

Suppos.

109. AC = the Base of a Triangle is given.
 110. DN = the Perpendicular is given.
 111. R and S two right lines expressing the Reason of the legs are given.
 112. $R \propto S$.

Req. to find the \triangle .



Construction.

113. By Probl. 7. Chap. 5. let it be made as R to S , so S to a third Proportional, suppose it be found T , therefore, $R : S :: S : T$.
 114. By Probl. 8. Chap. 5. let it be made as $R-T$ to R , so AC to a fourth Proportional, suppose it to be AD , therefore, $R-T : R :: AC : AD$.
 Which fourth Proportional AD shall necessarily be greater than AC , because R is greater than $R-T$.
 115. Make

115. Make $DN \perp AD$ in the point D , then from A and C , the ends of the given Base AC , draw the right lines AN and CN to meet with the top of the Perpendicular DN in N ; so shall ACN be the Triangle required. For first, the Base AC is equal to the given Base; also the Perpendicular ND is equal to the given Perpendicular. But that the legs AN and CN are in the given Reason of R to S , it may easily be demonstrated by a backward repetition of the Steps of the foregoing Resolution, in like manner as before in the Composition of Case 1; with this Caution, That as often as a is found in the Resolution, f must be taken instead of a , because in this second Case f is equal to a ; for since by Supposition in 106th, $f = \sqrt{pp + fb}$: it will be evident from 35th, that $f = a$. But in regard the Demonstration of this second Case differs not from that of the following Probl. 16. I shall wave it here.

COROLLARY.

116. From the premises it follows, that the Perpendicular DN of the Triangle ACN formed in Case 2. (before express'd in 106th, 107th and 108th;) is a mean Proportional between AD and DC the distances from D the foot of the Perpendicular falling without the Triangle to the ends of the Base; and consequently, (per prop. 6. Elem. 6.) the Triangles ADN and CDN are equiangular. See the preceding Diagram, and compare it with this following Demonstration.

117. . . . Req. demonstr. . . . $\square AD \cdot DN :: DN \cdot DC$. Also, $\triangle ADN$ and $\triangle CDN$ are equiangular.

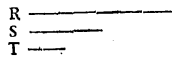
Demonstration.

118. By Suppos. in 108th, . . . $\square AD = \square DN + \square AD, AC$.
 119. Therefore by subtracting $\square AD, AC$ from each part, . . . $\square AD - \square AD, AC = \square DN$.
 120. And from 119th, (per prop. 14. Elem. 6.) these are Proportionals, viz. . . . $\square AD \cdot DN :: DN \cdot AD - AC$.
 121. But 'tis evident by the last preceding Diagram, that . . . $DC = AD - AC$.
 122. Therefore from 120th and 121st, . . . $\square AD \cdot DN :: DN \cdot DC$.
 123. Therefore from 122nd, (per prop. 6. Elem. 6.) $\triangle ADN$ and $\triangle CDN$ are equiangular. Which was to be Demonstr.

From the preceding Corollary and Construction of Case 2. the following Probl. 16. is deducible.

Probl. XVI.

To find a plain Triangle obtuse-angled at the Base, and that the Base may be equal to a right line given. Also, that the Perpendicular falling upon the Base continued may be a mean Proportional between the distances from the foot of the Perpendicular to the ends of the Base: And that the legs of the Triangle may be in a given Reason, suppose as R to S .



Suppos.

1. $\triangle ACN$ is obtuse-angled at C .
 2. AC the Base is given.
 3. ACD is a right line.
 4. $DN \perp AD$.
 5. $AD \cdot DN :: DN \cdot DC$.
 6. R and S are right lines given.
 7. $R : S :: AN : CN$.

Req. to make the $\triangle ACN$.

Construction.

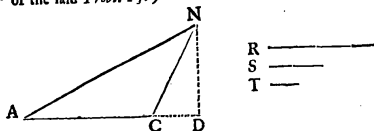
8. Making R the first of three Proportionals, and S the second, find a third, (per 7. Probl. 5. Chap.) let it be T , therefore $R : S :: S : T$.
 9. Also

9. Also making $R-T$ the first of four Proportionals, R the second, and the given Base AC the third, find a fourth, (per 8. Probl. 5. Chap.) suppose it be AD , therefore $R-T :: AC : AD$.

Which fourth Proportional AD shall necessarily be greater than AC , because the second Proportional R is greater than the first $R-T$.

10. Find a mean Proportional, as DN , between AD and DC , (per 9. Probl. 5. Chap.) therefore, $AD : DN :: DN : DC (= AD - AC)$.

11. Make $DN \perp AD$ in the point D ; then draw the right lines AN and CN , so shall ACN be the Triangle required. Now we must shew that it will satisfy the Problem: First then, AC the Base is equal to the right line prescribed for the Base, and from the 9th step it is less than AD ; therefore the angle ACN is obtuse: Secondly, the Perpendicular ND , (by Construction in 10th), is a mean Proportional between AD and DC , (to wit, the two distances from D the foot of the Perpendicular ND , to A and C the ends of the Base AC .) It remains only to prove, That the legs AN and CN of the Triangle ACN , are in such proportion one to the other, as R to S ; which Analogy I shall make manifest by the following Demonstration, formed out of the Resolution of the preceding Probl. 15. by a backward repetition of the steps of the said Resolution, in Case 2. (but respect must be had to the Caution given in 11th of the said Probl. 15.)



12. . . . Req. demonstr. $R : S :: AN : CN$.

Demonstration.

13. Forasmuch as by Constr. in 10th, . . . $AD : DN :: DN : DC (= AD - AC)$
 14. Therefore (per 17. prop. 6. Elem.) . . . $\square AD - \square AD, AC = \square DN$.
 15. And from 14th, by equal addition of $\square AD, AC$, $\square AD = \square DN + \square AD, AC$.
 16. And from 15th, by equal addition of $\square AD$, $2\square AD = \square AD + \square DN + \square AD, AC$.
 17. And from 16th, by equal subtraction of $\square AD, AC$, $2\square AD - \square AD, AC = \square AD + \square DN$.
 18. And because in the following Analogy the first and third Terms are one and the same, and the second and fourth equal one to the other, (as hath been proved in 17th) therefore (per 7. prop. 5. Elem.)

$$\left. \begin{array}{l} 2\square AC, AD - \square AC \\ \square AD + \square DN :: \\ 2\square AC, AD - \square AC \\ 2\square AD - \square AD, AC \end{array} \right\} \text{Proportionals.}$$

19. And the subsequent Analogy, by reason of the common altitude $2AD - AC$ in the two latter Terms, will be manifest, (per 1. prop. 6. Elem.) viz.

$$AC : AD :: 2\square AC, AD - \square AC : 2\square AD - \square AD, AC.$$

20. And because the two latter Terms of the Analogy in 19th are the same and in the same order with the two latter Terms of the Analogy in 18th, therefore from 18th and 19th; (per 11. prop. 5. Elem.) these shall be Proportionals, viz.

$$AC : AD :: 2\square AC, AD - \square AC : \square AD + \square DN.$$

21. But by Construction in 9th,

$$AC : AD :: R - T : R.$$

22. Therefore from 20th and 21th, (per 11. prop. 5. Elem.)

$$R - T : R :: 2\square AC, AD - \square AC : \square AD + \square DN.$$

23. And from 22th, by Reason inverse,

$$R : R - T :: \square AD + \square DN : 2\square AC, AD - \square AC.$$

24. And

24. And from 23th, by Conversion of Reason, these shall be Proportionals; viz.

$$\left. \begin{array}{l} R : R - T :: \square AD + \square DN : 2\square AC, AD - \square AC \\ \square AD + \square DN + \square AC - 2\square AC, AD :: \end{array} \right\} \text{Proportionals.}$$

25. And because by Constr. in 8th, . . . $R : S :: S : T$.

26. And consequently, (per Coroll. of 20. prop. 6. Elem.) $R : T :: \square R : \square S$.

27. Therefore from 24th and 26th, (per 11. prop. 5. Elem.) these shall be Proportionals, viz.

$$\left. \begin{array}{l} \square R : \square S :: \\ \square AD + \square DN + \square AC - 2\square AC, AD :: \end{array} \right\} \text{Proportionals.}$$

28. And because by Constr. in 11th, . . . $DN \perp AD$.

29. Therefore, per 47. prop. 1. Elem. (respect being had to the Diagram,) $\square AN = \square AD + \square DN$.

30. Likewise, $\square CN = \square DC + \square DN$.

31. Again, by Constr. in 9th, and by view of the Diagram, $DC = AD - AC$.

32. And consequently, (per Theor. 5. Chap. 4.) $\square DC = \square AD - \square AC$.

33. Therefore if instead of $\square DC$ in 30th, we set that which in 32th is found equal to $\square DC$, the Equation in 30th will be reduced to this, viz.

$$\square CN = \square AD + \square AC - 2\square AD, AC + \square DN.$$

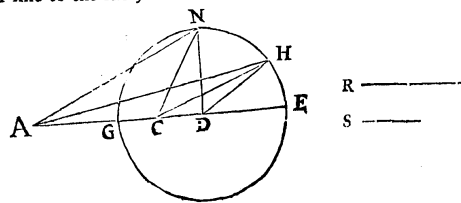
34. Likewise, if instead of the third and fourth Proportionals in 27th, we take those Squares which are found equal to them respectively in 29th and 33th, the Analogy in 27th will be reduced to this, $\square R : \square S :: \square AN : \square CN$.

35. Wherefore . . . $R : S :: AN : CN$.

Which was to be demonstrated. Therefore that is done which the Problem required.

Probl. XVII. (Probl. Apollon. Pergai.)

Two points (A and C) being given in a Plane, to describe a Circle in the same Plane, that two right lines drawn from those points to concur in any point of the Circumference may have a given Reason; suppose the greater line to the less, as R to S .



Construction.

1. Upon the given line AC as a Base, to wit, the shortest distance between the given points A and C , make (by the preceding Probl. 16.) a Triangle ACN obtuse-angled at C , and such, that the Perpendicular ND falling upon AC produced, may be a mean Proportional between AD and DC ; also, that the legs AN and CN may be in the given Reason of R to S . Therefore by that Construction these are Proportionals, viz.

$$AD : DN :: DN : DC.$$

$$R : S :: AN : CN.$$

2. Then from the Center D , at the distance of the Perpendicular DN , describe the Circle $DGNEQ$, which shall necessarily cut AC ; for by Construction DN is a mean Proport.

Proportional between DA and DC, which DC being but part of DA is less than DA, therefore the mean, or Semidiameter DN or DG is less than DA, but greater than DC. Now I say the Circle D G N E Q is that which is required by the Problem, and therefore we must shew that if two right lines be drawn from the given points A and C to meet in any point of the Circumference of that Circle, those right lines shall have such proportion one to the other as the given lines R and S; the demonstration whereof I shall divide into three Cases, in regard there may be a threefold position of the point taken in the Circumference; for the point may be either E, or else G, to wit, the ends of that Diameter which lies in the same straight line with the given line AC; or lastly, the point may be taken in any other part of the Circumference, as H; which Cases I shall demonstrate in their order.

Preparat.

3. Forasmuch as in the Triangles ADN and CDN the angle at D is common, and the sides about that angle are Proportionals, for by *Construction* in 1^o it hath been made, as $AD : DN :: DN : DC$, therefore (per *prop. 6. Elem. 6.*) $\triangle ADN$ and $\triangle CDN$ are equiangular.

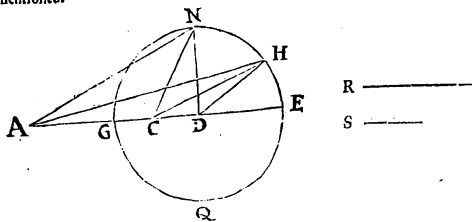
4. But we must enquire which angles in those like Triangles are equal one to the other. First then, because the angle at D is common, the angle CND in $\triangle CDN$ must be equal either to the angle AND , or to the angle NAD in $\triangle ADN$; but the angle CND being but part of the angle AND cannot be equal to it, therefore $\angle CND = \angle NAD$. Also, $\angle NCD = \angle AND$.

5. In like manner, because by *Constr.* in 1^o and 2^o, $AD : DN$ (or DH) :: DN (or DH) . DC .

6. Therefore, (per *prop. 6. Elem. 6.*) $\triangle ADH$ and $\triangle CDH$ are equiangular.

7. And for the like reason as before in 4^o, $\angle CHD = \angle HAD$. Also, $\angle HCD = \angle AHD$.

These things premised, I shall proceed to the Demonstration of the three CASES before mentioned.



8. . . . I. *Req. Demonstr.* $R . S :: AE . CE$.

Demonstration.

9. Because by *Constr.* in 1^o, $AD . DN :: DN . DC$.
 10. Therefore by Composition of Reason, $AD + DN . DN :: DN + DC . DC$.
 11. And because (per *defn. 15. Elem. 1.*) $DE = DN$.
 12. Therefore from 10^o and 11^o, $AD + DE . DN :: DE + DC . DC$.
 13. That is, as is evident by the *Diagram*, $AE . DN :: CE . DC$.
 14. Therefore alternly, $AE . CE :: DN . DC$.
 15. Again, it hath been proved in 3^o, that $\triangle ADN$ and $\triangle CDN$ are equiangular.
 16. And in 4^o, that $\angle NCD = \angle AND$.
 17. Therefore from 15^o and 16^o, (per *prop. 4. Elem. 6.*) $AN . DN :: CN . DC$.
 18. Therefore alternly, $AN . CN :: DN . DC$.
 19. But by *Constr.* in 1^o, $AN . CN :: R . S$.
 20. Therefore from 18^o and 19^o, (per *11. prop. 5. Elem.*) $R . S :: DN . DC$.

21. But

21. But it hath been proved in 14^o, that $AE . CE :: DN . DC$.
 22. Therefore from 20^o and 21^o, (per *11. prop. 5. Elem.*) $R . S :: AE . CE$.

Which was to be demonstr.

23. . . . II. *Req. Demonstr.* $R . S :: AG . CG$.

Demonstration.

24. Forasmuch as by *Constr.* in 1^o, $AD . DN :: DN . DC$.
 25. Therefore by Division of Reason, $AD - DN . DN :: DN - DC . DC$.
 26. And because (per *defn. 15. Elem. 1.*) $DG = DN$.
 27. Therefore from 25^o and 26^o, $AD - DG . DN :: DG - DC . DC$.
 28. That is, (as is evident by the *Diagram*), $AG . DN :: CG . DC$.
 29. Therefore alternly, $AG . CG :: DN . DC$.
 30. But before in 10^o, it hath been proved that $R . S :: DN . DC$.
 31. Therefore from 29^o and 30^o, (per *11. prop. 5. Elem.*) $R . S :: AG . CG$.
 Which was to be demonstrated in the second place.

32. . . . III. *Req. Demonstr.* $R . S :: AH . CH$.

Demonstration.

33. It hath before been proved in 6^o, that $\triangle ADH$ and $\triangle CDH$ are equiangular.
 34. And in 7^o, that $\angle HCD = \angle AHD$.
 35. Therefore from 33^o and 34^o, (per *4. prop. 6. Elem.*) $CH . CD :: AH . HD$.
 36. And alternly, $CH . AH :: CD . HD$ (or DN).
 37. Therefore inversly, $AH . CH :: DN . CD$.
 38. But it hath been proved in 20^o, that $R . S :: DN . DC$.
 39. Therefore from 37^o and 38^o, (per *11. prop. 5. Elem.*) $R . S :: AH . CH$.
 Which was to be demonstrated in the last place. Therefore that is done which the Problem required.

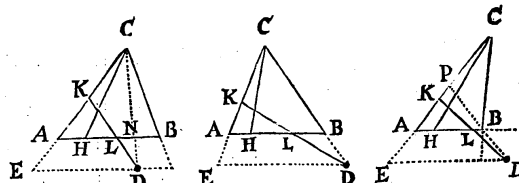
Probl. XVIII.

To divide a given Triangle ABC into two parts which shall be in a given Reason, suppose as AH to HB, by a right line DK drawn from a given point D without the Triangle.

FIG. 1.

FIG. 2.

FIG. 3.



Prepar.

1. By the given point D draw DE parallel to the Base AB, and continue the legs CA, CB beneath the Base; then the point D will either lye between the Increases of the legs, as in Fig. 1. or else in one of the said Increases, as in Fig. 2. or lastly, between the Increases of the Base and one of the legs, as in Fig. 3.
 2. Divide the Base AB in H in the given Reason, and draw CH, therefore (per *prop. 1. Elem. 6.*) $\triangle ACH . \triangle HCB :: AH . HB$.
 3. In Fig. 1. let a line be drawn from the given point D to C the angle opposite to the Base AB, as the line DC, which will either cut the Base AB in the point H, in which

the Base is divided in the given Reason, in which Case the Problem is evidently satisfied; or else in some other point N, and then the point H will either lye between N and A, or between N and B; if H lye between N and A, then the desired line of partition to be drawn from D will cut AB and AC; but if H lye between N and B, then the said line of partition will cut AB and BC.

4. In Fig. 2. where the given point D lyes in CB increased, 'tis evident that the line of partition to be drawn from D shall necessarily cut AB and AC.
5. In Fig. 3. the line of partition to be drawn from D will sometimes cut AB and AC, sometimes it may pass by the angular point B and cut AC only, and sometimes it will cut BC and AC; but which of these lines will happen to be cut when the given point D is posited according to the Definition in 1°, relating to Fig. 3. may be discovered by the Rule hereafter given in 34° of this Problem.
6. In every one of those three Cases before defined in 1°, which may happen by the various position of the given point D, the Resolution of the Problem propos'd will be one and the same. Supposing then it be discovered, that the line of partition to be drawn from D must cut AB and AC in each of the three preceding Figures, the Scope of the Resolution is to find a point in AC, as K, to which a right line being drawn from D, as DK, this line DK may cut the Base AB between H and N in Fig. 1. or between H and B in Fig. 2. likewise in Fig. 3. between H and B, (or else pass by the angular point B,) so as to make the Triangle AKL equal to the Triangle ACH, whence it evidently follows that $LKCB = \triangle HCB$, and $\triangle AKL : LKCB :: AH : HB$. These things premised, the Resolution of the Problem propounded may be formed in manner following.

Suppos.

7. ABC is a \triangle given in Fig. 1.
 8. D is a point given without the $\triangle ABC$.
 9. AH and HB are in a given Reason.
 10. $b = AC$ is given.
 11. $c = AH$ is given.
 12. $g = AE$ is given.
 13. $h = ED$ ($\parallel AB$) is given.

Req. to find

14. AK such a segment of AC, that DK being drawn, it may make
 15. $\triangle ALK : LKCB :: AH : HB :: \triangle ACH : \triangle HCB$.

Resolution.

16. Suppose that done which is required, and put $a = AK$.
 17. Then by considering well what is required, and by viewing Fig. 1. it will appear that $\triangle ALK = \triangle ACH$, and that $\angle CAL$ is common to both Triangles, therefore (per prop. 15. Elem. 6.)
 18. That is, in the letters of the Resolution, $a \cdot c :: b \cdot \frac{bc}{a}$.
 19. And because $\triangle AKL$ and $\triangle EKD$ are equiangular, (for by Constr. in 1° $ED \parallel AL$), therefore (per prop. 4. Elem. 6.)
 20. That is, in the letters of the Resolution, $a + g \cdot a :: b \cdot \frac{ha}{a + g}$.
 21. And because the fourth Proportional in 17° is the same with the fourth in 19°, therefore the fourth Proportionals in 18° and 20° shall also be equal to one another, viz.

$$\frac{bc}{a} = \frac{ha}{a + g}$$

 22. Now to avoid an Equation between Solids, let it be made as b to b , so c to a fourth Proportional, which may be called m ; therefore,

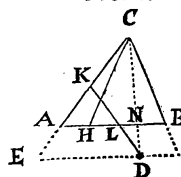
$$b \cdot b :: c \cdot m$$

 23. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, viz.

$$bm = bc$$

24. There-

FIG. 1.



24. Therefore from 21° and 23°, by exchanging equal quantities, $\frac{bm}{a} = \frac{ba}{a+g} = \frac{bc}{a}$
25. Whence 'tis easie to infer that these are Proportionals, viz. $a \cdot m :: b \cdot \frac{ha}{a+g} (= \frac{bc}{a})$
26. But it hath been shewn in 20°, that $a + g \cdot a :: b \cdot \frac{ha}{a+g}$
27. Therefore from 25° and 26°, (per prop. 11. Elem. 5.) $a \cdot m :: a + g \cdot a$
28. And from 27°, by comparing the Rectangle of the extremes to the Rectangle of the means, $aa = ma + mg$.
29. And from 28°, by subtracting ma from each part, $aa - ma = mg$.
30. Which last Equation may be resolved into these Proportionals, viz. $a - m \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$
31. Of which three Proportionals, the mean, to wit, \sqrt{mg} is given, as also m the difference of the extremes a and $a - m$; therefore the extremes shall be given severally, (per Probl. 12. Chap. 5.) the greater whereof is equal to the desired line AK, which (by the Theorem in 24° of the said Probl. 12. Chap. 5.) will be found equal to this right line, (or number), viz.

$$\frac{1}{2}m + \sqrt{\frac{1}{4}mm + mg} = AK = a$$

From which Equation and premisses we may deduce this following

CANON.

32. Let it be made as ED to AC, so AH to a fourth Proportional, which may be called M, then to the half of M add the Square Root of the sum of the Square of half M and the Rectangle of M into AE, so shall the sum of that addition be the value of AK sought.
33. This Canon serves to find out the value of the line AK in every one of the three preceding Figures, and when the given point D is posited according to the Definition of the first and second Cases in 1°, as in Fig. 1. and 2. it is easie to discover from what hath been said in 3° and 4°, which of the sides of the given Triangle ABC will be cut by the line of partition to be drawn from D. But when the point D is posited according to the Definition of the third Case in 1°, as in Fig. 3. then it may be doubtful which of the sides are to be cut; to remove therefore this ambiguity, observe the following Directions, viz. First, draw a right line from the given point D (in Fig. 3.) to pass by the angular point B, as DBP; then is EABD a Trapezium, having (by Construction) two parallel sides AB and ED, and the other two sides EA and DB which are not Parallel, being continued will meet in some point in AC, as in P, for (by Construction) EAC is a right line. Now if AB, ED and EA be severally given in numbers, the line AP shall be also given in number, for putting $g = AE$, and $h = ED$, (as before in the Resolution,) also $k = AB$, the line AP (by the Theorem in 9° of Probl. 18. Chap. 7.) will be found equal to $\frac{gk}{b-k}$. It is also manifest, that if a right line be drawn from any point in AC, between P and C, to the given point D, the line so drawn must necessarily cut BC, for the line PBD is supposed to pass by the angular point B; but if a right line be drawn from any point in AC between P and A to the point D, the line so drawn will evidently cut AB. From the premisses therefore we may infer this following

RULE.

34. If $\frac{1}{2}m + \sqrt{\frac{1}{4}mm + gm}$ the value of AK, be not greater than $\frac{gk}{b-k}$ the value of AP in Fig. 3. then the line of partition to be drawn from the given point D, shall either pass by the angular point B, as the line DBP, or else cut AB in some point between B and A, and AC in some point between P and A. But if the said value of AK be greater than the said value of AP, then the line of partition will cut BC and AC, and in this latter Case a Parallel is to be drawn by the point D to BC, which is to be esteemed the Base, and then the given lines being respectively

A a a

AHC, whence it evidently follows that $LKCB = \triangle HCB$, and $\triangle AKL$. $LKCB :: AH : HB$. These things being premised, the Resolution of the Problem propounded may be formed in manner following.

Suppos.

4. $\triangle ABC$ is given.
5. D is a point given within the $\triangle ABC$.
6. AH and HB are in a given Reason.
7. $b = AC$ is given.
8. $c = AH$ is given.
9. $g = AE$ is given.
10. $h = ED$ ($\parallel AB$) is given.

Req. to find

- Req. to find
11. AK such a segment of AC, that KDL being drawn, it may make
12. $\triangle AKL \sim \triangle LCB :: AH \sim HB :: \triangle ACH \sim \triangle HCB$.

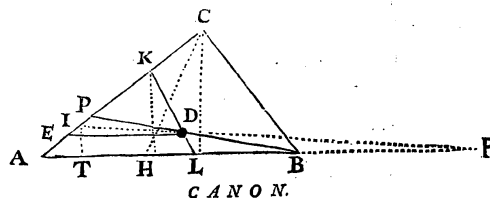
Resolution.

13. Suppose that none which is required, and put
14. Then by considering well what is required, and
viewing the Diagram, it will appear that $\triangle AKL$
 $= \triangle ACH$, and that the $\angle CAL$ is common to
both those Triangles, therefore (per prop. 15. El. 6.)
15. That is, in the letters of the Resolution, $a : c :: b : \frac{bc}{a}$.
16. And because $\triangle AKL$ and $\triangle EKD$ are like,
(for by *Confr.* in 1°, $\angle DELL$.) therefore
(per prop. 4. Elem. 6.) $AK : EK :: AL : ED$.
17. That is, in the letters of the Resolution, $a - g : a :: h : \frac{ba}{a - g}$.
18. And because the fourth Proportional in 14°
is the same with the fourth in 16°, therefore the fourth
Proportionals in 15° and 17° shall be equal to one
another, *viz.*
19. Now to avoid an Equation between Solids, let
it be made as h to b , so c to a fourth Proportional,
which may be called m , therefore
20. Whence, by comparing the Rectangle of the
extremes to the Rectangle of the means, this
Equation is produced, *viz.*
21. Therefore from 18° and 20°, by exchanging
equal Rectangles, 'tis evident that
22. Whence 'tis easie to infer that these are Proportional, *viz.* $a - g : a :: h : \frac{ba}{a - g} (= \frac{bc}{a})$.
23. But it hath been shewn in 17°, that $a - g : a :: h : \frac{ba}{a - g}$.
24. Therefore from 22° and 23°, (per prop 11. El. 5.)
25. And from 24°, by comparing the Rectangle of
the extremes to the Rectangle of the means,
26. From which Equation, after due transposition,
this ariseth, *viz.*
27. Which last Equation may be resolved into these
Proportionals, *viz.* $m - a : \sqrt{mg} :: \sqrt{mg} : a$.
28. Of which three Proportionals, the mean, to wit, \sqrt{mg} is given, as also m the sum
of the extremes, therefore the extremes shall be given severally (per Probl. 13. Chap. 5.)
the values whereof, by the Theorem in 21° of the said Probl. 13. Chap. 5. will be found
equal to these right lines, (or numbers,) *viz.*

$$\left. \begin{aligned} \frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg} : \\ \frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg} : \end{aligned} \right\} \text{the two Roots of the Equation in } z^6.$$

Hence this

CANON.



Let it be made as ED to AC, so AH to a fourth Proportional, which may be called M. Then to and from the half of M, add and subtract the square Root of the excess of the Square of half M above the Rectangle of M into AE, so shall the sum and remainder made by that Addition and Subtraction be the two values of AK sought, (represented by a in the foregoing Resolution.)

29. But to the end the said values of AK may be Real, that is, greater than nothing, the given lines must be subject to this

Determination. g not $\leq \frac{1}{4}m$, or $\frac{bc}{4b}$; that is, in words,

The line AE must not exceed the right line arising by the Application of the Rectangle made of the lines AC and AH to the quadruple of the line ED.

The truth of this Determination, which is discovered both by the Analogy in 27° , and by the values of n in 28° , may be proved thus,

It hath been discovered in 27^o, that these are Proportio- } $m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$
nals, viz. }

Of which Proportionals the sum of the extremes is
evidently m , and the mean is \sqrt{mg} ; therefore (as hath
been shewn in 20^o of *Probl. 13. Chap. 5.*)

Whence, by squaring each part it follows, that . . . } $mg \text{ not } \leq \frac{1}{4}mm.$
And by Application of each part to m , . . . } $g \text{ not } \leq \frac{1}{4}m.$

But by *Confr.* in 19°, $\left\{ \frac{bc}{b} = m. \right.$

And consequently , by taking $\frac{1}{4}$ of each part, . . . $\left\{ \begin{array}{l} bc \\ 4ab \end{array} \right. = \frac{1}{4} m$

Therefore from the three last steps, . . . g not $\leq \frac{1}{4}m$, or $\frac{bc}{4b}$.
Which was to be Demonstr.

Which was to be Demonstr.

- Which was to be Demonstr.*
- 30^e. It is also easie to perceive by the two Roots or extreme Proportionals found out in 8°, that if $g = \frac{1}{m}$, and consequently $mg = \frac{1}{mm}$, then those Roots will be equal to one another, viz. each Root equal to $\frac{1}{m}$; which, if it fall within the limits hereafter declared in the Rule in 31° of this Problem, shall be equal to the line A K sought; But if $g > \frac{1}{m}$, then the said Roots will be unequal, and the Equation in 26° may be expounded by each of those Roots; and sometimes either of them may be taken for the value of the line A K sought, but sometimes only one of the said Roots, and sometimes neither of them. To discover therefore whether there be a possibility of effecting the Problem or not, by the lines given in such manner as before is exprest, the value of the line AP must be enquired; to which end, first, draw a right line that may pass by the line A P itself be required; to which end, first, draw a right line that may pass by the line A P must be enquired; to which end, first, draw a right line that may pass by the angular point B and the given point D, (in the precedent Diagram,) then is EABD a Trapezium having (by Constr. in 1^a) two parallel sides AB and ED, and the other two sides AE and BD, which are not Parallels, being continued will meet in some point in AC, as in P. Now if AB, ED and AE be severally given in numbers, the line EP shall be also given in number, for putting $g = AE$, and $b = ED$, (as before in the Resolution,) also $k = AB$; the line EP (by the Theorem in 9° of Probl. 18. Chap. 7.) will be found equal to $\frac{gb}{k - b}$, to which adding g , that is, AE, it makes $\frac{gb}{k - b} + g$ for the value of AP.

It is

It is also evident, that if from any point in AC between E and P, a right line be drawn to pass by the given point D, as IDF, it shall necessarily cut the Base AB continued without the Triangle ABC, as in F; but if a right line be drawn from any point in AC between P and C to pass by the point D, as KDL, it shall necessarily cut the Base AB between A and B. From the premises we may infer the following

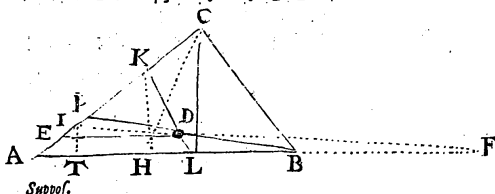
R U L E.

31. First, if $\frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg}$: and $\frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg}$: (the Roots before found out in 28° for the values of AK,) be unequal, and each of them less than AC, but neither of them less than $\frac{gk}{k-h}$, (the value of AP,) then two right lines equal to those Roots or values of AK being set from A towards C will end in two points, from which if two right lines be drawn to pass by the given point D, each of them shall divide the given Triangle ABC into two parts in the given Reason.

Secondly, if only one of those Roots or values of AK happens to be less than AC, and not less than the said value of AP, then the given Triangle ABC can be divided only by one right line passing by the given point D, so as to cut both AB and AC, or AC only, to divide the said Triangle into two parts in the given Reason.

Thirdly and lastly, if neither of the said Roots or values of AK happens to be within the limits above express'd, then 'tis impossible to draw a right line that shall pass by the given point D, and cut both AB and AC, or AC only, and pass by the angular point B, to divide the given Triangle into two parts in the given Reason: And if the like impossibility happen after a Parallel is drawn by the given point D to each of the other two sides AC, BC; and trial made as before, it shall be impossible to perform what the Problem requires; but when there is a possibility, then oftentimes which two sides you please may be cut by the line of partition.

The Composition of the foregoing Probl. 19.



- Suppos.
32. $\triangle ABC$ is given.
33. AH and HB are in a given Reason.
34. D is a point given within the $\triangle ABC$.
35. DE ($\parallel AB$) is given.
36. AE is given, and not greater than $\frac{\square AC, AH}{4 ED}$, agreeable to the preceding Determination in 29°.

Req. to find

37. AK such a segment of AC, that KDL being drawn, it may make
38. $\triangle ALK \cdot LKCB :: AH \cdot HB :: \triangle ACH \cdot \triangle HCB$.

Construction.

39. By the given point D draw $DE \parallel AB$.
40. Then supposing it be discovered by the preceding Rule in 31°, that the line of partition which is to pass by the given point D may cut AB and AC, let it be made (by Probl. 8. Chap. 5.) as ED to AC, so AH to a fourth proportional line, which may be called M, therefore, $ED \cdot AC :: AH \cdot M$.
41. Find a mean proportional line, as S, between M and AE, therefore $M \cdot S :: S \cdot AE$.
42. Divide the line M (before found) into two such parts, that the line S may be a mean Proportional

Proportional between the parts, which may be done (by Probl. 14. Chap. 5.) if S be not greater than $\frac{1}{2}M$; but that S is not greater than $\frac{1}{2}M$, I prove thus:

From the foregoing Constr. in 40°, it is easy to per-
ceive that $M = \frac{\square AC, AH}{ED}$.

Whence, by taking $\frac{1}{2}$ of each part, $\frac{1}{2}M = \frac{\square AC, AH}{4 ED}$.

By the Determination in 36°, AE not $\leq \frac{\square AC, AH}{4 ED}$.

Therefore from the two last steps, (per Ax. 3. Chap. 2.) AE not $\leq \frac{1}{2}M$.

Therefore by drawing M into each part, $\square M, AE$ not $\leq \frac{1}{4} \square M$.

And because from 41°, (per prop. 17. Elem. 6.) $\square S = \square M, AE$.

Therefore from the two last preceding steps, (per Ax. 4. Chap. 2.) $\square S$ not $\leq \frac{1}{4} \square M$.

But if one Square exceeds another, the side of the former exceeds also the side of the latter, therefore S not $\leq \frac{1}{2}M$.

from the last step,
Which was to be Demonstr. Therefore 'tis possible to cut the line M into two such parts, that the line S may be a mean Proportional between them; suppose then the right line R be found the greater part, and T the lesser, therefore these are Proportionals, viz.

$$\begin{aligned} \frac{1}{2}M - R &:: S :: S \cdot R \\ \frac{1}{2}M - T &:: S :: S \cdot T \end{aligned}$$

That is, in 27°, $m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$.
Which two lines R and T do answer to the two Roots or values of AK before express'd in the Canon in 28°, viz.

$$\begin{aligned} R &= \frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg} \\ T &= \frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg} \end{aligned} \quad \text{the values of AK.}$$

43. Now supposing that by the limits in 31° it be discovered that the greater Root or line R is less than AC, and not less than AP, let the line R be set from A towards C, as to K, viz. make AK = R, and draw the right line KDL; then shall $\triangle AKL$ be equal to $\triangle ACH$, and consequently $\triangle ALK \cdot LKCB :: AH \cdot HB :: \triangle ACH \cdot \triangle HCB$, as was required.

And if the lesser Root or line T happens to be less than AC, but not less than AP, let the line T be set also from A towards C, (as to I in the following Diagram belonging to the latter Example of this Problem,) and then a right line being drawn from the point in AC where T doth end, to pass by the given point D, it shall likewise divide the given Triangle ABC into two parts in the given Reason.

But if either of the said lines R and T, which are to be set (as before) from A towards C, happens to fall between E and P, then the line of partition to pass by D will cut the Base AB continued without the Triangle ABC, as, if AI be supposed equal to T, and the line ID be drawn and continued till it concur with the Base AB continued, as in F, then although $\triangle AIF$ be equal to $\triangle ACH$, yet it solves not the Problem, in regard part of $\triangle AIF$ lies without the given $\triangle ABC$.

It remains to prove that $\triangle ALK \cdot LKCB :: AH \cdot AB$, but this will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward (not direct) order.

44. . . . Req. demonstr. $\triangle ALK \cdot LKCB :: AH \cdot HB$.

Demonstration.

45. By Constr. in 42°, $M - R \cdot S :: S \cdot R$.
That is, in 27° (the last step of the Resolution,) $m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$.

46. Therefore from 45° (per prop. 17. Elem. 6.) $\square M - \square R = \square S$.

47. And because by Constr. in 43°, $AK = R$.

48. And from the Constr. in 41° (per prop. 17. Elem. 6.) $\square M, AE = \square S$.

49. Therefore from 46°, 47° and 48°, by exchanging equal quantities, $\square M, AK - \square AK = \square M, AE$.

- That is, in 26°, $ma - aa = mg$.

B b b

50. There-

Sixthly, because $\triangle CDI$ and $\triangle CAH$ are like, these are Proportionals, viz.

$$\begin{cases} CD : CI :: CA : CH (= 52 \frac{24602}{10000}) \\ CD : DI :: CA : AH (= 180 \frac{10224}{10000}) \end{cases}$$

Seventhly, $CH + CB = HB = 198 \frac{4104}{10000}$.
Lastly, in the $\triangle AHB$ right-angled at H , $\sqrt{AH^2 + HB^2} = AB = 268 \frac{7000}{10000}$, &c.

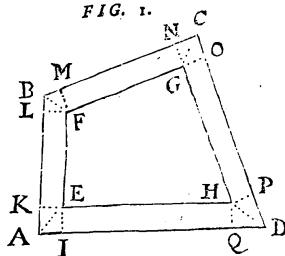
Note. When the angle ACB happens to be a right angle, then after EB and DA are found out in like manner as before in 38° and 42° of this Problem, there will be given CA and CB the sides about the right angle ACB of $\triangle ACB$; therefore (per prop. 47. Elem. 1.) the square Root of the sum of the Squares of those sides shall be the quantity of the Hypotenuse AB , to wit, the distance sought.

LEMMA 1. leading to Probl. 21.

Suppos.

FIG. 1.

1. $ABCD$ is a Trapezium.
2. $EFGH$ is a Trapezium.
3. $EF \parallel AB$.
4. $FG \parallel BC$.
5. $GH \parallel CD$.
6. $EH \parallel AD$.
7. AE, BF, CG, DH are right lines.
8. EI and $HQ \perp AD$.
9. EK and $FL \perp AB$.
10. FM and $GN \perp BC$.
11. GO and $HP \perp CD$.
12. $EI = EK = FL = FM = GN = GO = HP = HQ$.



Req. demonstr.

13. $AI = AK$. $BL = BM$. $CN = CO$. $PD = DQ$.
14. $\angle EAI = \angle EAK$. $\angle FBL = \angle FBM$. $\angle GCN = \angle GCO$. $\angle HDP = \angle HDQ$.

Demonstration.

15. By Suppos. in $8^\circ EI \perp AD$, therefore $\angle AIE = 90^\circ$.
(per defn. 10. Elem. 1.) $\square AI + \square IE = \square AE$.
16. Therefore (per prop. 47. Elem. 1.) $\square AK + \square KE = \square AE$.
17. Likewise, $\square AI + \square IE = \square AK + \square KE$.
18. Therefore from 16° and 17° , (per Ax. 1. Chap. 2.) $\square AI = \square AK$.
19. By Suppos. in $12^\circ EI = KE$, and consequently $\square AI = \square AK$.
20. Therefore from 18° and 19° , (per Ax. 9. Chap. 2.) $\square AI = \square AK$.
21. Therefore from 20° , (per Schol. prop. 48. Elem. 1.) $\square AI = \square AK$.

Which was to be Demonstr.

22. Again, because AE is common to the Triangles AIE and AKE , and the other sides of those Triangles are correspondently equal to one another, (as appears in 12° and 21°), therefore (per prop. 8. Elem. 1.) $\angle EAI = \angle EAK$.

Which was also to be Demonstr.

And by the like argumentation the truth of the rest of the Equations in 13° and 14° may be demonstrat.

LEMMA 2.

Let the same things be supposed here as before in Lemma 1. then, (respect being had to Fig. 1.)

Req.

1. . . . Req. demonstr. $\left\{ \begin{array}{l} \text{As Radius is to the sum of the Tangents of} \\ \text{So EI (or EK) to AI + LB + NC + PD.} \end{array} \right. \left\{ \begin{array}{l} AEI, \\ BFL, \\ CGN, \\ DHP; \end{array} \right.$

Demonstration.

2. By Suppos. in 12° of Lemma 1. $EI = FL = GN = HP$.
3. Therefore these four following Analogies will be manifest by a vulgar Axiom in the Doctrine of plain Triangles, viz.

$$\begin{array}{ll} \text{Radius} & EI :: \text{Tangent} < AEI & AI \\ \text{Radius} & FL, \text{ or } EI :: \text{Tangent} < BFL & LB \\ \text{Radius} & GN, \text{ or } EI :: \text{Tangent} < CGN & NC \\ \text{Radius} & HP, \text{ or } EI :: \text{Tangent} < DHP & PD \end{array}$$

4. And from those four Analogies this that follows is deducible, (per Schol. prop. 12. Elem. 5.) viz.

$$\begin{array}{ll} \text{As} & 4 \text{ Rad.} \\ \text{To} & 4 EI, \end{array} \left\{ \begin{array}{l} AEI, \\ BFL, \\ CGN, \\ DHP; \end{array} \right.$$

So is the sum of the Tangents of these angles, to wit,

$$\text{To} \dots \dots \dots AI + LB + NC + PD.$$

5. But (by prop. 15. Elem. 5.) $4 \text{ Rad.} : 4 EI :: \text{Rad.} : EI$.

6. Therefore from the two last preceding Analogies in 4° and 5° this ariseth, (per prop. 11. Elem. 5.) viz.

$$\begin{array}{ll} \text{As} & \text{Rad.} \\ \text{To} & EI, \end{array} \left\{ \begin{array}{l} AEI, \\ BFL, \\ CGN, \\ DHP; \end{array} \right.$$

So is the sum of the Tangents of

$$\text{To} \dots \dots \dots AI + LB + NC + PD.$$

7. Therefore alternately, As Radius is to the sum of the Tangents of $\left\{ \begin{array}{l} AEI, \\ BFL, \\ CGN, \\ DHP; \end{array} \right.$
So EI to $AI + LB + NC + PD$.
Which was to be Demonstr.

LEMMA 3.

Let the same things be supposed here as before in Lemma 1. then (respect being had to Fig. 1.)

Req. demonstr.

1. $\left\{ \begin{array}{l} \square EI \times AD - AI + \square EI \times AB - LB + \\ \square EI \times BC - NC + \square EI \times CD - PD \end{array} \right\} = \text{Space } AEFBCGHDA.$

That is, in words,

The Rectangle made of the parallel distance EI , ($= EK$), and the excess by which the sum of the four sides AD, AB, BC, CD of the Trapezium $ABCD$, exceeds the sum of the four segments AI, LB, NC, PD , is equal to the Interval or Space $AEFBCGHDA$.

Demonstration.

2. By Suppos. in 8° and 12° of Lemma 1. EI and $HQ \perp IQ$. Also, $EI = HQ$.
3. Therefore (per prop. 27, 33, 34. Elem. 1.) $EH = IQ$.
4. It is evident by Fig. and Lemma 1. respect being had to the last Equation, that $IQ = EH = AD - AI - QD$ (PD.)
5. And by adding AD to each part of the last Equation, $AD + EH = 2AD - AI - QD$ (PD.)
6. And by arguing as in 4° and 5° , this Equation will be manifest, viz. $AB + EF = 2AB - AK (AI) - LB$.
7. Likewise,

C c c

7. Likewise,

7. Likewise, $BC + FG = 2BC - BM(LB) - NC.$
 8. Likewise, $CD + GH = 2CD - CO(NC) - PD(QD).$
 9. And by comparing the half sum of the first parts of the four last preceding Equations to the half sum of the latter parts, this Equation aritheth, viz.

$$\left. \begin{aligned} \frac{1}{2}AD + \frac{1}{2}EH \\ \frac{1}{2}AB + \frac{1}{2}EF \\ \frac{1}{2}BC + \frac{1}{2}FG \\ \frac{1}{2}CD + \frac{1}{2}GH \end{aligned} \right\} = \left. \begin{aligned} AD - AI \\ AB - LB \\ BC - NC \\ CD - PD \end{aligned} \right\}$$

 10. By Suppos. the Trapezium AEHD hath two parallel sides AD and EH, and $EI \perp AD$, therefore (per Theor. 2. in 13° of Probl. 18. Chap. 7.)

$$\square EI \times \frac{1}{2}AD + \frac{1}{2}EH = \text{Trapez. AEHD.}$$

 11. In like manner, (respect being had to the Suppos. in 12° of Lemma 1.)

$$\square EI \times \frac{1}{2}AB + \frac{1}{2}EF = \text{Trapez. AEFB.}$$

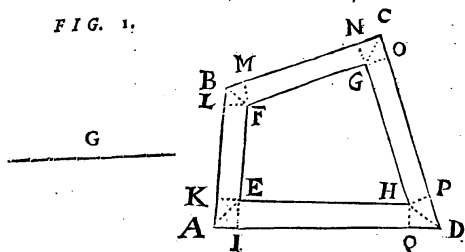
 12. Likewise, $\square EI \times \frac{1}{2}BC + \frac{1}{2}FG = \text{Trapez. BFGC.}$
 13. Likewise, $\square EI \times \frac{1}{2}CD + \frac{1}{2}GH = \text{Trapez. DHGC.}$
 14. By viewing Fig. 1. it will be evident, that the sum of the four Trapezia express'd in the four last preceding Equations is equal to the Interval or Space AEFB CGHDA, therefore from 9°, 10°, 11°, 12° and 13°, by exchanging equal quantities this Equation aritheth, viz.

$$\left. \begin{aligned} \square EI \times AD - AI + \square EI \times AB - LB + \\ \square EI \times BC - NC + \square EI \times CD - PD \end{aligned} \right\} = \text{Space AEFB CGHDA.}$$

 Which was to be Demonstr.

L E M M A 4.

FIG. 1.



Let the same things be suppos'd here as in Lemma 1. and let the line G be found out by this Analogy, viz.

1. As the sum of the Tangents of
 To
 So
 To a fourth Proportional
 2. Req. demonstr. $\frac{1}{2}G \square EI$, (the parallel distance.)

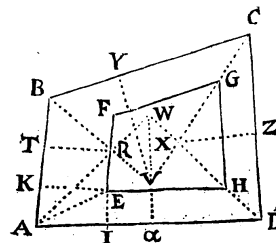
Demonstration.

3. By inverting the Terms of the Analogy demonstrated in 7° of the foregoing Lemma 2. these are Proportionals, viz.
 4. As the sum of the Tangents of
 To
 So is
 To the parallel distance
 5. Therefore

5. Therefore from 1° and 4°, (per prop. 11. Elem. 5.)
 $AB + BC + CD + DA : G :: AI + LB + NC + PD : EI.$
 6. But by prop. 15. Elem. 5.
 $AB + BC + CD + DA : G :: \frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA : \frac{1}{2}G.$
 7. Therefore from 5° and 6°, (per prop. 11. Elem. 5.)
 $\frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA : \frac{1}{2}G :: AI + LB + NC + PD : EI.$
 8. In which last Analogy the first Term is greater than the third, for 'tis evident by Fig. 1. and by what hath been demonstrated in Lemma 1. that the double of the first Term is greater than the double of the third, in regard the double of the third is but part of the double of the first: Therefore (per prop. 14. Elem. 5.) the second Term is greater than the fourth, viz.
 $\frac{1}{2}G \square EI$ (the parallel distance.)
 Which was to be Demonstr.

L E M M A 5.

FIG. 2.



If the angles A, B, C, D of the Trapezium ABCD be severally divided into two equal parts by the right lines AW, DW, BV, CV; and if from the points W, R, V and X where every two of the said lines that lyx next to one another do intersect, four right lines W_a , RT, VY and XZ be drawn perpendicularly upon AD, AB, BC and CD; and if the Perpendicular RT be shorter, or, if not shorter, yet not longer than any one of the other three Perpendiculars VY, XZ and W_a : And lastly, if a Trapezium be made within the before-mentioned Trapezium ABCD, so, as that the sides of the one are parallel to the sides of the other, and every where separated by an equal parallel distance: Then I say that the said parallel distance shall be less than the said Perpendicular RT.

For if it be said that the parallel distance is equal to the Perpendicular RT, then by the point R let ERF be drawn parallel to AB, and finish the Trapezium EFGH so as FG may be parallel to BC, likewise GH || CD and EH || AD; draw also AE: Now if the sides of the interior Trapezium EFGH be every where distant from the sides of the exterior Trapezium by an equal parallel distance, then by Lemma 1. the angle EAI is equal to the angle EAK; but by Supposition in this Lemma 5. $\angle RAI = \angle RAK$, and how thortfover the parallel line ERF be drawn, the angle EAI will be but part of, and consequently less than $\angle RAI$, ($= \angle RAK$), as is evident by Fig. 2. therefore the angle EAI cannot be equal to the angle EAK, and consequently it contradicts Lemma 1. before demonstrated. The like contradiction will ensue if a Parallel be drawn to AB at a distance greater than RT. Wherefore I conclude, that if a Trapezium be made within a Trapezium in such manner as is above suppos'd, the parallel distance shall be less than RT, which is suppos'd to be shorter, or, if not shorter, yet not longer than any one of the three Perpendiculars VY, XZ and W_a before mentioned.

Probl. XXI.

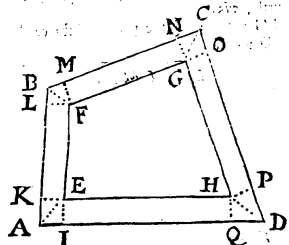
A Trapezium being given by Position, as also the quantities of all its sides and angles severally, to make a Trapezium within the former, in such manner that the sides of the one may be every where separated from

Ccc 2.

from the sides of the other by an equal parallel distance, and that the Space lying between the sides of both the Trapezia may be equal to any possible Space (or Figure) given.

Note. This Problem hath two Cases, viz. First, when each of the Diagonals of the given Trapezium lyes within the same; Secondly, when one of the Diagonals lyes without. I shall handle the first Case only, for this well understood, will be a sufficient light to shew the industrious Analyst how to solve the latter Case, which is briefly done by the Learned *Fran. à Schooten*, in his *Traicté de conimensables Demonstrationibus ex Entenlo Algebraico*.

FIG. 1.



Suppos.

1. ABCD is a Trapezium given, whose Diagonals AC and BD do lye within the same.
 2. e = a right line given, equal to the sum of AB, BC, CD, DA.
 3. r = the Radius, or Semidiameter of a Circle is given.
 4. s = a right line given equal to the sum of the Tangents of the angles AEI, BFL, CGN and DHP, agreeable to the Radius r ; which angles are the complements of the halves of the given angles of the Trapezium ABCD.
 5. g = the side of a given Square equal to the Interval or Space lying between the sides of the given Trapezium ABCD, and the sides of the Trapezium required to be made within the former.
- Req. to make*
6. EFGH a Trapezium, such that EF || AB, FG || BC, GH || CD, and EH || AD. Also,
 7. Space AEFBCGHDAE = □ g , (or ab .) Also,
 8. The Perpendiculars EI, FL, GN and HP to be equal between themselves, that there may be an equal parallel distance between the sides of the Trapezium given, and of that required.

Resolution of CASE 1.

9. Suppose that done which is required, and put a for \square EI = EK = FM = GO, the parallel distance, viz.
10. Then by the foregoing Lemma 2. this Analogy is manifest, viz.

As Radius, to the sum of the Tangents of

$\begin{matrix} \angle AEI, \\ \angle BFL, \\ \angle CGN, \\ \angle DHP; \end{matrix}$

So is the parallel distance EI, to . . . AI + LB + NC + PD.

11. Therefore in the letters of the Resolution,

$$r : s :: a : \frac{se}{r} \quad (= AI + LB + NC + PD.)$$

12. By the preceding Lemma 3. this Equation is manifest, viz.

$$\begin{aligned} \square EI \times AD - AI + \square EI \times AB - LB + \square EI \times BC - NC + \square EI \times CD - PD &= \text{Space AEFBCGHDA.} \end{aligned}$$

13. Therefore in the letters of the Resolution,

$$\frac{se}{r} \times e - \frac{se}{r} = bb \quad (= \text{Space AEFBCGHDA.})$$

14. Which

14. Which Equation in 13° may be resolved into these Proportionals, viz. $b \cdot c - \frac{sa}{r} :: a \cdot b$.
15. And by drawing r as a common Factor into each of the two first Terms of the last preceding Analogy, this aritheth, viz. $br \cdot cr - sa :: a \cdot b$.
16. Now to avoid an Equation between Solids, which would arise by comparing the Rectangle of the extremes to the Rectangle of the means of the last Analogy, let it be made, as s to r , so b to a fourth Proportional, which may be called d , therefore $s : r :: b : d$.
17. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, viz. $sd = br$.
18. Again, let it be made, as s to r , so c to a fourth Proportional, call it g , therefore $s : r :: c : g$.
19. Therefore from the last Analogy, by comparing the Rectangle of the extremes to the Rectangle of the means, $sg = cr$.
20. Therefore from 15° , 17° and 19° , by exchanging equal Rectangles, this Analogy aritheth, viz. $sd \cdot sg - sa :: a \cdot b$.
21. Therefore from 20° , by casting the common Factor s out of the two first Terms, $d \cdot g - a :: a \cdot b$.
22. From which last Analogy, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation is produced, viz. $ga - aa = bd$.
23. Which Equation may be resolved into these Proportionals, viz. $g - a : \sqrt{bd} :: \sqrt{bd} : a$.
24. But of those three continual Proportionals, the mean, to wit, \sqrt{bd} is given, as also the sum of the extremes $g - a$ and a , therefore (per Probl. 13. Chap. 5.) the extremes shall be given severally, which by the Theorem in 21° of the said Probl. 13. are equal to these right lines, (or numbers,) viz.

$$\frac{1}{2}g + \sqrt{\frac{1}{4}gg - bd} : \frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd} :: \text{the extreme Proportionals in } 23^{\circ}.$$

Which extreme Proportionals, (or Roots of the Equation in 23° .) are equal one to the other when $\frac{1}{2}g = \sqrt{bd}$, in which Case each of the said extremes is evidently equal to $\frac{1}{2}g$; but if $\frac{1}{2}g < \sqrt{bd}$, then the said extremes are unequal, as they happen to be in the Resolution of this Problem, but the lesser of them only (for the reason hereafter given in 26°) shall be the parallel distance sought. Hence this

CANON.

25. $\frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd} = EI = EK$ the parallel distance. That is, in words,

Let it be made, As (s) the sum of the Tangents of the complements of the halves of the angles of the given Trapezium ABCD, is to the Radius (r); So (b) the given side of a Square equal to the prescribed Interval or Space between the sides of both the Trapezia, to a fourth Proportional (d); and so (c) the sum of the four sides of the given Trapezium to a fourth Proportional (g). Then subtract the Square Root of the excess whereby the Square of half g exceeds the Rectangle made of b and d , from half g ; the remainder shall be EI (= EK) the parallel distance sought.

Which Canon may be propounded in the form of a Theorem, the Demonstration whereof may be easily framed by a repetition of the steps of the foregoing Resolution, by proceeding in a direct order from the beginning to the end thereof, for the Argumentation therein used is clearly Geometrical, as well as Arithmetical: But waving the Demonstration of the Canon, I shall in the next place shew what Determinations are necessary for limiting the given lines, that there may be a possibility of effecting the Problem propounded.

Determinat. 1.

$$26. \dots \frac{1}{2}g < \sqrt{bd}; \text{ that is, } \frac{cr}{2s} < \sqrt{b \times \frac{br}{s}};$$

Although

Although there be a possibility to find out the extreme Proportionals in 24° , (or the two Roots of the Equation in 22.) if $\frac{1}{2}g$ be not less than \sqrt{bd} , yet the parallel distance is not possible unless $\frac{1}{2}g$ be greater than \sqrt{bd} ; for if $\frac{1}{2}g = \sqrt{bd}$, then each of the said extreme Proportionals is equal to $\frac{1}{2}g$; but by the preceding Lemma 4. $\frac{1}{2}g$ is greater than E I or E K (the parallel distance, therefore neither of those equal extremes can be the parallel distance: But if $\frac{1}{2}g < \sqrt{bd}$, as Determinat. 1. requires, then the two extreme Proportionals before mentioned are unequal; for if $\frac{1}{2}g$, that is, half the sum of the Proportionals of three Proportionals be greater than the mean, to wit, \sqrt{bd} , then the extremes are unequal, and consequently in such Case $\frac{1}{2}g$ is greater than the lesser extreme; therefore agreeable to Lemma 4. the lesser extreme shall be the parallel distance required; but the greater extreme is to be neglected, because 'tis greater than $\frac{1}{2}g$, and consequently doth contradict the said Lemma 4.

Determinat. 2.

$$27. \dots \frac{1}{2}g - \sqrt{\frac{1}{2}gg - bd} = RT.$$

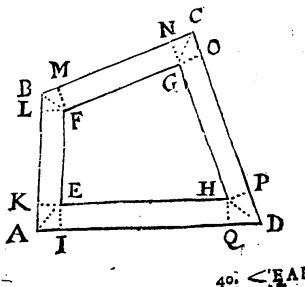
The perpendicular line R T in Fig. 2. prefix'd before the precedent Lemma 5. is supposed to be shorter, or if not shorter, yet not longer than any one of the other three Perpendiculars V Y, X Z and W a. But the quantities of the said four Perpendiculars may be found out in numbers by the Doctrine of Plain Triangles; for in each of the Triangles ARB, BVC, CXD and AWD, the Base is given, as also the angles at the Base, to find the Perpendicular. Which Bases are the given sides of the Trapezium ABCD, and the angles at the Bases are the halves of the given angles of the said Trapezium, therefore the said Perpendiculars are given also, whence it may be known which of them is the shortest. Now because by the foregoing Canon in 25° , and by what hath been said in 26° , the parallel distance is found equal to $\frac{1}{2}g - \sqrt{\frac{1}{2}gg - bd}$: and because by the preceding Lemma 5. the said parallel distance must be less than R T, which is supposed to be the shortest of the said four Perpendiculars, or if not the shortest, yet not longer than any one of the other three, therefore $\frac{1}{2}g - \sqrt{\frac{1}{2}gg - bd}$ must be less than R T; otherwise that cannot be done which the Problem requires.

But if the given lines be qualified according to the import of the precedent Determinations, then the parallel distance E I or E K shall be given by the foregoing Canon; and then, after the angles A, B, C, D of the given Trapezium ABCD in Fig. 1. are severally divided into two equal parts by the right lines A E, B F, C G and D H; and after right lines are drawn parallel to the sides A B, B C, C D, D A, at the distance of the said E I or E K, the points where those Parallels do concur in the lines bisecting the said angles will form the desired Trapezium EFGH: And it will not be difficult to demonstrate, by form a repetition of the steps of the foregoing Resolution in a backward order, (in like manner as in divers preceding Problems of this Chapter,) that the Area of the Interval or Space lying between the sides of both the Trapezia is equal to the Square of the given side b . But leaving the Composition of this Problem to the Learners practice, I shall prove the truth of its Solution by an Example in Numbers.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 21.

Suppos.

28. ABCD is a Trapezium.
29. EFGH is a Trapezium.
30. $\begin{cases} EH \parallel AD. EF \parallel AB. \\ FG \parallel BC. GH \parallel CD. \end{cases}$
31. A E, B F, C G, D H are right lines.
32. E I and H Q \perp A D.
33. E K and F L \perp A B.
34. F M and G N \perp B C.
35. G O and H P \perp C D.
36. $\begin{cases} AD = 40 \\ AB = 24 \\ BC = 29 \\ CD = 36 \end{cases}$ are given.



$$40. \angle EAI$$

	Gr.	Min.	
40. $\angle EAI =$	$\angle EAK =$	$44 : 6.$	Are given.
41. $\angle FBL =$	$\angle FBM =$	$55 : 56.$	
42. $\angle GCN =$	$\angle GCO =$	$44 : 43.$	
43. $\angle HDP =$	$\angle HDQ =$	$35 : 15.$	
44. $\angle AEI =$	$\angle AEK =$	$45 : 54.$	These are consequently given, because the angles at I, K, L, M, N, O, P, Q are right angles.
45. $\angle BFL =$	$\angle BFM =$	$34 : 4.$	
46. $\angle CGN =$	$\angle CGO =$	$45 : 17.$	
47. $\angle DHP =$	$\angle HDQ =$	$54 : 44.$	

$$48. c = 129 = AD + AB + BC + CD \text{ is given; (from } 36^\circ, 37^\circ, 38^\circ, 39^\circ)$$

$$49. bb = 4507000 \text{ is given for the Area of the Space AEFB CGHDA.}$$

$$50. r = 100000 \text{ the Radius of a Circle is given.}$$

Req. to find

$$51. EI = FL = GN = HP, \text{ (the parallel distance.)}$$

Solution Arithmetical.

Gr. Min.

52. 103192 = Tangent of $45 : 54 =$	$\angle AEI =$	$\angle AEK.$
53. 67620 = Tangent of $34 : 04 =$	$\angle BFL =$	$\angle BFM.$
54. 100994 = Tangent of $45 : 17 =$	$\angle GCN =$	$\angle CGO.$
55. 141409 = Tangent of $54 : 44 =$	$\angle DHP =$	$\angle HDQ.$

$$56. 413215 = s = \text{the sum of those four Tangents.}$$

Then according to the foregoing Canon in 25° , let it be made,

$$57. \text{As } \begin{matrix} s \\ 413215 \end{matrix} : \begin{matrix} r \\ 100000 \end{matrix} :: \begin{matrix} b \\ 4507000 \end{matrix} : \begin{matrix} d \\ 570000, \&c. \end{matrix}$$

$$58. \text{As } \begin{matrix} s \\ 413215 \end{matrix} : \begin{matrix} r \\ 100000 \end{matrix} :: \begin{matrix} c \\ 129 \end{matrix} : \begin{matrix} E. \\ 317000, \&c. \end{matrix}$$

Lastly, by the precedent Canon in 25° ,

$$59. \frac{1}{2}g - \sqrt{\frac{1}{2}gg - bd} = 4 \text{ (very near)} = EI = EK.$$

The Proof.

To r , s and EI find a fourth Proportional, (agreeable to the foregoing Lemma 2.) viz.

$$60. \text{As } \begin{matrix} r \\ 100000 \end{matrix} : \begin{matrix} s \\ 413215 \end{matrix} :: \begin{matrix} EI \\ 4 \end{matrix} : \begin{matrix} AI + LB + NC + PD. \\ 16710000. \end{matrix}$$

Then (by Lemma 3.)

$$EI \times \begin{Bmatrix} AD - AI + AB - LB + \\ BC - NC + CD - PD \end{Bmatrix} = \text{Space AEFB CGHDA.}$$

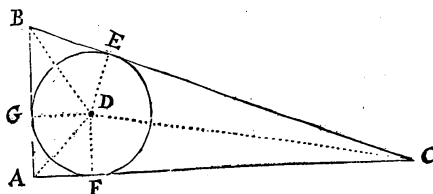
$$\text{That is, } 4 \text{ into } 129 - 16710000 = 4497000, \&c.$$

Which Area doth not want $\frac{1}{10}$ of 4507000 the prescribed Area of the Interval AEFB CGHDA, the defect arising from this, that such numbers as were found out near the truth, were assumed to be exactly true, to avoid tediousness of Calculation. Therefore the truth of the Solution of the Problem propounded is evident.

Probl. XXII.

In a right-angled plain Triangle, the Area being given, as also the Perimeter, (that is, the sum of all the three sides,) to find out the Triangle. But the quadruple Area must be less than the Square of the Perimeter.

Suppos.



Suppos.

1. $\triangle ABC$ is right-angled at A.
2. $\square^{\frac{1}{2}} AC, AB$ = the Area of $\triangle ABC$ is given.
3. $AC + AB + BC$ = the Perimeter is given.

Req. to find out AC, AB, BC severally.

Prepar.

4. Supposing ABC to be the Triangle sought, bisect (by *Probl. 9. Elem. 1.*) the angles BAC and ABC by the lines AD and BD meeting in D , and make $DF \perp AC$, $DG \perp AB$, $DE \perp BC$; and draw DC : Then is $DF (= DG = DE)$ the Semidiameter of the inscribed Circle $DFGE$, which toucheth the Triangle in the points F , G and E , (per *prop. 4. Elem. 4.*)

Resolus. I.

5. From the premises 'tis easie to perceive, that in any plain Triangle, the Semiperimeter multiplied by the Semidiameter of the inscribed Circle produceth the Area of the Triangle, and consequently the Area divided by the Semiperimeter gives the Semidiameter of the inscribed Circle: As in $\triangle ABC$ before expos'd, the Rectangle (or Product) of $\frac{1}{2} AC$ into DF is equal to the $\triangle ADC$, (per *prop. 4. Elem. 1.*) Likewise, $\square^{\frac{1}{2}} AB, DG (DF) = \triangle ABD$: Also, $\square^{\frac{1}{2}} BC, DE (DF) = \triangle BCD$; and those three Triangles are evidently equal to $\triangle ABC$.
6. Moreover, in every right-angled plain Triangle, if the Diameter of the inscribed Circle be subtracted from the Perimeter, the remainder is equal to the double of the Hypotenuse: As in $\triangle ABC$ before expos'd, if $AF + AG, (= DF + DG =$ the Diameter of the inscribed Circle $DFGE,)$ be subtracted from the Perimeter $AC + AB + BC$, there evidently remains $FC + GB + BC = 2 BE + 2 EC = 2 BC$; for $FC = EC$, and $GB = BE$. Therefore, From 5° and 6° we may deduce

CANON 1.

7. Divide the given Area by the Semiperimeter, and subtract the double of the Quotient from the whole Perimeter; the half of the remainder shall be the Hypotenuse, which subtracted from the Perimeter leaves the sum of the sides about the right angle. Then the Hypotenuse being given, as also the sum of the sides about the right angle, the sides shall be given severally, both Geometrically and Arithmetically, by *Probl. 4. Chap. 8.*

Another way to find out the Hypotenuse.

Suppos.

8. $\triangle ABC$ is right-angled at A.
9. $cc = \square^{\frac{1}{2}} AC, AB = \triangle ABC$ is given.
10. $b = AC + AB + BC$ is given.

Req. to find BC the Hypotenuse.

Resolus. II.

11. Put a for the Hypotenuse, viz. $a = BC$.
12. Therefore from 10° and 11° , the sum of the sides about the right angle is $b - a (= AC + AB)$.
13. And from 11° the Square of the Hypotenuse is aa .
14. And from 12° , the Square of the sum of the sides about the right angle is $aa - 2ba + bb$.

15. Therefore

15. Therefore from $9^\circ, 13^\circ$ and 14° , (by *Theor. 1. in 12^\circ* of *Probl. 15. Chap. 8.*) this Equation ariseth, viz. $aa - 2ba + bb = aa + 4cc$.
16. From which Equation reduced, the Hypotenuse will be made known, viz. $a = \frac{1}{2}b - \frac{2cc}{b}$.

Hence

CANON 2.

17. From the Semiperimeter subtract the Quotient of the double Area divided by the Perimeter, and the remainder shall be the Hypotenuse. Then the sides about the right angle shall be given as before in Canon 1.
18. But to the end there may be a possibility of finding out a right-angled Triangle to solve the Problem propounded, it is evident by Canon 2. that the given quantities b and cc must be such, that $\frac{1}{2}b < \frac{2cc}{b}$.
19. Whence, by doubling each part, $b < \frac{4cc}{b}$.
20. Therefore from 19° , by multiplying each part into b , $bb < 4cc$.
21. And consequently, $4cc > bb$.

Therefore the reason of the Determination added to *Probl. 22.* is manifest, and the Canons may be exemplified by any right-angled Triangle in Rational numbers.

CHAP. X.

The fourth Class of Examples of the Resolution and Composition of Plane Problems.

IN which Examples, the Resolution ends in an Analogy consisting of three Squares in continual Proportion, whereof the Mean is given; as also a Square equal either to the Difference, or else to the Summ of the Extremes; and therefore the Extremes shall be given severally, by *Probl. 15, or 16. of Chap. 5.*

Probl. I.

The difference of the Squares and the Rectangle of two right lines being given severally, to find out those lines.

Suppos.

1. d = the given side of a Square equal to the difference of the Squares of two right lines.
2. m = the given side of a Square equal to the Rectangle of the same lines.

Req. to find out the lines.

Resolution.

3. For the lesser of the two right lines sought put a .
4. Therefore the Square of the lesser line is aa .
5. And from 1° and 4° , the Square of the greater line is $aa + dd$.
6. Therefore from 5° the greater line is $\sqrt{aa + dd}$.
7. And from 3° and 6° the Rectangle made of the two right lines sought is $a \times \sqrt{aa + dd}$.
8. Which Rectangle (according to the import of the Problem) must be equal to the Square of the given right line m ; therefore $a \times \sqrt{aa + dd} = mm$.
9. And that Equation may be resolved into these Proportionals, viz. $\sqrt{aa + dd} : m :: m : a$.

Which Analogy is qualified in every respect like that in 60° of *Probl. 15. Chap. 5.* therefore the lines sought, which are represented by a and $\sqrt{aa + dd}$, shall be given by the Geometrical Construction of the said *Probl. 15.* and their quantities in Numbers shall be given also by the Canon in 77° of the same Problem.

D d d

Probl. II.

28. And by resolving the last Equation into Proportionals, this Analogy is manifest, viz.
 $\sqrt{\frac{1}{2}} \square AR - \frac{1}{2} \square AG : M :: M : \sqrt{\frac{1}{2}} \square AD - \frac{1}{2} \square AR$:
 29. The Squares of which Proportionals are also Proportionals, (per prop. 22. Elem. 6.) viz.
 $\frac{1}{2} \square AR - \frac{1}{2} \square AG : \square M :: \square M : \frac{1}{2} \square AD - \frac{1}{2} \square AR$:
 30. And by quadrupling all in 29,
 $\square AR - \square AG : 4 \square M :: 4 \square M : \square AD - \square AR$:
 31. But the sides of proportional Squares are also Proportionals, therefore from 30,
 $\sqrt{\frac{1}{2}} \square AR - \square AG : 2 M :: 2 M : \sqrt{\frac{1}{2}} \square AD - \square AR$:
 32. And because by Constr. in 23°, . . . $S = \sqrt{\frac{1}{2}} \square AR - \square AG$:
 33. Also by Constr. in 24°, . . . $T = \sqrt{\frac{1}{2}} \square AD - \square AR$:
 34. Therefore from 31°, 32° and 33°, by exchanging equal right lines, this Analogy is manifest, viz.

$$S : 2 M :: 2 M : T.$$

35. And because the last Analogy consists of three Proportionals, therefore by what hath been said in 20° of Probl. 13. Chap. 5.

$$4 M \text{ not } \leq S + T.$$

36. And by squaring each part in 35°,
 $16 \square M \text{ not } \leq \square S + \square T + 2 \square S, T$:
 37. And because by Constr. in 23°, . . . $\square AR - \square AG = \square S$:
 38. And by Constr. in 24°, . . . $\square AD - \square AR = \square T$:
 39. Therefore by adding the two last Equations together,
 $\square AD - \square AG = \square S + \square T$:
 40. Therefore from 36° and 38°, by exchanging equal Squares,
 $16 \square M \text{ not } \leq \square AD - \square AG + 2 \square S, T$:
 41. But from 34°, (per 17. prop. Elem. 1.) . . . $4 \square M = \square S, T$:
 42. And consequently, . . . $8 \square M = 2 \square S, T$:
 43. Therefore from 40° and 42°,
 $16 \square M \text{ not } \leq \square AD - \square AG + 8 \square M$:
 44. Wherefore from 43°, by subtracting $8 \square M$ from each part,
 $8 \square M \text{ not } \leq \square AD - \square AG$.

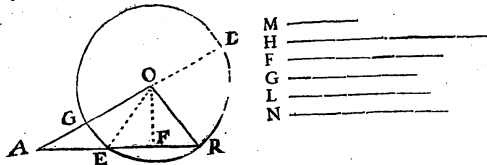
Which was to be Demonstr. Therefore the truth both of the Lemma and Determination is manifest.

The Composition of Probl. 2.

Suppos.

45. AO and OR the legs of the Triangle ARO are given severally.
 46. AO = OR.
 47. AD = AO + OR is given.
 48. AG = AO - OR is given.
 49. M = the given side of a Square equal to $\triangle ARO$.
 50. $8 \square M \text{ not } \leq \square AD - \square AG$. (Determination.)

Req. to find out the Triangle.



Construction.

51. By Probl. 4. Chap. 5. find a right line H, such, that its Square may be equal to $\square AD - \square AG$; therefore,
 $\square H = \square AD - \square AG$.

52. Then

53. Then making H to be the Hypothenusal of a right-angled Triangle, and 2M a mean Proportional between the Base and Perpendicular, find out (per Probl. 16. Chap. 5.) the Base and Perpendicular; but such a Triangle cannot be found out unless it be proved that $\frac{4 \square M}{H}$ is not greater than $\frac{1}{2} H$; for, (agreeable to the Determination annexed to the said Probl. 16.) the right line arising by the Application of the Square of the given mean Proportional to the given Hypothenusal, must not be greater than half the Hypothenusal; I shall therefore first prove that $\frac{4 \square M}{H}$ is not greater than $\frac{1}{2} H$.

$$\text{Req. demonstr.} \dots \dots \dots \frac{4 \square M}{H} \text{ not } \leq \frac{1}{2} H.$$

Demonstration.

- By the Determination in 50°, . . . $8 \square M \text{ not } \leq \square AD - \square AG$.
 And by Constr. in 51°, . . . $\square H = \square AD - \square AG$.
 Therefore, (per Ax. 3. Chap. 2.) . . . $8 \square M \text{ not } \leq \square H$.
 And consequently, . . . $4 \square M \text{ not } \leq \frac{1}{2} H$.
 Wherefore by Application of each part to H, $\frac{4 \square M}{H} \text{ not } \leq \frac{1}{2} H$.

Which was to be Demonstr.

53. Having proved that 'tis possible to find out a right-angled Triangle which shall have the right line H for a Hypothenusal, and the double of the right line M for a mean Proportional between the Base and Perpendicular, suppose that (by Probl. 16. Chap. 5.) the right lines F and G are found equal to the said Base and Perpendicular; therefore by such Construction,

$$F : 2 M :: 2 M : G.$$

$$\text{Also,} \dots \dots \dots \square F + \square G = \square H.$$

54. By Probl. 2. Chap. 5. find out a right line L, such, that its Square may be equal to $\square AG + \square G$; therefore, $\square L = \square AG + \square G$.
 55. Likewise, find out a right line N, such, that its Square may be equal to $\square AG - \square F$; therefore, $\square N = \square AG - \square F$.
 56. Now either of the said right lines N and L, (being the two values of α in the foregoing Resolution,) shall be the Base of a Triangle, to satisfy the Problem propounded; therefore let a Triangle, as ARO be made, so, that its Base AR may be equal to the right line N, and the legs equal to the given right lines AO, OR, which may be done, (by prop. 22. Elem. 1.) if those three right lines AR, AO and OR be such that every two taken together as one right line, is longer than the third; but that the sum of every two of the said right lines is longer than the third, I demonstrate thus:
 57. . . . I. Req. demonstr. . . . $AR + AO \leq OR$.

Demonstration.

58. Forasmuch as by Supposition in 46°, . . . $AO \leq OR$.
 59. Therefore much more . . . $AR + AO \leq OR$.
 60. . . . II. Req. demonstr. . . . $AR + OR \leq AO$.

Demonstration.

61. By Constr. in 55°, . . . $\square AG + \square F = \square N$.
 62. Also by Constr. in 56°, $AR = N$, and consequently, . . . $\square AR = \square N$.
 63. Therefore from 61° and 62°, (per Ax. 1. Chap. 2.) . . . $\square AR = \square AG + \square F$.
 64. Therefore from 63°, . . . $\square AR \leq \square AG$.
 65. And consequently, . . . $\square AR \leq \square AG$.
 66. But by Suppos. in 48°, . . . $\square AG = \square AO - \square OR$.
 67. Therefore from 65° and 66°, (per Ax. 3. Chap. 2.) . . . $\square AR \leq \square AO - \square OR$.
 68. Wherefore by adding OR to each part in 67°, $\square AR + \square OR \leq \square AO$.

Which was to Demonstr.

69. III. Req.

69. . . . III. *Req. demonstr.* $AO + OR \sqsubset AR.$

Demonstration.

70. By *Construction* in 51° , $\square AD - \square AG = \square H.$
 71. Also by *Constr.* in 53° , $\square F + \square G = \square H.$
 72. Therefore from 70° and 71° , (*per Ax. 1.*) . . . $\square AD - \square AG = \square F + \square G.$
Chap. 2. }
 73. And from 72° , by adding $\square AG$ to each } $\square AD = \square F + \square G + \square AG.$
part, }
 74. And from 73° , by equal subtraction of $\square G$, } $\square AD - \square G = \square F + \square AG.$
 75. But by *Constr.* in 55° , $\square N = \square F + \square AG.$
 76. Therefore from 74° and 75° , (*per Ax. 1.*) . . $\square AD - \square G = \square N.$
Chap. 2. }
 77. And because by *Constr.* in 56° , $AR = N$, } . . . $\square AR = \square N.$
 and consequently, }
 78. Therefore from 76° and 77° , (*per Ax. 1.*) . . $\square AD - \square G = \square AR.$
Chap. 2. }
 79. And by adding the $\square G$ to each part in 78° , } $\square AD = \square AR + \square G.$
 80. By 79° 's evident, (*per Ax. 25. Chap. 2.*) } $\square AD \sqsubset \square AR.$
that, } $AD \sqsubset AR.$
 81. And consequently, $AD = AO + OR.$
 82. And by *Suppos.* in 47° , $AO + OR \sqsubset AR.$
 83. Wherefore from 81° and 82° , (*per Ax. 4.*) . . $AO + OR \sqsubset AR.$
Chap. 2. }

Which was to be Demonstr.

84. Having proved that of the said three right lines AR , AO and OR every two taken together as one right line is longer than the third, it is possible to make a Triangle of them, (by *prop. 22. Elem. 1.*) suppose it then to be $\triangle ARO$; I say the Triangle so made is that which was required. Now we must shew that it will satisfy the Problem propounded.
 85. First then, by *Construction* in 84° , the legges AO and OR are equal to the two right lines given in 45° to be the legges of the Triangle sought; and that the $\triangle ARO$ is equal to the Square of the given right line M , the following Demonstration, formed by a retrograde repetition of the steps of the Resolution foregoing, will make manifest.

86. . . . *Req. demonstr.* $\triangle ARO = \square M.$

Demonstration.

87. By *Construction* in 53° , $F \cdot 2M :: 2M \cdot G.$
 88. Therefore, (*per 22. prop. 6. Elem.*) $\square F \cdot 4 \square M :: 4 \square M \cdot \square G.$
 89. By *Constr.* in 55° , $\square N = \square AG + \square F.$
 90. Also by *Constr.* in 56° , $N = AR$, and consequently, $\square N = \square AR.$
 91. Therefore from 89° and 90° , (*per Ax. 1.*) . . $\square AR = \square AG + \square F.$
 92. And from 91° , by equal subtraction of $\square AG$, } $\square AR - \square AG = \square F.$
 93. It hath been proved in 79° , that $\square AD = \square AR + \square G.$
 94. And consequently, by subtracting $\square AR$ from each part, $\square AD - \square AR = \square G.$
 95. Therefore from 88° , 92° and 94° , by exchanging equal Planes, this Analogy will be manifest, *viz.* $\square AR - \square AG \cdot 4 \square M :: 4 \square M \cdot \square AD - \square AR.$
 That is, in 11° , $aa - dd \cdot 4mm :: 4mm \cdot cc - aa.$
 96. And by taking a quarter of all in 95° , . . $\frac{1}{4} \square AR - \frac{1}{4} \square AG \cdot \square M :: \square M \cdot \frac{1}{4} \square AD - \frac{1}{4} \square AR.$
 That is, in 10° , $\frac{1}{4} aa - \frac{1}{4} dd \cdot mm :: mm \cdot \frac{1}{4} cc - \frac{1}{4} aa.$
 97. And because (*per prop. 22. Elem. 6.*) the sides of proportional Squares are also Proportionals, therefore from 96° ,
 $\sqrt{\frac{1}{4} \square AR - \frac{1}{4} \square AG} : M :: M : \sqrt{\frac{1}{4} \square AD - \frac{1}{4} \square AR}.$
 That is, in 9° , $\sqrt{\frac{1}{4} aa - \frac{1}{4} dd} : m :: m : \sqrt{\frac{1}{4} cc - \frac{1}{4} aa}.$

98. And

98. And from 97° , (*per 17. prop. 6. Elem.*)

$$\square \text{ of } \sqrt{\frac{1}{4} \square AR - \frac{1}{4} \square AG} : \sqrt{\frac{1}{4} \square AD - \frac{1}{4} \square AR} :: \square M.$$

That is, in 8° ,

$$\sqrt{\frac{1}{4} aa - \frac{1}{4} dd} : \sqrt{\frac{1}{4} cc - \frac{1}{4} aa} :: mm.$$

99. But by *Theor. 5.* in 69° of *Probl. 8. Chap. 8.* respect being had to the last preceding Diagram,

$$\square \text{ of } \sqrt{\frac{1}{4} \square AR - \frac{1}{4} \square AG} : \sqrt{\frac{1}{4} \square AD - \frac{1}{4} \square AR} :: \triangle ARO.$$

100. Therefore from 98° and 99° , (*per Axiom. 1.*) . . $\triangle ARO = \square M.$
Chap. 2. }

Which was to be Demonstr.

Therefore one Triangle is found out to solve the Problem; and by the like *Construction* and *Argumentation* another Triangle may be made upon the right line L as a *Bafe*, which latter Triangle shall have its Area and leggs correspondently equal to the Area and leggs of the first Triangle.

101. But in order to raise an Arithmetical Canon for the finding out of the third side (or *Bafe*) of the Triangle sought, first let us suppose the given quantities to be express'd by numbers, and then let the Analogy in 11° be here repeated, *viz.*

$$aa - dd \cdot 4mm :: 4mm \cdot cc - aa.$$

102. Which Analogy is reducible to this following Equation, *viz.*

$$ccaa + ddmm - aaaa = 16mmmm + ccdd.$$

103. Now if that Equation be resolved according to the Canon in 55° of *Probl. 16. Chap. 5.* this following Canon will thence arise, whereby the *Bafe* of the Triangle sought may be found out Arithmetically.

C A N O N.

$$\sqrt{\frac{1}{2}cc + \frac{1}{2}dd + \sqrt{\frac{1}{2}cccc + \frac{1}{2}dddd - \frac{1}{2}ccdd - 16mmmm}} = a.$$

Also,

$$\sqrt{\frac{1}{2}cc + \frac{1}{2}dd - \sqrt{\frac{1}{2}cccc + \frac{1}{2}dddd - \frac{1}{2}ccdd - 16mmmm}} = a.$$

An Example in Numbers, to illustrate the foregoing Resolution of *Probl. 2.*

Suppos.

104. $b = 17$ } the leggs of a plain Triangle are given; whence,
 105. $k = 10$ }
 106. $c = 27$ } the sum of the leggs is given, and
 107. $d = 7$ } the difference of the leggs is given.
 108. $mm = 84$ } the Area of the same Triangle is given.

Req. to find out the *Bafe* or third side.

Solution Arithmetical.

109. By the help of the numbers given in 106° , 107° , 108° , and of the Canon in 103° , the numbers 21 and $\sqrt{337}$ will be found out, either of which may be taken for the value of a in the foregoing Resolution, that is, the *Bafe* of a Triangle which shall have 17 and 10 for its leggs, and 84 for its Area: For as well from these three sides 21, 17 and 10, as from these, $\sqrt{337}$, 17 and 10, the Area will be found 84, (by *Canon 1.* in 77° of *Probl. 8. Chap. 8.*) And therefore the Problem propounded is solved both Arithmetically and Geometrically.

Probl. III.

The *Bafe*, Perpendicular and Rectangle of the leggs of a plain Triangle being given severally, to find out the Triangle. But the given quantities must be subject to the Determinations hereafter express'd.

Suppos.

1. b = the *Bafe* of a Triangle is given.
 2. p = the Perpendicular is given.
 3. r = the side of a Square equal to the Rectangle of the leggs is given.

Req. to find the Triangle.

4. *Vina,*

'tis possible to make a Triangle of those three lines; (per prop. 20. Elem. 1.) Suppose then it be made, and that the Triangle found out is ABC , having its Base $AB = B$, (the Base prescribed in the Problem,) and the greater leg $AC = \frac{1}{2}C + \frac{1}{2}A$, and the lesser leg $BC = \frac{1}{2}C - \frac{1}{2}A$: I say the Triangle ABC will satisfy the Problem propounded. But to render the Demonstration thereof the more clear and easie, I shall premise a few things in seven steps, next following.

27. If the given Quantities be express'd by numbers, the kind of the Triangle sought in Case 1. when the legs are unequal, as they were supposed in the foregoing Resolution, may be discovered by the help of the Canon in 19° of this Problem, and of Theor. 3. in 35° of *Prop. 8. Chap. 8.* Supposing then it be discovered that the Perpendicular falls without the Triangle ABC upon A B increased, from C as a Center at the distance of CB describe the Circle $CBGD$ cutting CA in G ; then produce AC and A B to the Circumference in D and E ; draw also the Semidiameter CE , and from the Center C make $CF \perp BE$, whence $FE = EB$, (per prop. 3. Elem. 3.)

28. Then because (per Defn. 15. Elem. 1.) $CB = CD$,
29. It follows, (by adding CA to each part,) that $CA + CB = CD + CA = AD$.
30. And because by Constr. in 26°, $\frac{1}{2}C + \frac{1}{2}A = CA$,
31. Also by Constr. in 26°, $\frac{1}{2}C - \frac{1}{2}A = CB = CD = CG$.
32. Therefore the sum of the Equations in 30° and 31° gives
33. And by subtracting the Equation in 31° from that in 30°,
34. Now I shall shew that the Triangle ABC made as before will satisfy the Problem. First then by Constr. in 26° the Base AB is equal to the prescribed Base B . Secondly, that the Rectangle of the legs AC and BC is equal to the Square of the given line R , (to wit, the prescribed Rectangle,) I shall here demonstrate.

35. . . . Reg. demonstr. $\square AC, BC = \square R$.
Demonstration.

36. By Constr. in 24° and 26°, $\square A = \square AB - 4 \square P$.
37. And from 33°, $\square A = \square AG$.
38. Therefore from 36° and 37°, (per Axiom. 1. Chap. 2.) $\square AG = \square AB - 4 \square P$.
39. And by adding $4 \square P$ to each part in 38°, $4 \square P + \square AG = \square AB$.
40. And by adding $\square AB$ to each part of the last Equation, $\square AB + 4 \square P + \square AG = 2 \square AB$.
41. But from 32°, 33° and 26°, $\square AD = 2 \square AB$.
42. Therefore from 40° and 41°, (per Axiom. 1. Chap. 2.) $\square AB + 4 \square P + \square AG = \square AD$.
43. And by subtracting $\square AG$ from each part in 42°, $\square AB + 4 \square P = \square AD - \square AG$.
44. From 32° and 33°, AD is the sum, and AG the difference of CA and CB , therefore, (per Theor. 7. Chap. 4.)
45. Therefore from 43° and 44°, (per Axiom. 1. Chap. 2.) $\square AB + 4 \square P = 4 \square AC, BC$.
46. But from 23° and 26°, $\square AB + 4 \square P = 4 \square R$.
47. Therefore from 45° and 46°, (per Ax. 1.) $4 \square AC, BC = 4 \square R$.
48. Therefore, (per Ax. 21. Chap. 2.) $\square AC, BC = \square R$.
Which was to be Demonstr.

Thirdly and lastly, that the Perpendicular CF is equal to the given Perpendicular P , I shall here demonstrate by a retrograde repetition of the steps of the preceding Resolution.

49. . . . Reg. demonstr. $CF = P$.
Demonstration.

50. By Constr. in 24°, $A = \sqrt{\square B - 4 \square P}$.
51. And it hath been shewn in 33°, that $A = AG$.
52. There-

52. Therefore from 50° and 51°, (per Axiom. 1. Chap. 2.) $AG = \sqrt{\square B - 4 \square P}$.

53. And because by Constr. in 26° $AB = B$, there fore from 52°, $AG = \sqrt{\square AB - 4 \square P}$.
That is, in 15°, $a = \sqrt{bb - 4pp}$.

54. And from 53°, by comparing the Squares of each part one to another, $\square AG = \square AB - 4 \square P$.
That is, in 14°, $aa = bb - 4pp$.

55. And by adding $4 \square P$ to each part of the Equation in 54°, $\square AG + 4 \square P = \square AB$.
That is, in 13°, $aa + 4pp = bb$.

56. And from 55°, (per prop. 7. Elem. 5.) this Analogy aritheth, viz.

$$\square AG + 4 \square P : \square AG :: \square AB : \square AG.$$

That is, in 12°,

$$aa + 4pp : aa :: bb : aa.$$

57. It is manifest that the Analogy in 56° is equal to this that follows, (for $\square AB - \square AB = 0$. Also, $4 \square P - 4 \square P = 0$.) viz.

$$\begin{array}{l} \square AG + \square AB + 4 \square P - \square AB \\ \square AG + \square AB + 4 \square P - \square AB - 4 \square P :: \end{array} \left. \begin{array}{l} \square AB \\ \square AB \\ \square AG \end{array} \right\} \text{Proportionals.}$$

That is, in 11°,

$$\begin{array}{l} aa + bb + 4pp - bb \\ aa + bb + 4pp - bb - 4pp :: \end{array} \left. \begin{array}{l} bb \\ bb \\ aa \end{array} \right\} \text{Proportionals.}$$

58. From 23° and 26°, $4 \square R = \square AB + 4 \square P$.
That is, in 10°, $4rr = bb + 4pp$.

59. Therefore from 57° and 58°, by exchanging equal quantities, this Analogy aritheth,

$$\begin{array}{l} \square AG + 4 \square R - \square AB \\ \square AG + 4 \square R - \square AB - 4 \square P :: \end{array} \left. \begin{array}{l} \square AB \\ \square AB \\ \square AG \end{array} \right\} \text{That is, in 8°, } \left\{ \begin{array}{l} aa + 4rr - bb \\ aa + 4rr - bb - 4pp :: \\ bb \end{array} \right.$$

60. But it hath been shewn in 47°, that $4 \square AC, BC = 4 \square R$.

61. Therefore by setting $4 \square AC, BC$ in the place of $4 \square R$ in the Analogy in 59°, this aritheth, viz.

$$\begin{array}{l} \square AG + 4 \square AC, BC - \square AB \\ \square AG + 4 \square AC, BC - \square AB - 4 \square P :: \end{array} \left. \begin{array}{l} \square AB \\ \square AB \\ \square AG \end{array} \right\} \text{are Proportionals.}$$

62. And because from 32° and 33°, AD is the sum, and AG the difference of AC and BC , therefore by Theor. 7. Chap. 4.

$$\square AD = \square AG + 4 \square AC, BC.$$

63. Therefore from 61° and 62°, by exchanging equal quantities, this Analogy aritheth, viz.

$$\square AD - \square AB : \square AD - \square AB - 4 \square P :: \square AB : \square AG.$$

64. And from 63°, by Conversion of Reason,
 $\square AD - \square AB : 4 \square P :: \square AB : \square AB - \square AG$.

65. And from 65°, by altern and inverse Reason,
 $\square AB : \square AB - \square AG :: \square AD - \square AB : 4 \square P$.

66. But by Theor. 4. in 68° of Probl. 8. Chap. 8.
 $\square AB : \square AB - \square AG :: \square AD - \square AB : 4 \square CF$.
Ecc 2 67. There-

27. By *Probl. 9. Chap. 5* find a mean proportional line M between the given Base B and the line D, (found out in 26°;) therefore,
 $B : M :: M : D.$

28. Then esteeming the line C to be the Base of a right-angled Triangle, and the line M to be a mean Proportional between the Hypotenuse and Perpendicular, find out (by *Probl. 15. Chap. 5.*) the Perpendicular it self, suppose it be the right line A: Then if A be the Perpendicular, and C the Base, the Hypotenuse shall be equal to $\sqrt{C^2 + A^2}$: and from that Effection this Analogy ariseth, *viz.*
 $\sqrt{C^2 + A^2} : M :: M : A.$

Which line A shall be equal to the difference of the leggs of the Triangle sought.

29. By *Probl. 2. Chap. 5.* find a right line F, such, that its Square may be equal to $4QR + CA$; therefore,
 $\square F = 4QR + CA.$

Which line F shall be equal to the sum of the leggs of the Triangle sought.
 30. Then let a Triangle be made of these three right lines, *viz.* B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$; which may be done (by *Prop. 22. Elem. 1.*) if F be greater than A; and the sum of every two of those three lines be greater than the third: First, that $F > A$, is evident from the *Constr.* in 29°. Secondly, it is also evident that the sum of B and $\frac{1}{2}F + \frac{1}{2}A$ exceeds $\frac{1}{2}F - \frac{1}{2}A$. Thirdly, that the sum of $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ exceeds the Base B, *viz.* that $F > B$, I demonstrate thus;

By *Constr.* in 29°, $4QR + CA = \square F.$
 And from the *Constr.* in 25°, $2QR - \frac{1}{2}CA = \square B.$
 But 'tis evident (per *Axiom. 25. Chap. 2.*)
 $4QR + CA < 2QR - \frac{1}{2}CA.$
 that $\square F < \square B.$
 Therefore from the three last preceding steps, (per *Axi. 3. Chap. 2.*) $F < B.$
 And consequently, $F < B.$

Which was to be Demonstr.

Fourthly and lastly, that the sum of B and $\frac{1}{2}F - \frac{1}{2}A$ is greater than $\frac{1}{2}F + \frac{1}{2}A$; *viz.* that $B < A$, the following Demonstration, formed by a repetition of the steps of the foregoing Resolution in a retrograde order, will make manifest.

31. *Req. demonstr.* $B < A.$

Demonstration.

32. Because by *Constr.* in 28°, $\sqrt{CA + C^2} : M :: M : A.$
 That is, in 18°, (the last step of the Resolution,) $\sqrt{aa + cc} : m :: m : a.$
 33. And by *Constr.* in 27°, $M : B :: D : M.$
 34. Therefore from 32° and 33°, by exchanging the mean Proportionals, (according to *Defin. 8. Chap. 3.*) this Analogy ariseth, *viz.*
 $\sqrt{CA + C^2} : B :: D : A.$
 That is, in 16°, $\sqrt{aa + cc} : b :: d : a.$
 35. But the Squares of proportional lines are also Proportionals, therefore from 34°, (per *prop. 22. Elem. 6.*)
 $\square A + \square C : \square B :: \square D : \square A.$
 That is, in 15°, $aa + cc : bb :: dd : aa.$
 36. And because by *Constr.* in 25°, $4QR - 2CB = \square C.$
 37. Also by *Constr.* in 26°, $4QR - \square B - 4QP = \square D.$
 38. Therefore from 35°, 36° and 37°, by exchanging equal quantities, this Analogy ariseth, *viz.*
 $\square A + 4QR - 2CB : \square B :: 4QR - \square B - 4QP : \square A.$

That is, in 12°, $aa + 4rr - 2bb : bb :: 4rr - bb - 4pp : aa.$

39. There-

39. Therefore from 38°, by Composition of Reason,

$\square A + 4QR - \square B : \square B :: \square A + 4QR - \square B - 4QP : \square A.$
 That is, in 9°,
 $aa + 4rr - bb : bb :: aa + 4rr - bb - 4pp : aa.$

40. But the first Proportional in 39° is evidently greater than the third, therefore (per *prop. 14. Elem. 5.*) the second shall be greater than the fourth, *viz.*
 $\square B < \square A$; therefore $B < A.$

Which was to be Demonstr.

41. Now because it hath been demonstrated that $F < A$; and consequently $\frac{1}{2}F - \frac{1}{2}A$ is equal to a real right line, and that the sum of every two of these three right lines, to wit, B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ is longer than the third, 'tis possible to make a Triangle of them (per *prop. 22. Elem. 1.*): Suppose then it be done, and that the Triangle so made is ABC, having its Base AB equal to the right line B; (the Base prescribed in the Problem;) also $AC = \frac{1}{2}F - \frac{1}{2}A$, and $BC = \frac{1}{2}F + \frac{1}{2}A$, I say the Triangle ABC will satisfy *Cafe 2. Probl. 3.* before propounded. But to render the Demonstration thereof the more easie, I shall premise a few things in seven steps next following.

42. First, if the quantities given in the Problem be express'd by numbers, the kind of the Triangle sought in *Cafe 2.* shall be known: For the Base was first given, and by what hath been said in 19°, the leggs are given also, therefore by the Corollary in 45° of *Probl. 10. Chap. 7.* it may be discovered whether the Triangle sought be obtuse-angled, acute-angled, or right-angled at the Base. Supposing then it be found that the Perpendicular falls upon the Base AB within the Triangle, from the Center C, at the distance of CB, ($= \frac{1}{2}F - \frac{1}{2}A$), the lesser leg of the Triangle ABC made as before, describe the Circle CBGD cutting CA in G; then produce AC to the Circumference in D, and draw the Semidiameter CE; and from the Center C let fall CF ⊥ EB, therefore $FE = FB$, (per *prop. 3. Elem. 3.*) Then,

43. Because (per *defn. 15. Elem. 11.*) $CD = CB = CG.$
 44. Therefore by adding AC to each part, $AD = AC + CB.$
 45. By *Constr.* in 41°, $AC = \frac{1}{2}F - \frac{1}{2}A.$
 46. Also by *Constr.* in 41°, $CB = \frac{1}{2}F + \frac{1}{2}A.$
 47. Therefore the sum of the Equations in 45° and 46° gives $AD = F = AC + CB.$
 48. And the Equation in 46° being subtracted from that in 45°, leaves $AG = A \pm AC - CB.$

Now I shall shew that the Triangle ABC, formed as before is express'd in 41°, will satisfy *Cafe 2. Probl. 3.* First then, by *Constr.* in 41° the Base AB is equal to the prescribed Base B. Secondly, that the Rectangle made of the leggs AC and BC is equal to the given Rectangle, that is, the Square of the right line R, I shall here-under demonstrate.

49. *Req. demonstr.* $\square AC, BC = \square R.$

Demonstration.

50. By *Constr.* in 29°, $\square F = 4QR + CA.$
 51. And because from 47°, $\square AD = \square F.$
 52. And from 48°, $\square AG = \square A.$
 53. Therefore from 50°, 51° and 52°, by exchanging equal quantities, $\square AD = 4QR + CA - \square AG.$
 54. And from 53°, by subtracting $\square AG$ from each part, $\square AD - \square AG = 4QR.$
 55. It hath been shewn in 47° and 48° that AD is the sum, and AG the difference of AC and BC, therefore (per *Theor. 7. Chap. 4.*)
 $\square AD - \square AG = 4 \square AC, BC.$
 56. And from 55°, by subtracting $\square AG$ from each part, $\square AD - \square AG = 4 \square AC, BC.$

57. There-

57. Therefore from 54° and 56° , (*per Axiom. 1.*) $\} 4 \square AC, BC = 4 \square R$.
Chap. 2.)
 58. Therefore from 57° , (*per Ax. 21. Chap. 2.*) $\} \square AC, BC = \square R$.
 Which was to be Demonstr.

Thirdly and lastly, that the Perpendicular CF is equal to the given Perpendicular P, the following Demonstration, form'd out of the steps of the preceding Resolution by a repetition of its steps in a backward order, will make manifest.

59. . . . *Req. demonstr.* CF = P.

Demonstration.

- It hath been proved in 39° , that
 60. $\square A + 4 \square R - \square B = \square B :: \square A + 4 \square R - \square B = 4 \square P = \square A$.
 That is, in 9° ,
 $aa + 4rr - bb :: bb :: aa + 4rr - bb - 4pp :: aa$.
 61. And because from 41° and 48° , . . . $\square AB = \square B$; and $\square AG = \square A$.
 62. Therefore from 60° and 61° , by exchanging equal quantities,
 $\square AG + 4 \square R - \square AB = \square AB :: \square AG + 4 \square R - \square AB = 4 \square P = \square AG$.
 63. Therefore from 62° , by alternate Reason,
 $\square AG + 4 \square R - \square AB = \square AG + 4 \square R - \square AB = 4 \square P :: \square AB = \square AG$.
 That is, in 8° ,
 $aa + 4rr - bb :: aa + 4rr - bb - 4pp :: bb :: aa$.
 64. It hath been proved in 53° , that $\square AD = \square AG + 4 \square R$.
 65. Therefore from 63° and 64° , by exchanging equal quantities,
 $\square AD - \square AB = \square AD - \square AB - 4 \square P :: \square AB = \square AG$.
 66. Therefore from 65° , by Conversion of Reason,
 $\square AD - \square AB = 4 \square P :: \square AB = \square AG - \square AG$.
 67. And from 66° , by Altern and Inverse Reason,
 $\square AB = \square AG - \square AG :: \square AD - \square AB = 4 \square P$.
 68. But by *Theor. 4.* in 68° of *Probl. 8.*
 $\square AB = \square AG - \square AG :: \square AD - \square AB = 4 \square CF$.
 69. Therefore from 67° and 68° , (*per prop. 11. Elem. 5.*)
 $\square AD - \square AB = 4 \square CF :: \square AD - \square AB = 4 \square P$.
 70. Therefore from 69° , (*per prop. 14. Elem. 5.*) . . . $\} 4 \square CF = 4 \square P$.
 71. Therefore from 70° , (*per Ax. 21. Chap. 2.*) . . . $\} \square CF = \square P$.
 72. But the sides of equal Squares are also equal, therefore from 71° , $\} CF = P$.
 Which was to be Demonstr.

73. An Arithmetical Canon to find out the difference of the legs of the Triangle sought in *Case 2.* may be deduced from the Analogy in 12° , for by comparing the Rectangle of the extremes to the Rectangle of the means of that Analogy, this ariseth, *viz.*

$$aaaa + 4rraa - 2bbaa = 4rrbb - bbbb - 4ppbb.$$

Which Equation being resolv'd according to the Canon in 77° of *Probl. 15. Chap. 5.* will give this following

C A N O N.

$$74. . . . \sqrt{(2)} : \sqrt{4rrrr - 4bbpp + bb - 4rr} = a.$$

An Example in Numbers, to illustrate the foregoing Resolution of *Case 2. Probl. 3.*

Suppos.

75. $b = 14$ the Base of a Triangle is given.
 76. $p = 12$ the Perpendicular is given.
 77. $rr = 195$ the Rectangle of the legs is given.
 78. $rr \square = 4bb + pp$ } (*per Suppos. in Case 2.*)
 79. $rr \square = 4bb$,

Req. to find the Triangle.

Solution Arithmetical.

80. 2 = the difference of the legs; is found out of 74° , 75° , 76° and 77° .
 81. 28 = the sum of the legs, is found out of 7° , 77° and 80° .
 82. 15 and 13 = the legs are found out of 80° and 81° , by *Theor. 9. Chap. 4.*

The

The Proof.

83. $15 \times 13 = 195$ the given Rectangle; and if

$$84. \left. \begin{array}{l} 14 = \text{the Base} \\ 13 = \text{the legs} \end{array} \right\} \text{ of a Triangle, then}$$

85. $12 =$ the Perpendicular will be found out, (*per Theor. 4. in 68° of Probl. 8. Chap. 8.*) which is the same with the given Perpendicular in 76° .

The Resolution of CASE 3. Probl. 3.

Suppos.

1. $b =$ the Base of a Triangle is given.
 2. $p =$ the Perpendicular is given.
 3. $r =$ the side of a Square equal to the Rectangle of the legs is given.
 4. $rr \square = 4bb + pp$, } (*per Suppos. in Case 3.*)
 5. $rr \square = 4bb$,

Req. to find the Triangle.

- The legs of the Triangle sought in this *Case 3.* are unequal, for the reason before given in 6° of the Resolution of *Case 2.* therefore,
 6. For the difference of the legs put a .
 7. Therefore from 3° and 6° , the Square of the sum of the } $aa + 4rr$.
 legs (*per Theor. 7. Chap. 4.*) shall be
 8. And from 1° , 2° , 4° , 6° and 7° , (*per Theor. 2. in 34° of Probl. 8. Chap. 8.*) this Analogy ariseth, *viz.* $aa + 4rr - bb :: aa + 4rr - bb - 4pp :: bb :: aa$.
 9. Therefore alternately,

$$aa + 4rr - bb :: bb :: aa + 4rr - bb - 4pp :: aa$$

10. From 4° 'tis evident that $4rr - bb - 4pp \square = a$.
 11. Whence it follows, by adding aa to each part, . . . $aa + 4rr - bb - 4pp \square = aa$.
 12. Therefore from the Analogy in 9° , by Division of Reason, (which is evidently possible from the Equation in 11°), there are Proportionals, *viz.*
 $aa + 4rr - 2bb :: bb :: 4rr - bb - 4pp :: aa$.
 13. It is evident by the first Term of the last Analogy, that $\} aa + 4rr \square = 2bb$.
 14. And 'tis easie to infer from the *Suppos.* in 5° , that . . . $\} 2bb \square = 4rr$.
 15. Therefore from 13° , by subtracting $4rr$ from each part, $\} aa \square = 2bb - 4rr$.
 16. It is evident, that if $2bb - 4rr$ be subtracted from aa , the remainder $aa + 4rr - 2bb$ is the same with the first Term of the Analogy in 12° ; therefore if cc be put equal to $2bb - 4rr$, and $aa - cc$ be taken instead of the said first Term $aa + 4rr - 2bb$ that Analogy will be converted into this, *viz.*

$$aa - cc :: bb :: 4rr - bb - 4pp :: aa$$

17. And by putting $dd = 4rr - bb - 4pp$, the last preceding Analogy will be converted into this that follows, *viz.* $aa - cc :: bb :: dd :: aa$.
 18. But the sides of proportional Squares are Proportionals also, (*per prop. 22. Elem. 6.*) therefore,
 $\sqrt{aa - cc} :: b :: d :: a$.
 19. Between b and d find a mean Proportional, which may be called m ; therefore,
 $b :: m :: m :: d$.
 20. Therefore from 18° and 19° , by exchanging the mean Proportionals, according to *Defin. 8. Chap. 3.* concerning Inordinate Proportion, this Analogy ariseth, *viz.*

$$\sqrt{aa - cc} :: m :: m :: a$$

21. Which three continual Proportionals last express'd being well examin'd, it will appear that the greater extreme a may be esteem'd the Hypotenusal of a right-angled Triangle whose Base is e , and the Perpendicular is $\sqrt{aa - cc}$: Now in that right-angled Triangle the Base e is given, as also m , a mean Proportional between the Hypotenusal a and the Perpendicular $\sqrt{aa - cc}$: therefore the Hypotenusal, which is equal to the difference of the legs of the Triangle sought in *Case 3.* shall be given also, (*per Probl. 15. Chap. 5.*) Then the Rectangle and difference of the legs being given severally, the sum of the legs shall be given also, (*by Probl. 1. Chap. 9.*) And lastly, the sum and difference of the legs being given, the legs shall be given severally, (*by Theor. 9. Chap. 4.*)

F i f

This

This third Case needs not any other Determination than what is implied in the Suppositions, and the Composition may be formed (by the help of *Probl. 15. Chap. 5.*) in like manner as before in *Case 2.*

22. But for the Learner's fuller satisfaction, an Arithmetical Canon to find out the value of a , to wit, the difference of the legs of the Triangle sought in *Case 3.* is deducible from the premises: For the Rectangle of the extremes of the Analogy in 12° being compared to the Rectangle of the means, and it being observed (as is easie to infer from the *Suppos.* in 5°) that $2bb \square 4rr$, this following Biquadratick Equation ariseth, *viz.*

$$aaaa - 2bb = 4rr \times aa = 4rrbb - bbbb - 4ppbb.$$

Which Equation, if it be resolved according to the Canon in 58° of *Probl. 15. Chap. 5.* gives this

$$C A N O N.$$

$$23. \dots a = \sqrt{bb - 2rr + \sqrt{4rrrr - 4ppbb}} = \text{the difference of the legs.}$$

An Example in Numbers, to illustrate the precedent Resolution of *Case 3. Probl. 3.*

Suppos.

24. $b = 10185$ the Base of a Triangle is given.
 25. $p = 4752$ the Perpendicular is given.
 26. $rr = 50307696$ the Rectangle of the legs is given.
 27. $rr \square \frac{1}{2}bb + pp$, } agreeable to the *Suppos.* in *Case 3.*
 28. $rr \square \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

29. 5529 = the difference of the legs is found out of 23° , 24° , 25° and 26° .
 30. 15225 = the sum of the legs is found out of 7° , 26° and 29° .
 31. $\begin{cases} 10377 \\ 48485 \end{cases}$ = the legs are found out of 29° and 30° , (by *Theor. 9. Chap. 4.*)

The Proof.

32. $10377 \times 4848 = 50307696$ the given Rectangle: And, if
 $\begin{cases} 10185 \\ 10377 \end{cases}$ = the Base } of a Triangle; then, by those three sides,
 33. $\begin{cases} 10377 \\ 48485 \end{cases}$ = the legs }
 34. 4752 = the Perpendicular will be found out, (per *Theor. 4.* in 68° of *Probl. 8.* *Chap. 8.*) which is the same with the Perpendicular given in 25° .

The Resolution of CASE 4. Probl. 3.

Suppos.

1. b = the Base of a Triangle is given.
 2. p = the Perpendicular is given.
 3. r = the side of a given Square equal to the Rectangle of the legs.
 4. $rr \square \frac{1}{2}bb + pp$, } (per *Suppos.* in *Case 4.*)
 5. $rr \square \frac{1}{2}bb$, }
Req. to find the Triangle.
 6. For the difference of the legs sought put $\dots > a$.
 7. Then proceed as before in the Resolution of *Case 3.* from the 6^{th} step to the 12^{th} , and let the Analogy in that 12^{th} step be here repeated, *viz.*
 $aa + 4rr - 2bb : bb :: 4rr - bb - 4pp : aa$.
 8. Now because by *Suppos.* in 5° , $\dots > 4rr = 2bb$.
 7. And consequently from 8° , by subtracting bb from each part, $> 4rr - bb = bb$.
 10. Therefore 'tis evident from 8° and 9° , that by exchanging equal quantities the Proportionals in 7° are reducible to these, *viz.*
 $aa : bb :: bb - 4pp : aa$.
 11. But the sides of proportional Squares are also Proportionals, therefore from 10° ,
 $a : b :: \sqrt{bb - 4pp} : a$.

12. There-

12. Therefore from 11° , by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation ariseth, *viz.*

$$aa = b\sqrt{bb - 4pp}$$

13. Therefore from 12° , by extracting the square Root out of each part, the difference of the legs sought is made known, *viz.*

$$a = \sqrt{b \times \sqrt{bb - 4pp}}$$

14. And because by *Theor. 7. Chap. 5.* the Square of the sum of any two right lines (or numbers) is equal to four times the Rectangle together with the Square of their difference; therefore from 3° , 6° , 12° and 13° , the sum of the legs sought is also made known, *viz.*

$$\sqrt{aa + 4rr} = \sqrt{b\sqrt{bb - 4pp} + 4rr}$$

The two last preceding Equations give this following

$$C A N O N.$$

15. $\dots \begin{cases} \sqrt{b\sqrt{bb - 4pp}} \\ \sqrt{b\sqrt{bb - 4pp} + 4rr} \end{cases}$ = the difference of the legs.
 = the sum of the legs.

Then the sum and difference of the legs being given, the legs shall be given severally by *Theor. 9. Chap. 4.*

By which Canon and Resolution foregoing, the Geometrical Effect and Demonstration of *Case 4.* is very easie to be made, and therefore I shall leave the same to the Learner's practice.

An Example in Numbers, to illustrate the foregoing Resolution of *Case 4. Probl. 3.*

Suppos.

16. $b = 5$ the Base of a Triangle is given.
 17. $p = 2$ the Perpendicular is given.
 18. $rr = 12\frac{1}{2}$ the Rectangle of the legs is given.
 19. $rr \square \frac{1}{2}bb + pp$, } agreeable to the *Suppos.* in *Case 4.*
 20. $rr \square \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

21. $\sqrt{15}$ = the difference of the legs is found out of 15° , 16° and 17° .
 22. $\sqrt{65}$ = the sum of the legs is found out of 15° , 16° , 17° and 18° .
 23. $\begin{cases} \sqrt{15} + \sqrt{65} \\ \sqrt{15} - \sqrt{65} \end{cases}$ = the legs, found out of 21° and 22° , (per *Theor. 9. Chap. 4.*)

The Proof.

24. $\dots \sqrt{15} + \sqrt{65}$ into $\sqrt{15} - \sqrt{65} = 12\frac{1}{2}$ the given Rectangle.
 5 = the Base of a Triangle.
 25. And, if $\begin{cases} \sqrt{15} + \sqrt{65} \\ \sqrt{15} - \sqrt{65} \end{cases}$ the Legs,
 26. Then 2 = the Perpendicular will be found out, (per *Theor. 4.* in 68° of *Probl. 8.* *Chap. 8.*) which is the same with the given Perpendicular in 17° .

The Resolution of CASE 5. Probl. 3.

Suppos.

1. b = the Base of a Triangle is given.
 2. p = the Perpendicular is given.
 3. r = the side of a Square equal to the Rectangle of the legs is given.
 4. $rr \square \frac{1}{2}bb + pp$, agreeable to the preceding *Suppos.* in *Case 5.*
Req. to find the Triangle.

FFF 2

S. For

5. For the difference of the legs sought put a .
 6. Then from 3° and 5° , (*per Theor. 7. Chap. 4.*) the } $aa + 4rr$.
 Square of the sum of the legs is
 7. And from 1° , 2° , 5° and 6° , (*per Theor. 2. in 34° of Probl. 8. Chap. 8.*) this Analogy
 arith. *viz.*
 $aa + 4rr - bb : aa + 4rr - bb - 4pp :: bb : aa$
 8. Therefore alternly,
 $aa + 4rr - bb : bb :: aa + 4rr - bb - 4pp : aa$
 9. Therefore inverfly,
 $aa : aa + 4rr - bb - 4pp :: bb : aa + 4rr - bb$
 10. By quadrupling all in 4° , and subtracting $4bb + 4pp$ } $4rr - bb - 4pp \Rightarrow 0$.
 from each part, it will be evident that
 11. And by adding aa to each part in 10° , } $aa + 4rr - bb - 4pp \Rightarrow aa$
 12. Therefore from 9° , by Conversion of Reason, (which the 11° step shews is possible,) these are Proportionals, *viz.*
 $aa : bb + 4pp - 4rr :: bb : 2bb - 4rr - aa$
 13. It is evident from the *Suppos.* in 4° , that } $bb + 4pp \sqsubset 4rr$.
 14. Therefore for the excess whereby $bb + 4pp$ ex- } $dd = bb + 4pp - 4rr$.
 ceeds $4rr$ we may put dd , whence
 15. By viewing the 11° step it will appear that the } $aa + 4rr - bb \Rightarrow bb$.
 third Proportional in 8° is less than the fourth, there-
 fore the first is less than the second, *viz.*
 16. Whence, by adding bb to each part, } $aa + 4rr \Rightarrow 2bb$.
 17. And from 16° , by comparing the latter part to } $2bb \sqsubset 4rr$.
 the first, 'tis easie to infer that
 18. Therefore for the excess whereby $2bb$ exceeds $4rr$, } $cc = 2bb - 4rr$.
 we may put cc , whence
 19. Then from the Analogy in 12° , by exchanging } $aa : dd :: bb : cc - aa$.
 equal quantities according to the Positions in 14°
 and 18° , this Analogy arith. *viz.*
 20. And because (*per prop. 22. Elem. 6.*) the sides of } $a : d :: b : \sqrt{cc - aa}$.
 proportional Squares are also Proportionals, there-
 fore from 19° ,
 21. Between b and d find a mean Proportional, which } $b : m :: m : d$.
 may be called m , therefore
 22. Therefore from 20° and 21° , by exchanging the } $a : m :: m : \sqrt{cc - aa}$.
 mean Proportionals according to *Defin. 8. Chap. 3.*,
 this Analogy arith. *viz.*
 23. Which last Analogy doth evidently consist of three continual Proportionals, whereof
 the extremes a and $\sqrt{cc - aa}$ may be esteem'd the Base and Perpendicular of a right-
 angled Triangle whose Hypotenuse c is given, as also m a mean Proportional between
 the Base and Perpendicular; therefore (*per Probl. 16. Chap. 5.*) the Base and Perpen-
 dicular shall be given severally, either of which may be taken for the difference of the legs
 of a Triangle to satisfy the Problem propounded in *Case 5*. For here, two different
 Triangles may be always found out that shall have these three things common, to wit,
 the Base, the Perpendicular, and Rectangle of the legs, except when it happens that
 $\frac{bp}{r} = r$, for then the two values of a will be equal to one another, in which Case there
 can only one Triangle be found out to agree with the preceding *Suppos.* in 1° , 2° , 3°
 and 4° .

But that there may be a possibility of finding out a Triangle to solve *Probl. 3.* in *Case 5*.
 the right line represented by $\frac{bp}{r}$ must not be longer than r . Now to make it evident
 that this Determination is necessary in *Case 5*,

24. *Req. demonstr.* $\frac{bp}{r}$ not $\sqsupset r$. *Demonstr.*

Demonstration.

25. First, if c be given for the Hypotenuse, and
 m for a mean Proportional between the Base
 and Perpendicular of a right-angled Triangle,
 (as before is supposed in 23°), then to find out
 that Triangle by *Probl. 16. Chap. 5.* this De-
 termination is necessary, *viz.* $\frac{mm}{c}$ not $\sqsupset \frac{1}{2}c$.
 26. Therefore from 25° , by multiplying each } mm not $\sqsupset \frac{1}{2}c$.
 part into c , } $bd = mm$.
 27. But from 21° , (*per prop. 17. Elem. 6.*) } bd not $\sqsupset \frac{1}{2}c$.
 28. Therefore from 26° and 27° , (*per Ax. 4.*) } $\sqrt{bb + 4pp - 4rr} = d$.
 29. And because (as is evident in 14°), } $\sqrt{bb + 4pp - 4rr} : not \sqsupset \frac{1}{2}c$.
 30. Therefore from 28° and 29° , by exchanging } $bb - 2rr = \frac{1}{2}cc$.
 equal Factors, } $\sqrt{bb + 4pp - 4rr} : not \sqsupset \frac{1}{2}c$.
 31. And because from 18° , } $\sqrt{bb + 4pp - 4rr} : not \sqsupset \frac{1}{2}c$.
 32. Therefore from 30° and 31° , (*per Ax. 3.*) } $\sqrt{bb + 4pp - 4rr} : not \sqsupset \frac{1}{2}c$.
 33. And from 32° , (by dividing each part by b), } $\sqrt{bb + 4pp - 4rr} : not \sqsupset \frac{1}{2}c$.
 34. And from 33° , by squaring each part, } $bb + 4pp - 4rr$ not $\sqsupset \frac{1}{2}c$.
 35. And from 34° , by adding $4rr$ unto; and sub- } $4pp$ not $\sqsupset \frac{4rrrr}{bb}$.
 tracting bb from each part, } $2p$ not $\sqsupset \frac{2rr}{b}$.
 36. And from 35° , by extracting the square Root } $2p$ not $\sqsupset \frac{2rr}{b}$.
 out of each part, } $2p$ not $\sqsupset \frac{2rr}{b}$.
 37. And from 36° , by multiplying each part by b , } bp not $\sqsupset rr$.
 38. And by halving each part in 37° , } bp not $\sqsupset rr$.
 39. Therefore from 38° , by dividing each part } $\frac{bp}{r}$ not $\sqsupset r$. Which was to be Dem.
 by r ,
 40. Supposing then that $\frac{bp}{r}$ is not greater than r , the Composition of *Case 5. Probl. 3.*
 may be easily formed out of the last preceding Resolution, by the help of *Probl. 16.*
Chap. 5. in like manner as before in *Case 2.* and an Arithmetical Canon to find the
 difference of the legs of the Triangle sought in the said *Case* may be deduced from
 the Analogy in the foregoing 12° step: For the Rectangle of the extremes of that
 Analogy being compared to the Rectangle of the means, this following Biquadratick
 Equation arith. *viz.*
 $2bb - 4rr \times aa - aaaa = bbbb + 4ppbb - 4rrbb$

Which Equation being resolved according to the Canon in 55° of *Probl. 16. Chap. 5.*
 gives this

$CANON$

41. $a = \sqrt{bb - 2rr + \sqrt{4rrrr - 4ppbb}}$: Also, $a = \sqrt{bb - 2rr - \sqrt{4rrrr - 4ppbb}}$:
 42. Now either of those Roots or values of a may be taken for the difference of the legs
 of a Triangle to solve *Case 5. Probl. 3.* before propounded, and if it happens that $\frac{bp}{r}$
 $\Rightarrow r$, (and consequently, $bp \Rightarrow rr$) then those values of a are unequal, and there
 may be two different Triangles found out to satisfy the said *Case 5*. But when
 $\frac{bp}{r} = r$, and consequently, $bp = rr$, then the said values of a are equal to one another,
 each being equal to $\sqrt{bb - 2rr}$: which shall be the difference of the legs of a Tri-
 angle having a right angle opposite to the Base, (as appears by Canon 1. in 21° of
Probl. 14. Chap. 8.) Which Triangle, when $\frac{bp}{r} = r$, is the only Triangle that can
 be found to solve *Case 5. Probl. 3.*

Exemplus

Examples in Numbers, to illustrate the preceding Resolution of Case 5. Probl. 3.

Examp. 1. Where two Triangles are found out to solve Case 5.

Suppos.

43. $b = 51$ the Base of a Triangle is given.
 44. $p = 12$ the Perpendicular is given.
 45. $rr = 740$ the Rectangle of the legs is given.
 46. $rr \rightarrow \frac{1}{2}bb + pp$, agreeable to the Suppos. in Case 5.
 Req. to find the Triangle.

Solution Arithmetical.

47. $17 =$ the difference of the legs is found out of $43, 44$ and 45 , by the lesser Root in 41° .
 48. $57 =$ the sum of the legs is found out of $6^\circ, 45^\circ$ and 47° .
 49. 37 and $20 =$ the legs are found out of 47° and 48° , (per Theor. 9. Chap. 4.)

The Proof.

50. $37 \times 20 = 740$ the given Rectangle.
 51. And if $\begin{cases} 51 = \text{the Base} \\ 37 = \text{the legs} \end{cases}$ of a Triangle; then,
 52. $12 =$ the Perpendicular will be found out of 51 , (per Theor. 4. in 68° of Probl. 8. Chap. 8.) which is the same with the given Perpendicular in 44° .
 Again, from the same things given as before in $43, 44, 45$, another Triangle may be found out by the greater Root in 41° , to solve Case 5. Probl. 3. the sides of which latter Triangle are here-under exprest, viz.

53. $\sqrt{1953} =$ the difference of the legs.
 54. $\sqrt{4913} =$ the sum of the legs.
 55. $\sqrt{\frac{1}{2}bb} + \sqrt{\frac{1}{2}pp} =$ the legs.
 56. $\sqrt{\frac{1}{2}bb} - \sqrt{\frac{1}{2}pp} =$

The Proof is easy to be made, in like manner as in Example 1.

Examp. 2. Where only one Triangle can be found out to solve Case 5. Probl. 3.

57. $b = 169$ the Base of a Triangle is given.
 58. $p = 60$ the Perpendicular is given.
 59. $rr = 10140$ the Rectangle of the legs is given.
 60. $rr \rightarrow \frac{1}{2}bb + pp$, agreeable to the Suppos. in Case 5.
 Req. to find the Triangle.

Solution Arithmetical.

61. $91 =$ the difference of the legs is found out of $57, 58$ and 59 , by either of the Roots in 41° .
 62. $221 =$ the sum of the legs is found out of $6^\circ, 59^\circ$ and 61° .
 63. 156 and $65 =$ the legs are found out of 61° and 62° , (per Theor. 9. Chap. 4.)

The Proof.

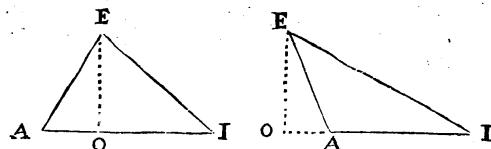
64. $156 \times 65 = 10140$ the Rectangle given in 59° .
 65. And if $\begin{cases} 169 = \text{the Base} \\ 65 = \text{the legs} \end{cases}$ of a Triangle, then,
 66. $60 =$ the Perpendicular will be found out of 65 , (per Theor. 4. in 68° of Probl. 8. Chap. 8.) which is equal to the given Perpendicular in 58° .

Note. This last Triangle hath a right angle opposite to the Base, agreeable to what was before hinted in 42° .

A L E M M A, leading to the following Probl. 4.

In any plain Triangle, As the Sine of an angle is to the Radius, (or total Sine;) So is the double Area of the Triangle, (that is, the Rectangle made of the Perpendicular and Base,) to the Rectangle of the legs containing the said angle.

Suppos.



Suppos.

1. EI the Base } of the oblique-angled $\triangle AIE$.
 2. $\begin{cases} AE \\ AI \end{cases}$ the legs }
 3. $\angle A$, (that is, $\angle EAI$) is contain'd under the legs AE, AI .
 4. $EO \perp AI$.
 5. $R =$ the Radius, (or total Sine.)
 6. $S \angle A =$ the Sine of the angle A .
 7. . . . Req. demonstr. $S \angle A . R :: EO, AI :: AE, AI$.

Demonstration.

8. By a vulgar Axiom in the Doctrine of plain } $S \angle A . R :: EO . AE$.
 Triangles this Analogy is manifest, viz. }
 9. Therefore by drawing AI into each of the two } $S \angle A . R :: EO, AI :: AE, AI$.
 latter Terms of that Analogy, this arithm. viz. }
 Which was to be Demonstr.

Probl. IV.

The Base of a plain Triangle being given, as also the Perpendicular, and angle opposite to the Base, to find the Triangle.

Construction.

Let a Circle be described by a Radius (or Semidiameter) taken at pleasure, and according to the Note at the beginning of Probl. 19. Chap. 8. find out a right line that shall be the Sine of an angle equal to the given angle. Then (by Probl. 9. Chap. 5.) find out a Square equal to a Rectangle made of the given Base and Perpendicular. That done, let it be made, (by Probl. 11. Chap. 5.) As the Sine, (found out as above,) to the Radius first assumed; So the said Square to another Square, which Square (or fourth Proportional) found out, shall be equal to the Rectangle of the legs containing the given angle, (as is evident by the foregoing Lemma.) Now there is given the Base and Perpendicular, as also a Square equal to the given Rectangle of the legs, to find out the Triangle; and therefore if those given quantities be exprest by numbers, this Probl. 4. may be solved both Geometrically and Arithmetically in all Cases, by the help of the preceding Probl. 3.

Note. Some subtil Geometrical Problems, wherein the measures of angles as well as of right lines are given in numbers, may be solved Arithmetically by the Doctrine of Plain Triangles, without the help of Algebra; of such kind is the following Problem, with which I shall conclude this Treatise.

Probl. V.

The distances AB, AC, BC between three Towers A, B and C not standing in a straight line, being given severally in Feet. Also a fourth Tower being suppos'd to stand within the Triangle ABC , as at D ; and the measures of the angles ADB, BDC and CDA being given severally in Degrees; to find the distance between the fourth Tower D and each of the other three, viz. the measures of the three right lines DB, DA and DC in Feet.

Prepar.

Forasmuch as by Suppos. the point D lies within the Triangle ABC , and consequently the three points A, D, B do not lye in a straight line; let the Circumference of a Circle be suppos'd to pass by those points, as ADB , whose Center is E ; suppose also the Semi-

Semi-

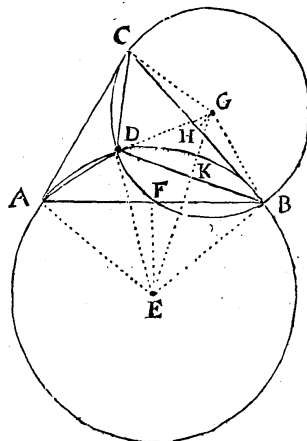
Semidiameters EA, ED, EB to be drawn, and $EF \perp AB$. In like manner, supposing G to be the Center of a Circle whose Circumference passeth by the points B, D, C, draw the Semidiameters GB, GD, GC, and make $GH \perp CB$.

Feet.

AB	=	20000
AC	=	15000
BC	=	18000
EB	=	12521.286 +
GB	=	9317.445 +
DB	=	13888.610 +
DA	=	8283.832 +
DC	=	8406.944 +

Gr. Min.

ADB	=	127: 0
BDC	=	105: 0
GDA	=	128: 0
FEB	=	53: 0
EBA	=	37: 0
HGB	=	75: 0
GBC	=	15: 0
CBA	=	46: 8
GBE	=	98: 8
GEB	=	33: 41
DBA	=	19: 19
DBC	=	26: 49



Solution Arithmetical, by the Doctrine of plain Triangles.

First, subtract the given $\angle ADB$ from two right angles, (*viz.* from 180. Degrees.) the remainder shall be the sum of the unknown angles DAB and DBA, (*per prop. 32. Elem. 1.*)

Secondly, forasmuch as (by *prop. 20. Elem. 3.*) $\angle DEB = 2\angle DAB$, and $\angle DEA = 2\angle DBA$; it follows that $\angle AEB = 2\angle DAB + 2\angle DBA$; therefore in $\triangle FEB$ right-angled at F, the $\angle FEB$ (that is, $\frac{1}{2}\angle AEB$) $= \angle DAB + \angle DBA$ is given; and by *Suppos.* $FB = \frac{1}{2}AB$ is given, therefore the Semidiameter $EB = ED = EA$ shall be given also.

Thirdly, by arguing as above in the first and second steps $\angle HGB = \angle HGC$ is given; also $GD = GC = GB$ the Semidiameter of the Circle GBC is given.

Fourthly, because $EF \perp AB$ and $\angle FEB$ is given as before, therefore $\angle EBA$ the Complement of the $\angle FEB$ to a right angle is given; likewise the $\angle GBC$ the Complement of $\angle HGB$ to a right angle is given; and the $\angle CBA$ is given, for it may be found out by the three given sides AB, AC, BC ; therefore $\angle GBE$ the sum of those three angles, EBA, CBA, GBC is given.

Fifthly, in $\triangle GBE$, the sides GB and EB , (to wit, the Semidiameters of the two Circles GBC and EAD), are given severally, as also the angle GBE comprehended by those sides, therefore the angle GEB is given also.

Sixthly, because the two Triangles EGB and EGD have two sides GB, EB equal to the two sides GD, ED , *viz.* $GB = GD$, and $EB = ED$, also the Base GE common to both those Triangles; the angles contain'd under equal right lines shall be equal, *viz.* $\angle GEB = \angle GED = \frac{1}{2}\angle DEB$: But $\angle GEB (= \frac{1}{2}\angle DEB)$ is given in the fifth step, and (*per prop. 20. Elem. 3.*) $\angle DAB$ is equal to $\frac{1}{2}\angle DEB (= \angle GEB)$, therefore $\angle DAB$ is given. Now in $\triangle ADB$ there is given $\angle DAB$, as also $\angle ADB$ and the side AB , therefore the sides DB and DA , (to wit, two of the Distances sought,) are given also.

Seventhly and lastly, in $\triangle DAC$ there is given DA , as also AC , and $\angle ADC$, therefore DC (the third Distance sought,) is given.

The End of the Fourth and Last BOOK.

S O L I D E O G L O R I A.